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ON THE INDEX OF REFRACTION OF SPATIALLY PERIODIC PLASMA

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ABSTRACT

A knowledge of the change produced in the index of refraction of a uniform plasma by the spontaneous generation of coagula or inhomogeneities is essential to the use of electromagnetic waves as a diagnostic tool. The general problem is a difficult one to handle, but certain non-trivial cases are mathematically tractable. One of these, which is also of some practical import, occurs when the inhomogeneities are periodically distributed throughout the plasma. Here this special case is analyzed within the framework of the theory of periodic structures. The problem is reduced by virtue of Floquet's theorem to an equivalent problem for the domain of a unit cell with periodic boundary conditions. An approximate solution is obtained by a simplified theory. As a specific application the calculation for a plasma with periodically spaced spherical inhomogeneities is worked out in detail.

1. INTRODUCTION

When a macroscopic electromagnetic wave propagates through a uniform plasma it behaves as though the plasma were a homogeneous dielectric medium. In an otherwise uniform plasma if there is a coagulum of finite extent, the effects of such an inhomogeneity on an incident wave can be accounted for by considering the coagulum as a scatterer. Likewise if there are several coagula of this type, located arbitrarily, the effects may in principle be calculated by again considering the inhomogeneities as scattering centers. In these cases the effects of the coagula on the wave propagation are best described in terms of scattering cross-sections. However, if the plasma is spatially periodic throughout a region large compared to the wavelength, a more convenient measure of the effects of the inhomogeneities is the equivalent index of refraction, which is not a measure of the local properties of the plasma but in a sense an over-all macroscopic average of the individual scattering effects.

In this paper we shall briefly outline a general method of deducing the equivalent index of refraction of a spatially periodic plasma which is an adaptation of the techniques used in the theory of wave propagation in periodic structures. Also we shall use the method to determine the equivalent index of refraction of a plasma with periodically spaced spherical coagula.

2. SPATIALLY PERIODIC PLASMA

In the M.K.S. system of units the dielectric constant of a uniform plasma is given by

$$\epsilon = \epsilon_o(1 - \omega_p^2/\omega^2) \tag{1}$$

where ω is the angular frequency of a plane electromagnetic wave being propagated through the plasma, and ω_p is the plasma frequency defined by

$$\omega_p^2 = Ne^2/(m\epsilon_o) \tag{2}$$

where N denotes the number of electrons per unit volume, e the electronic charge, m the electron mass, and ϵ_o the dielectric constant of free space. The permeability μ of the plasma is equal to the permeability μ_o of free space. Hence the index of refraction of the uniform plasma is given by

$$\eta = \sqrt{\epsilon/\epsilon_o} = \sqrt{1 - Ne^2/(m\epsilon_o\omega^2)} \tag{3}$$

For an inhomogeneous plasma, i.e., a plasma for which N varies from point to point, these expressions are only locally valid. However, if N is a periodic function of the space coordinates, the plasma in a certain mean again behaves as a uniform dielectric medium and is describable by a spacewise constant index of refraction called the "equivalent index of refraction".

As an example of a spatially periodic plasma consider

$$N = N_o \sin^2(\pi x/T_x) \sin^2(\pi y/T_y) \sin^2(\pi z/T_z) \tag{4}$$

where N_o is a constant and T_x, T_y, and T_z are the spatial periods respectively in the x, y and z directions of a cartesian coordinate system. Then from equations 1 and 2 it follows that

$$\epsilon(xyz) = \epsilon_o \left[1 - \frac{N_o e^2}{m\epsilon_o \omega^2} \sin^2(\frac{\pi x}{T_x}) \sin^2(\frac{\pi y}{T_y}) \sin^2(\frac{\pi z}{T_z}) \right] \quad . \tag{5}$$

This expression gives the local dielectric constant as a periodic function of the spatial coordinates. That is,

$$\epsilon(x + mT_x, y + nT_y, z + pT_z) = \epsilon(x,y,z) \tag{6}$$

where n, m, p = 0, 1, 2, 3, 4, etc. The periodic distribution (4) divides the space into identical cells. Indeed, when N is any spatially periodic function, ϵ is spatially periodic and space can be divided into identical cells with one coagulum to each cell. That is, the plasma constitutes a periodic medium.

