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Dependence on Reynolds Number of High-Order Moments of Velocity Derivatives in Isotropic Turbulence

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The results of a heuristic model of the fine scale structure of homogeneous turbulence are presented. Predictions are made about the way in which flatness and skewness factors of arbitrary order depend upon the Reynolds number of the turbulence.

It has been recognized for some time that the fine scale structure of high Reynolds number turbulence is spatially intermittent,[1] and that the statistical properties of the fine scale motion may be dependent on the Reynolds number. Several heuristic models have been proposed to predict the average values of turbulent energy fluctuations and similar quantities.[2-5] Experimental evidence is now beginning to accumulate which might soon enable a choice to be made between the various heuristic models, and the purpose of this note is to place on record the predictions of a model constructed several years ago by the author[6] in an attempt to develop some ideas by Townsend.[7]

This model, which assumes that intermittency is vital to an understanding of the fine scale structure and that a physical model can be given for the dissipation process, leads to the predictions that for $r \ll \lambda$,

$$(\Delta u_p)_{2n} \equiv \langle (u_p' - u_p)^{2n} \rangle = u^{2n} f_{2n}(r/l) R_\lambda^{-1}, \qquad (1)$$

$$(\Delta u_p)_{2n+1} \equiv \langle (u_p' - u_p)^{2n+1} \rangle = u^{2n+1}(r/\lambda) f_{2n+1}(r/l) R_\lambda^{-1}, \qquad (2)$$

for the difference between the components u_p', u_p of velocity in the parallel direction at two points a distance r apart. Here, u denotes $\langle \mathbf{u}^2 \rangle^{1/2}$, $\lambda = (5\, \nu u^2/\epsilon)^{1/2}$ is the Taylor microscale, ϵ is the mean rate of energy dissipation per unit mass, $R_\lambda = u\lambda/\nu$, and $l = (\nu^3/\epsilon)^{1/4}$ is the Kolmogoroff length scale. The functions $f_{2n}(r/l)$ and $f_{2n+1}(r/l)$ may depend on the lack of isotropy of the large scale motion, but should be roughly the same for all kinds of turbulence. Both of these functions are proportional to $(r/l)^{2n}$ as $r \to 0$. If isotropy does hold for $r \ll \lambda$, it follows from the Kármán–Howarth equation that $f_3(r/l) \to$ const as $r/l \to \infty$. If, in addition, the skewness factor $(\Delta u_p)_3/[(\Delta u_p)_2]^{3/2}$ is independent of r when $r \gg l$, the well known $\epsilon^{2/3} r^{2/3}$ dependence of $(\Delta u_p)_2$ follows immediately.

In the model, it is asserted that the intermittent structure is related to the generation of vorticity by the stretching of vortex lines under random motion. Since solutions of $d\omega/dt = \alpha(t)\omega$, where $\alpha(t)$ is a stationary random function, have a log normal distribution as $t \to \infty$, it is expected that the vorticity strength in the regions of large dissipation will be distributed according to a log normal distribution. This suggests that the functions f_{2n} will be related to one another as for a log normal distribution, and similarly for the functions f_{2n+1}, i.e.,

$$\frac{f_{2n}}{(f_2)^n} = \exp\left[\tfrac{1}{2}(n^2 - n)\beta\right], \quad \frac{f_{2n+1}}{(f_3)^n} = \exp\left[\tfrac{1}{2}(n^2 - n)\gamma\right], \qquad (3)$$

where β and γ are functions of r/l, and $\beta = \mathrm{var}\,[\log (u_p' - u_p)^2]$. The argument for the odd moments is more subtle, and a simple interpretation of γ cannot be given.

We define

$$S_m = \frac{\langle (\partial u_1/\partial x_1)^m \rangle}{\langle (\partial u_1/\partial x_1)^2 \rangle^{m/2}}.$$

Then, it follows from (1) and (2) that

$$S_{2n} = \{f_{2n}(z)/[f_2(z)]^n\}_0 R_\lambda^{n-1}, \qquad (4)$$

$$S_{2n+1} = \{z f_{2n+1}(z)/[f_2(z)]^{n+1/2}\}_0 R_\lambda^{n-1}, \qquad (5)$$

where suffix 0 denotes that quantities are to be evaluated at $z = 0$. In particular, S_3 is the skewness and S_4 the kurtosis or flatness factor; and according to (4) and (5), S_3 is independent of Reynolds number while S_4 grows like R_λ. These predictions were also made by Tennekes[8] from a modification of Corrsin's model,[3] and are not inconsistent with the available data.

The model lacks the detail necessary to make predictions about β and γ. However, it is a plausible guess that the functions f_{2n} should be algebraic for large r, in which case β is proportional to log r for $r \gg l$. In this respect the model has properties like that of Gurvich and Yaglom.[5] However, the latter model implies that

$$S_{2n} \propto R_\lambda^{(3/4)\mu n(n-1)}, \qquad (6)$$

where μ is some constant (estimated to be about 0.4). Thus, measurements of the power law dependence of high order flatness factors should enable a choice to be made between the models.

