THE PHYSICS OF FLUIDS

VOLUME 11, NUMBER 12 DECEMBER 1968

Conservation of Circulation Theorem for Superfluids

P. G. SAFFMAN
California Institute of Technology, Pasadena, California
(Received 3 June 1968)

The reversible two-fluid equations of helium II are shown to possess two integrals of circulation. The first, which expresses conservation of circulation in the superfluid, is well known, but the second, which expresses conservation of the line integral of $\rho_n(\mathbf{u}_n - \mathbf{u}_s)/(\rho s)$ around a circuit moving with the normal fluid, appears to be new.

The Landau equations for the two-fluid model of nondissipative motion of liquid helium II can be written in the form

$$\frac{\partial \rho_n}{\partial t} + \operatorname{div}(\rho_n \mathbf{u}_n) = \Gamma; \qquad \frac{\partial \rho_s}{\partial t} + \operatorname{div}(\rho_s \mathbf{u}_s) = -\Gamma; \tag{1a, b}$$

$$\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s = -\nabla\mu + \mathbf{g}; \tag{2}$$

$$\frac{\partial \mathbf{u}_n}{\partial t} + (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n = -\nabla\mu - \frac{\rho s}{\rho_n}\nabla T - \tfrac{1}{2}\nabla(\mathbf{u}_n - \mathbf{u}_s)^2 - \frac{\Gamma}{\rho_n}(\mathbf{u}_n - \mathbf{u}_s) + \mathbf{g}; \tag{3}$$

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \mathbf{u}_n) = 0. \tag{4}$$

For derivations of these equations, see Landau and Lifshitz[1] for an outline, or Donnelly,[2] or Khalatinikov[3] for details. Here, ρ_n and ρ_s are the densities of the normal and superfluids, respectively; $\rho = \rho_n + \rho_s$; $\mathbf{u}_n$ and $\mathbf{u}_s$ are their velocities; μ and s are the chemical potential and specific entropy of the entire fluid and T is the temperature. These dynamic relations are supplemented by thermodynamic relations

$$\mu = e + \frac{p}{\rho} - Ts - \frac{\rho_n}{\rho}(\mathbf{u}_n - \mathbf{u}_s)^2, \tag{5}$$

$$d\mu = \frac{dp}{\rho} - s\,dT - \frac{1}{2}\frac{\rho_n}{\rho}\,d(\mathbf{u}_n - \mathbf{u}_s)^2, \tag{6}$$

where p is the pressure and e is the specific internal energy. All the physical quantities are to be regarded as known functions (to be determined by a microscopic theory) of *three* independent variables of state; say p, T, and $c^2 = (\mathbf{u}_n - \mathbf{u}_s)^2$ which is also a variable of state in the two fluid model. The vector g is the force per unit mass due to external causes; in particular, gravity.

Equation (2) is usually written in the form (neglecting gravity)

$$\frac{\partial \mathbf{u}_s}{\partial t} = -\nabla(\mu + \tfrac{1}{2}\mathbf{u}_s^2), \tag{7}$$

because the superfluid motion is supposed to be irrotational, i.e.,

$$\boldsymbol{\omega}_s = \operatorname{curl}\mathbf{u}_s \equiv 0, \tag{8}$$

except possibly at isolated singularities (quantized vortex lines). However, the pointwise irrotationality of the superfluid motion is not a necessary condition for Landau's derivation of the two-fluid equations. It is straightforward (although laborious) to show that the arguments work just the same if (7) is replaced by (2); and the irrotationality of the superfluid is not a macroscopic requirement on Landau's equations. Therefore, we shall suppose for the present that (8) is not necessarily satisfied. [The quantum-mechanical arguments adduced to justify (8) do not yet carry complete conviction.]

[1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Ltd., London, 1959), Sec. 130.
[2] R. J. Donnelly, *Experimental Superfluidity* (University of Chicago Press, Chicago, Illinois, 1967), Sec. 16.
[3] I. M. Khalatinikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), Sec. 8.

