Dependence of effective slip on wall roughness in dilute polymer solutions

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A hypothesis is made for the modification of a complete turbulence model to take account of a dilute polymer concentration. The equations are integrated in the wall layer and give the effective slip $u_* \Delta B$ as functions of wall roughness and empirical polymer properties, where u_* is the friction velocity and ΔB is the change of the constant in the logarithmic law of the wall. Results in qualitative agreement with experiment are found.

I. INTRODUCTION

It seems to be generally accepted that the reduction in drag when a dilute polymer solution is in turbulent flow over a wall is due to changes in the turbulence structure close to the wall, i.e., in the logarithmic wall layer, buffer layer, and viscous sublayer. In the outer part of the wall layer, where $y_+ = u_* y / \nu \gg 1$ (u_* is the friction velocity, y is the distance from the wall, ν is the kinematic viscosity) the mean velocity $U(y)$ is asymptotic to

$$U(y) \sim u_* \left[\frac{1}{k} \ln\left(\frac{u_* y}{\nu}\right) + B \right], \tag{1}$$

where k is Kármán's constant (≈ 0.42) and B is a function of the wall roughness and the polymer concentration and properties.[1]

For Newtonian flow over rough walls, B can be obtained from Nikuradse's data[2] as a function of $z_+ = u_* z_0 / \nu$, where z_0 is the equivalent sand roughness; see Fig. 1. For $z_+ < 20$ and $z_+ > 65$, the following formulas are a good approximation:

$$B \simeq 5.5 - 0.16 z_+, \qquad z_+ < 20$$
$$B \simeq 8.5 - (1/k) \ln z_+, \qquad z_+ > 65. \tag{2}$$

The effect of wall roughness can also be interpreted in terms of an effective (negative) slip velocity $\Delta U = u_* (B - B_0)$, where $B_0 (\approx 5.5)$ is the value for a smooth wall. Then, for example, the Prandtl law for fully developed turbulent flow of Newtonian fluid can be written

$$\bar{u} = \Delta U + u_* [2.46 \ln(u_* d / \nu) + 0.29],$$

where $\bar{u}$ is the average velocity, d is the pipe diameter, and $\Delta U / u_*$ is a function of the dimensionless wall roughness z_+.

Similarly, the quantitative measure of the effect of adding polymers to turbulent flow over walls can be described by the dependence of B or effective slip ΔU on polymer properties. Since polymer addition increases B, one could describe its effect as being akin to a negative roughness. By focusing attention upon the values of B, it does not matter if the motion in question is pipe flow, channel flow, or boundary layer flow.

Since we are unable to calculate the value of B for Newtonian fluid, or indeed the value of k, from first principles it is out of the question to find B for a polymer solution where, in addition, the constitutive equa-

tion is not known with any degree of certainty. The purpose of the present work is to put forward a hypothesis about the way addition of a polymer may alter a particular complete phenomenological model of turbulence and then calculate how B depends on roughness, given the dependence on polymer properties which, in principle, could be determined by experiments on dilute polymer flow over smooth walls.

In Sec. II we shall summarize the turbulence model[3] for Newtonian flow over rough walls. The modifications to incorporate polymer effect and the results are given in Sec. III.

II. TURBULENT NEWTONIAN FLOW OVER ROUGH WALLS

The turbulence model to be employed is that introduced by the present writer.[3] In the wall layer, it reduces to

$$\left(\frac{Ae}{\omega} + \nu\right) \frac{dU}{dy} = u_*^2, \tag{3}$$

$$\alpha' \omega^2 \frac{dU}{dy} - \beta' \omega^3 + \frac{d}{dy}\left[\left(\frac{A'e}{\omega} + \nu\right) \frac{d\omega^2}{dy}\right] = 0, \tag{4}$$

$$\alpha'' e \frac{dU}{dy} - e\omega + \frac{d}{dy}\left[\left(\frac{A''e}{\omega} + \nu\right) \frac{de}{dy}\right] = 0. \tag{5}$$