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[1] G. K. Batchelor and A. A. Townsend, Proc. Roy. Soc. (London) **A199**, 238 (1949).
[2] A. N. Kolmogoroff, J. Fluid Mech. **13**, 82 (1962).
[3] S. Corrsin, Phys. Fluid **5**, 1301 (1962).
[4] A. M. Yaglom, Dokl. Akad. Nauk. SSSR **166**, 49 (1966) [Sov. Phys. Dokl. **11**, 26 (1966)].
[5] A. S. Gurvich and A. M. Yaglom, Phys. Fluids Suppl. **10**, S59 (1967).
[6] P. G. Saffman, in *Topics in Non-Linear Physics*, edited by N. Zabusky (Springer-Verlag, Berlin, 1968), p. 485.
[7] A. A. Townsend, Proc. Roy. Soc. (London) **A208**, 534 (1951).
[8] H. Tennekes, Phys. Fluids **11**, 669 (1968).

Functional Dependence of Drag Coefficient of a Sphere on Reynolds Number

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An argument on the drag coefficient of a sphere results in the expression $C = C_0[1 + \delta_0/(R)^{1/2}]^2$, where R is Reynolds number, $C_0\delta_0^2 = 24$, and $\delta_0 = 9.06$. This expression is in remarkable agreement with experiment for a wide range of R, i.e., $0 \leq R \lesssim 5000$.

In this note we will present a heuristic argument on the drag of a blunt body; in particular, the sphere. The argument results in a relation between the drag coefficient C and Reynolds number R given by the expression

$$C = C_0[1 + \delta_0/(R)^{1/2}]^2,$$

where $C_0\delta_0^2 = 24$ and $\delta_0 = 9.06$. Figure 1 demonstrates that this expression is in remarkable agreement with experiment for a wide range of R, i.e., $0 \leq R \lesssim 5000$.

Whenever an object is placed in a moving fluid (or moves through a stationary fluid), it will experience a force in the direction of motion of the fluid relative to the object. Dimensional analysis can be utilized to make an important assertion about the drag force D. Assuming that D will depend on the linear size of the body d, the fluid velocity V, the fluid density ρ, and the fluid viscosity μ, and that it is of the functional form

$$D \sim d^\alpha V^\beta \rho^\gamma \mu^\Psi, \qquad (1)$$

we obtain from dimensional analysis

$$D \sim \rho V^2 d^2 \left(\frac{\mu}{\rho V d}\right)^\Psi, \qquad (2)$$

where Ψ is an undetermined exponent; e.g., Cole.[1] It immediately follows that if the drag is independent of the viscosity, then $\Psi = 0$ [Eq. (1)] and from Eq. (2),

$$D = \text{const}\,(\rho V^2 d^2), \qquad (3)$$

where the constant is a number depending only on the shape of the body. Such is the case for large Reynolds numbers ($R \equiv \rho V d/\mu$) where the whole flow pattern is almost independent of the viscosity (so long as the boundary layer remains laminar).[2]

At sufficiently large Reynolds numbers, the solutions of the fluid dynamics equations are constituted so as to permit a subdivision of the flow field. These subdivisions are an external region which is expected to satisfy the equations of frictionless flow ($\mu = 0$), and a region near the body together with a wake behind it in which vorticity is inherent ($\mu \neq 0$).[1,2] The second region is called the boundary layer, and its thickness is denoted by δ. From the boundary layer theory of Tomotika,[3] McDonald[4] estimates that for a point about 80 deg from the forward stagnation point of a sphere with radius a,

$$\delta/a = \delta_0/(R)^{1/2}, \qquad \delta_0 \cong 9.06. \qquad (4)$$

In our previous discussion, we have considered certain dynamical features associated with a rigid body passing through a viscous fluid at constant velocity. Notably, we learn that fluid motion around objects may be divided into two regions: a region close to the object where frictional effects are important, and an outer region where friction may be neglected. The boundary layer thickness δ acts as a dividing line between the potential flow region where friction is negligible and the rotational flow region.

The Model: If we picture the boundary layer as part of the body and, hence, traveling with it, this new system Σ,

$$\Sigma = \text{rigid body} + \text{boundary layer}, \qquad (5)$$

may be considered a "body" passing through an inviscid fluid at constant velocity. (With this model transformation, the new body Σ has, to a first approximation, a thin boundary layer which is independent of the velocity of the rigid body.)

For a sphere of radius a, the system Σ has a dimension $d = a + \delta$ at the waist, where δ is the boundary layer thickness at the waist of the sphere. For the system Σ, the drag D must be independent of the viscosity. From Eq. (3),

$$D = C_0 \frac{\rho V^2}{2} \pi d^2 = C_0 \left(1 + \frac{\delta}{a}\right)^2 \left(\frac{\rho \pi a^2}{2} V^2\right), \qquad (6)$$