Taking the curl of Eq. (2), we obtain

$$\frac{\partial \boldsymbol{\omega}_s}{\partial t} + (\mathbf{u}_s \cdot \nabla)\boldsymbol{\omega}_s + \boldsymbol{\omega}_s \operatorname{div} \mathbf{u}_s = (\boldsymbol{\omega}_s \cdot \nabla)\mathbf{u}_s, \qquad (9)$$

if the external force field is constant or conservative. It follows (see Whitham[4]) that if $\boldsymbol{\omega}_s \equiv 0$ at any instant, then $\boldsymbol{\omega}_s$ remains zero for all times. More generally, from (2) we can deduce by the standard argument used to prove Kelvin's circulation theorem, that

$$\frac{d}{dt} \oint_S \mathbf{u}_s \cdot d\mathbf{r} = 0, \qquad (10)$$

where S is any closed circuit moving with the superfluid, so that the flux of $\boldsymbol{\omega}_s$ threading S remains constant. These results are trivial and well known, although it does seem worth emphasizing again that the two-fluid equations, using (2) instead of (7), predict the irrotationality of the superfluid for motion started from rest by conservative forces, so that (7) may be regarded as a consequence of zero viscosity.

However, it seems to have escaped notice that the equation for the normal fluid gives rise to another circulation theorem. We introduce a vector $\mathbf{w}$ defined by

$$\mathbf{w} = \frac{\rho_n}{\rho S}(\mathbf{u}_n - \mathbf{u}_s), \qquad (11)$$

which is the momentum density of the normal fluid relative to the superfluid divided by the entropy density. By straightforward manipulation of Eqs. (1)–(4), we find

$$\frac{\partial \mathbf{w}}{\partial t} + (\operatorname{curl} \mathbf{u}_s) \times \mathbf{w} + (\operatorname{curl} \mathbf{w}) \times \mathbf{u}_s + \frac{\rho S}{\rho_n}(\operatorname{curl} \mathbf{w}) \times \mathbf{w} = -\nabla\left(T + \frac{\rho S}{\rho_n}\mathbf{w}^2 + \mathbf{u}_s \cdot \mathbf{w}\right). \qquad (12)$$

Equation (12) is obtained by subtracting (2) from (3) and using the result

$$\frac{\partial}{\partial t}\left(\frac{\rho S}{\rho_n}\right) = -\Gamma \frac{\rho S}{\rho_n^2} - \mathbf{u} \cdot \nabla\left(\frac{\rho S}{\rho_n}\right), \qquad (13)$$

which follows from (1a) and (4).

Taking the curl of (12) gives an equation for ∂ (curl $\mathbf{w})/\partial t$. This is a long equation which we do not write down, as it is clear from inspection that every term contains either curl $\mathbf{w}$ or $\boldsymbol{\omega}_s$. Thus, if $\boldsymbol{\omega}_s$ and curl $\mathbf{w}$ are both identically zero at any instant, then they remain zero at all subsequent times. In other words, for motion started from rest by conservative forces, there exist scalars α and β such that

$$\mathbf{u}_s = -\nabla\alpha, \qquad \frac{\rho_n}{\rho S}(\mathbf{u}_n - \mathbf{u}_s) = -\nabla\beta. \qquad (14)$$

The existence of the scalar β was predicted by Zilsel[5] (see also London[6]) in the course of a derivation of the Landau equations from a generalization of Eckart's variational principle, and provided the motivation of the present study. (The statement by London[6] that there is a slight difference between Zilsel's final equations and those of Landau appears to be without foundation, as the final equations are identical. It should be noted, however, that London's definitions of chemical potential and internal energy differ from Landau's by an amount $\rho_n c^2/2\rho$ for the chemical potential and $-\rho_n c^2/2\rho$ for the internal energy. It is clear that the variational principle does not necessarily give the most general hydrodynamic flow, but gives those that can be produced from rest by conservative forces. In some cases, the limitation can be removed by generalizing the variational principle, but such extensions are on shaky ground unless there is independent confirmation of the resulting equations.)

In motion inside containers, the result (14) is not useful because the effects of viscosity may be cumulative, and it is not a valid assumption to regard the motion as being started from rest by conservative forces. However, if the viscous terms are small, they are confined to boundary layers and shear layers, outside of which viscous or dissipative effects can be neglected. It is, therefore, worth looking for the analog of Kelvin's circulation theorem (10), since it is well known in classical fluid dynamics that Kelvin's theorem can often give useful results about the vorticity injected by boundary layers into an otherwise inviscid flow.