The quantities e and ω are a pseudo-energy and pseudo-vorticity, respectively. The constants will be given the following values:

$$A = 2A' = 2A'' = \alpha''^2, \qquad \alpha'' = \frac{1}{3}, \qquad \beta' = \frac{5}{3}, \qquad \alpha' = \alpha'' \beta' - 2k^2.$$

The boundary conditions for these equations are that

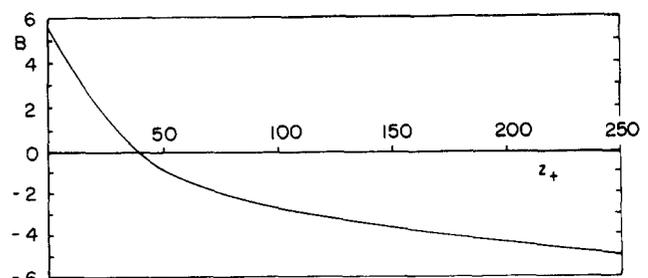$$e \sim u_*^2 / \alpha'', \qquad \omega \sim \alpha'' u_* / ky, \qquad y \to \infty$$

FIG. 1. Values of B vs nondimensional equivalent roughness height $z_+ = u_* z_0 / \nu$ from the data of Nikuradse.[2]
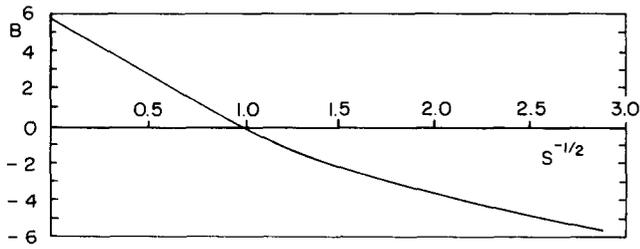
FIG. 2. Values of B vs $S^{-1/2}$ for a Newtonian fluid calculated from Eqs. (3)-(6).

together with Eq. (1). At the wall, $y = 0$,

$$U = 0, \qquad e = 0, \qquad \omega = \alpha''(u_*^2/\nu) S(z_+), \tag{6}$$

where the quantity S is a function of only the dimensionless wall roughness z_+.

These equations were solved by putting them in dimensionless form with the variables

$$\tilde{y} = \frac{ku_* y}{\nu}, \qquad \tilde{e} = \frac{\alpha'' e}{u_*^2}, \qquad \tilde{\omega} = \frac{\nu \omega}{\alpha'' u_*^2}, \qquad \tilde{U} = \frac{kU}{u_*},$$

then determining the asymptotic behavior as $\tilde{y} \to \infty$ which contains two arbitrary constants, one of which can be absorbed into the origin of $\tilde{y}$, and then shooting from a large value of $\tilde{y}$ (typically 100) toward $\tilde{y} = 0$, the wall being placed at the point where $\tilde{e} = 0$. By varying the free arbitary constant, solutions are obtained for various S and by interpolation and extrapolation the functional dependence of B on S is readily obtained.

Theoretical reasons were given earlier[3] (in terms of a "roughness layer") for believing that $S \propto z_+^{-1/2}$ as $z_+ \to 0$, and this is confirmed by the numerical solution. Similarly, the asymptotic behavior $\tilde{\omega} \sim \tilde{y}^{-1}$ as $\tilde{y} \to \infty$ suggests that $S \propto z_+^{-1}$ as $z_+ \to \infty$, and this is also found. Figure 2 shows a plot of B against $S^{-1/2}$ obtained in the way just described. The approximate formulae

$$B = 5.6 - 6S^{-1/2}, \qquad\qquad S^{-1/2} < 0.7,$$

$$B = -0.5 - (2/k)\ln S^{-1/2}, \qquad S^{-1/2} > 2.3,$$

are good descriptions for small and large $S^{-1/2}$. From Figs. 1 and 2, we can obtain an empirical relation between S and z_+, with properties that

$$S \simeq 1372 z_+^{-2} \quad \text{for } z^+ < 25$$

$$S \simeq 43.8 z_+^{-1} \quad \text{for } z^+ > 230.$$

For the ideal case of a perfectly smooth wall, $S = \infty$, the behavior of the pseudo-variables as $\tilde{y} \to 0$ is

$$\omega \sim 20k^2/\beta' \alpha'' \tilde{y}^2, \qquad \tilde{e} \propto \tilde{y}^p,$$

where $p = \frac{1}{2} + (\frac{1}{4} + 20/\beta')^{1/2} = 4$ if $\beta' = \frac{5}{3}$. We emphasize that the calculated value $B_0 = 5.6$ was found without varying any adjustable parameters.