3. BLOCH WAVES

Since the medium is periodic, Floquet's theorem is applicable, and the problem can be spacewise limited to the domain of one cell. Within the cell the fields must satisfy Maxwell's equations with $\mu = \mu_o$ and $\epsilon = \epsilon(x,y,z)$, and on the walls of the cell they must satisfy periodic boundary conditions. From Floquet's theorem it follows that the solution must have the form of a Bloch wave[1]

$$\psi = \exp(i\, \underline{k} \cdot \underline{r})\, \phi(\underline{k},\underline{r}) \tag{7}$$

where $\underline{k}$ is the wave vector and $\underline{r}$ the position vector. The function ϕ is spatially periodic and has the period of the medium. The equivalent

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[1] L. Brillouin, Wave Propagation in Periodic Structures (McGraw-Hill Book Company, Inc., New York, 1946).

index of refraction η_{eq} can be found simply from a knowledge of $\underline{k}$. In principle, this calculation can be performed exactly when $\epsilon(x,y,z)$ is such as to permit a separation of the variables. In practice, however, an exact solution is forbiddingly difficult to achieve without the aid of a high-speed computer. For most purposes an approximate solution obtained by a technique such as Slater's extension of the method of Wigner and Seitz is quite satisfactory[2]. In the case of present interest where the coagula are spherical (as coagula produced by shock waves sometimes are) and only the dominant-mode wave is of physical import, the technique can be further simplfied to the calculation of the dipole moment of a plasma sphere of a certain charge density surrounded by a uniform plasma of different charge density.

4. SIMPLIFIED THEORY

The dipole moment of a dielectric sphere of radius a and dielectric constant ϵ_1, when immersed in a uniform dielectric medium of dielectric constant ϵ_2 is

$$\underline{p} = 4\pi\epsilon_2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} a^3 \underline{E} \tag{8}$$

where $\underline{E}$ is the ambient electric field. If there are δ such spheres per unit volume, the dipole moment per unit volume is

$$\underline{P} = \delta\, \underline{p} = 4\pi\epsilon_2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} a^3 \delta\, \underline{E} \quad . \tag{9}$$

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[2] J. C. Slater, Physical Review 45, 794 (1934).

Hence the displacement vector $\underline{D}$ is given by

$$\underline{D} = \epsilon_2 \underline{E} + \underline{P} = \epsilon_2 (1 + 4\pi \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} a^3 \delta) \underline{E} \quad . \tag{10}$$

From this follows the equivalent dielectric constant ϵ_{eq} of the medium

$$\epsilon_{eq} = \epsilon_2 (1 + 4\pi \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} a^3 \delta) \tag{11}$$

and in turn the equivalent index of refraction

$$\eta_{eq} = \sqrt{\epsilon_{eq}/\epsilon_o} = \left[\frac{\epsilon_2}{\epsilon_o} (1 + 4\pi \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} a^3 \delta) \right]^{1/2} \quad . \tag{12}$$

Clearly, this simplified theory is not limited to spherical bodies.[3]

5. PERIODICALLY SPACED SPHERICAL COAGULA

Now suppose that the coagula are plasma spheres of charge density N_1 and radius a and the background is a uniform plasma of charge density N_2. Then

$$\epsilon_1 = \epsilon_o \left[1 - N_1 e^2/(m\epsilon_o \omega^2) \right] = \epsilon_o (1 - \omega_{p_1}^2/\omega^2) \tag{13}$$

and

$$\epsilon_2 = \epsilon_o \left[1 - N_2 e^2/(m\epsilon_o \omega^2) \right] = \epsilon_o (1 - \omega_{p_2}^2/\omega^2) \tag{14}$$

Substituting expressions (13) and (14) into relation (12) we thus find the equivalent index of refraction of spherical coagula periodically spaced in a uniform plasma

$$\eta_{eq} = \left[(1 - \frac{\omega_{p2}^2}{\omega^2})(1 + 4\pi \frac{\omega_{p_1}^2 - \omega_{p_2}^2}{2\omega_{p2}^2 + \omega_{p1}^2 - 3\omega^2} a^3 \delta) \right]^{1/2} \tag{15}$$

where $\omega_{p1} > \omega_{p2}$ or $N_1 > N_2$.

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[3] C. H. Papas, Lectures on Microwave Theory (California Inst. of Technology, Pasadena, 1952).