Consider the "circulation" N of the vector $\mathbf{w}$ around a circuit N which is moving with an arbitrary velocity $\mathbf{V}$,

$$N = \oint_N \mathbf{w} \cdot d\mathbf{r}. \qquad (15)$$

Then

$$\frac{dN}{dt} = \oint_N \left(\frac{d\mathbf{w}}{dt} \cdot d\mathbf{r} + \mathbf{w} \cdot \frac{d}{dt}(d\mathbf{r})\right), \qquad (16)$$

[4] G. B. Whitham, in *Laminar Boundary Layers*, L. Rosenhead, Ed. (Clarendon Press, Oxford, England, 1963), Chap. III.

[5] P. R. Zilsel, Phys. Rev. **79**, 309 (1950).

[6] F. London, *Superfluids* (Dover Publications, Inc., New York, 1954), Vol. II, Sec. 19.

where in the integrand

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla, \quad \text{and} \quad \frac{d}{dt}(d\mathbf{r}) = (d\mathbf{r}\cdot\nabla)\mathbf{V} \tag{17}$$

and $\mathbf{V}$ is so far an arbitrary function of position. Using (12), we have

$$\frac{d}{dt}N = -\oint_N \boldsymbol{\omega}_s \times \mathbf{w}\cdot d\mathbf{r} - \oint_N (\operatorname{curl}\mathbf{w}) \times \mathbf{u}_n \cdot d\mathbf{r} + \oint_N \mathbf{w}\cdot(d\mathbf{r}\cdot\nabla)\mathbf{V} + \oint_N d\mathbf{r}\cdot(\mathbf{V}\cdot\nabla)\mathbf{w}. \tag{18}$$

Now,

$$\mathbf{w}\cdot(d\mathbf{r}\cdot\nabla)\mathbf{V} + d\mathbf{r}\cdot(\mathbf{V}\cdot\nabla)\mathbf{w} - (\operatorname{curl}\mathbf{w})\times\mathbf{V}\cdot d\mathbf{r} \equiv d\mathbf{r}\cdot\nabla(\mathbf{V}\cdot\mathbf{w}). \tag{19}$$

Hence,

$$\frac{d}{dt}N = -\oint_N [\boldsymbol{\omega}_s \times \mathbf{w} - (\operatorname{curl}\mathbf{w})\times(\mathbf{V} - \mathbf{u}_n)]\cdot d\mathbf{r}. \tag{20}$$

In particular, if we choose a circuit which moves with the normal fluid velocity $\mathbf{u}_n$,

$$\frac{d}{dt}N = -\oint_N (\boldsymbol{\omega}_s \times \mathbf{w})\cdot d\mathbf{r}. \tag{21}$$

This is the analog for the normal fluid of the circulation theorem (10) for the superfluid. Remember that the velocity for the "normal circulation" is not $\mathbf{u}_n$ but the combination $\mathbf{w}$, which does not seem, however, to have any direct physical interpretation. [The velocity $\mathbf{V}$ can be chosen so that the square bracket in (20) is the gradient of a scalar and N is conserved, but this does not seem to be a natural approach.] Incidentally, Eq. (21) is the Kelvin theorem for motion of a classical fluid relative to a rotating coordinate system with angular velocity $\frac{1}{2}\boldsymbol{\omega}_s$.

Now if $\boldsymbol{\omega}_s \equiv 0$, as would be the case if superfluid vorticity could not be generated in any way, we would have that the circulation of $\mathbf{w}$ is conserved around circuits moving with the normal fluid, provided the circuits are outside boundary layers or other regions of high dissipative effects. If there are quantized vortex lines in the flow, (21) will hold only for circuits N that do not cross the quantized vortex lines in the superfluid.

There are two Bernoulli integrals if the flow is "doubly irrotational." From (7) and (14),

$$-\frac{\partial\alpha}{\partial t} + \mu + \tfrac{1}{2}(\nabla\alpha)^2 = F(t) \tag{22}$$

and from (12) and (14)

$$-\frac{\partial\beta}{\partial t} + T + (\mathbf{u}_n\cdot\mathbf{w}) = G(t), \tag{23}$$

where F and G are arbitrary functions of time.

These integrals can be used to simplify calculations in which the nonlinear terms in the inviscid Landau equations are taken into account. Examples are second sound in a heat current or wave motions of finite amplitude. For more general flows, one faces the difficulty of whether the Landau equations are valid when the nonlinear terms are not small, but this is outside the scope of the present discussion.