III. MODEL EQUATIONS FOR DILUTE POLYMER SOLUTIONS

We shall assume that the change in a constitutive relation in a dilute polymer solution is an elastic effect and has no direct effect on the dissipation of turbulent energy. It differs in this respect from, say, a dusty

gas.[4] A naive picture would be that of elastic bands threading a viscous fluid. We, therefore, make the hypothesis that the change increases or generates pseudo-vorticity, this being a measure of the rotation rate of the eddies which we suppose increased by the elasticity.

Thus, we add a generation term to the pseudo-vorticity rate equation, Eq. (4), leaving the other terms unaltered. Equation (4) is then replaced by

$$\alpha'\omega^2 \frac{dU}{dy} - \beta'\omega^3 + \omega^3 f(e, \omega) + \frac{d}{dy}\left[\left(\frac{A'e}{\omega} + \nu\right)\frac{d\omega^2}{dy}\right] = 0, \tag{7}$$

where f is a dimensionless function of the turbulence quantities, e and ω, and also depends on the concentration and properties of the polymer and the viscosity of the fluid. The further hypothesis to be made is that the polymer is characterized by its concentration c and a characteristic time τ, and that the polymer produces pseudo-vorticity when the dimensionless variables $\omega\tau$ and $e\tau/\nu$ are large. For simplicity and definiteness, we take

$$f(e, \omega) = p(c)g(e\omega\tau^2/\nu), \tag{8}$$

where p is a function only of the concentration, but can also be expected to depend on molecular weight, etc., and g is a step function, $g(\xi) = 0$ for $\xi < 1$, $g(\xi) = 1$ for $\xi > 1$. It follows that the polymer will change the dependence of B on wall roughness only if $\tilde{e}\tilde{\omega} > \nu^2/u_*^4\tau^2 = \tau_+^{-2}$.

For a Newtonian fluid, $\tilde{e}\tilde{\omega}$ is zero at the wall and vanishes like $\tilde{y}^{-1}$ as $\tilde{y} \to \infty$. Furthermore, the maximum value, $(\tilde{e}\tilde{\omega})_m$, and the value $\tilde{y}_m$ at which it occurs, are practically independent of z_+ until $z_+ \approx 20$. It is found that

$$(\tilde{e}\tilde{\omega})_m \simeq 0.2, \qquad \tilde{y}_m \simeq 2 \quad \text{for } z_+ < 20.$$

As z_+ increases, $(\tilde{e}\tilde{\omega})_m$ decreases leveling off at about 0.05 for $z_+ > 200$. The value of $\tilde{y}_m$ increases to about 5 as z_+ increases. Remember that $\tilde{y} = ky_+$, where y_+ is the conventional dimensionless distance from the wall. Identifying $\tilde{y}_m$ with the thickness of the sublayer, we see that the thickness increases from $5\nu/u_*$ to $12\nu/u_*$ as the roughness increases.

Since the action of the polymer is confined to those regions near the edge of the sublayer wherein $u_*^2\tau/\nu > (\tilde{e}\tilde{\omega})^{-1/2}$, the value of $(\tilde{e}\tilde{\omega})_m$ gives the onset value of $u_*^2\tau/\nu$. That is, for $u_*(\tau/\nu)^{1/2} < (\tilde{e}\tilde{\omega})_m^{-1/4}$, there is no change in flow properties. Thus, we conclude that the critical wall stress for the onset of drag reduction is unaffected by roughness provided $z_+ < 20$. Moreover, since in this range it is found that the profiles of $\tilde{e}\tilde{\omega}$ are insensitive to z_+ except very close to the wall (i.e., $\tilde{y} < 0.1$, $\tilde{e}\tilde{\omega} < 0.02$), we conclude that the changes in B due to polymer are also insensitive to roughness. Figure 3 also shows the dependence of onset values of the dimensionless wall stress $\hat{u}_*(\tau/\nu)^{1/2}$ on the dimensionless roughness $u_* z_0/\nu$. Figure 4 plots $\Delta B = B(\tau_+, z_+) - B(0, z_+)$ against z_+ for various values of $u_*^2\tau/\nu$. These plots are for $p = 1$; similar results are obtained for other values of p. In Fig. 5 we show the dependence of B for a smooth wall and two values of p against $(u_*^2\tau/\nu - \hat{u}_*^2\tau/\nu)^{1/2}$. Theoretical considerations suggest this dependence should be linear close to onset.
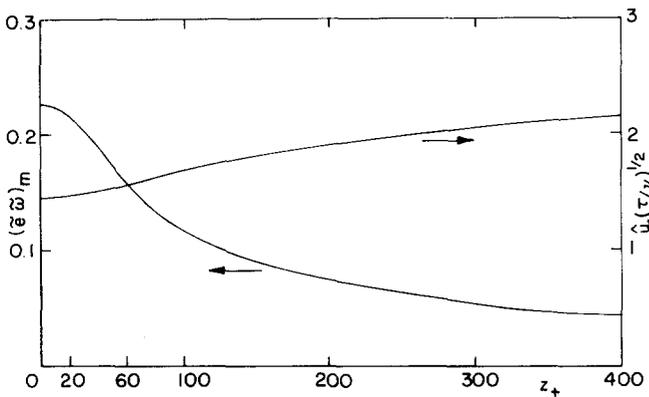
FIG. 3. Maximum value of product $\tilde{e}\tilde{\omega}$ of dimensionless pseudo energy and pseudo vorticity as function of z_+ for Newtonian fluid. Value of onset wall stress $\hat{u}_{*}$, nondimensionalized with $(\tau/\nu)^{1/2}$, vs z_+.

These results appear to be in qualitative agreement with experiment.[1] Thus, dimensionless onset shear stress depends weakly on roughness, increasing from 1.45 when $z_+ = 0$ to 1.7 for $z_+ = 100$. For values of $z_+ < 35$, the flow is effectively smooth for polymer addition, since ΔU varies by less than 10% in this range. As z_+ increases over this value, the effect of polymer becomes less, eventually disappearing at values of z_+ depending on $u_*^2\tau/\nu$. Virk[1] reports $z_+ \sim 50$ as the end of the effectively smooth regime, but refers to other workers who found values in the range 20–40. The results shown in Fig. 5 indicate that $p(c)$, the function that measures the effects of concentration, is not linear in c. In fact, the empirical observation that effective slip is proportional to $c^{1/2}$ indicates that $p \sim c^{0.2}$ is appropriate.

A feature of the numerical method employed to obtain the results obtained here was that it failed when p or τ_+ became too large, the critical values being just greater than $p = 1$ or $\tau_+ = 5$.

The system of equations is not simple, and it remains an open question whether the failure is due to the inadequacies of the numerical scheme or is a geniune mathematical property modeling a real physical phenomena,
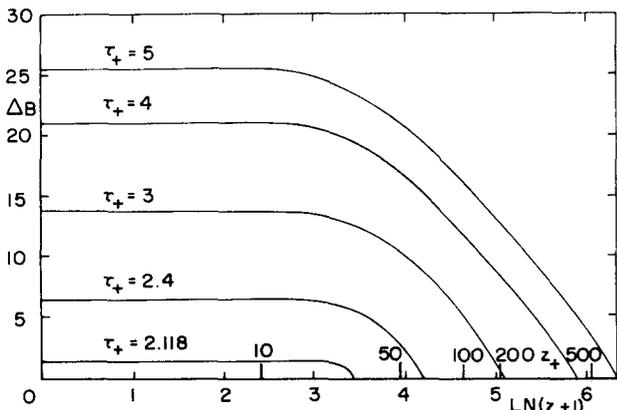

FIG. 4. ΔB vs $\ln(z_+ + 1)$ for dilute polymer solutions. The concentration function $p(c) = 1$ for all curves. Similar results are found for all $p < 1$. The curves are for constant $\tau_+ = u_*^2\tau/\nu$.
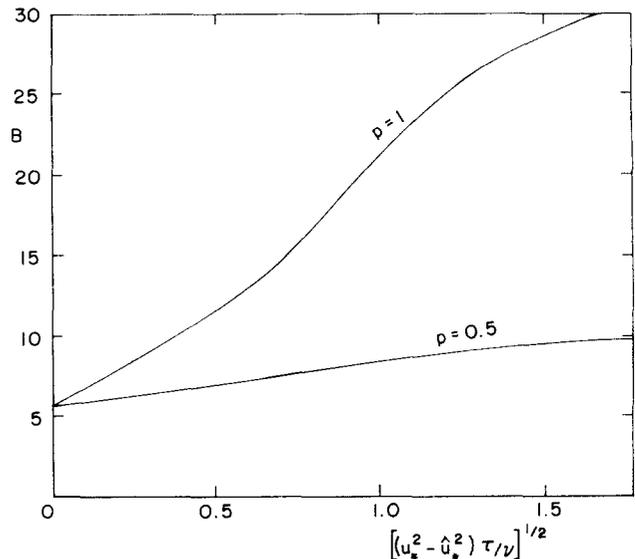

FIG. 5. B vs $[(u_*^2 - \hat{u}_*^2)\tau/\nu]^{1/2}$ for a dilute polymer solution over a perfectly smooth wall for $p = 0.5$ and $p = 1.0$.

namely, the existence of a maximum drag reduction asymptote. In this connection, we note that if $f \to f_\infty$ for large p and τ_+, then the law of the wall, Eq. (1), is replaced by

$$U \sim u_* \left(\frac{1}{k'} \ln \frac{u_* y}{\nu} + B' \right) \qquad k' = (k^2 - \tfrac{1}{2}\alpha'' f_\infty)^{1/2}. \qquad (9)$$

The equations were integrated with $f = f_\infty = 1$, and boundary condition (9), corresponding to $k' = 0.09$. B' is found to be approximately constant for $z_+ < 12$ with a value of around -9 and then starts to decrease, reaching -54 at $z_+ = 300$. Virk's[1] maximum drag asymptote for smooth walls is equivalent to experimental values of k' $= -0.08$ and $B' = -12$. The difference between theory and experiment is comparable to the uncertainty in the observed values. Note that under conditions of maximum drag reduction, the drag with large roughness may be comparable to, but will not be exactly equal to, that of the pure solvent; the change in the Kármán constant from k to k' implies a different Reynolds number dependence.

IV. REMARKS

The hypothesis that a dilute polymer increases the generation of pseudo-vorticity is intuitively consistent with the evidence presented at this Symposium that dilute polymer tends to destabilize laminar flows although contrary evidence exists. Fibrous materials, which also have drag reducing effects, probably act in a way similar to dust in a gas by increasing the dissipation, and their effect should be modeled by adding an extra dissipation term to the rate equation for pseudo-energy, Eq. (5). This would be consistent with the observation that drag reduction due to polymer and fibers is more effective when they are used in combination than when employed separately, and that addition of fibers to dilute polymer solutions gives drag reductions in excess of Virk's maximum asymptote. A test of the proposed difference between the mechanisms would be to determine

if the addition of fibers tended to stabilize laminar flows.

It should be noted that we have assumed that the surface roughness z_0 is unaffected by the polymer. The surface roughness may, in fact, be smoothed to some extent by polymer, the amount depending on the concentration but not on τ. Then, the polymer drag reduction with a small amount of roughness could actually be greater than for a smooth wall, because of the decrease in roughness in addition to the dynamical changes.

The present model does not describe the changes in turbulent intensity through the wall layer which have been observed.[5] For this purpose, it would be necessary to use a more sophisticated stress relaxation model which calculates all components of the Reynolds stress tensor.[3,6]

An empirical description of drag reduction for dilute polymer solutions in rough pipes has been given by Poreh.[7] The object of the present paper is completely different, being a fundamental investigation to see if the phenomenon of drag reduction can be predicted or included in a complete turbulence model (which does not make *ad hoc* assumptions about the variation of mixing length in the sublayer).

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