# Stability of a plane soliton to infinitesimal two-dimensional perturbations

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(Received 22 December 1977; final manuscript received 16 May 1978)

The stability of two-dimensional infinitesimal disturbances of soliton solutions of the two-dimensional nonlinear Schrödinger equation is considered. Previous results for small transverse wavenumber are extended numerically to arbitrary values.

We are interested in the evolution of a weakly nonlinear deep-water wavetrain in two space dimensions. The wavetrain has a carrier wave vector $\mathbf{k}_0 = (k_0, 0)$ and a carrier frequency $\omega_0$, and is subject to slowly varying, two-dimensional modulations in wave vector, frequency, and amplitude. Zakharov[1] showed that the slowly varying envelope of the modulated wavetrain satisfies the two-dimensional nonlinear Schrödinger equation

$$i\left(\frac{\partial\psi}{\partial t} + \frac{\omega_0}{2k_0}\frac{\partial\psi}{\partial x}\right) - \frac{\omega_0}{8k_0^2}\frac{\partial^2\psi}{\partial x^2} + \frac{\omega_0}{4k_0^2}\frac{\partial^2\psi}{\partial y^2} - \tfrac{1}{2}\omega_0 k_0^2|\psi|^2\psi = 0\,, \tag{1}$$

where $x$ and $y$ are horizontal spatial coordinates, $t$ is time, and the complex envelope $\psi$ is related to the elevation $\eta(x, y, t)$ of the free surface by the expression

$$\eta(x, y, t) = \mathrm{Re}[\psi(x, y, t)\exp(ik_0x - i\omega_0 t)]\,. \tag{2}$$

For oblique plane modulations, $\psi$ is a function of $z$ and $t$, where $z$ is defined by

$$z = x\cos\alpha + y\sin\alpha\,. \tag{3}$$

Equation (1) can then be written as

$$i\left(\frac{\partial\psi}{\partial t} + \frac{\omega_0}{2k_0}\cos\alpha\frac{\partial\psi}{\partial z}\right) - \frac{\omega_0}{8k_0^2}(1 - 3\sin^2\alpha)\frac{\partial^2\psi}{\partial z^2} - \tfrac{1}{2}\omega_0k_0^2|\psi|^2\psi = 0 \tag{4}$$

which is the one-dimensional nonlinear Schrödinger equation. For $\alpha < \sin^{-1}(1/3)^{1/2} = 35.26°$, (4) possesses soliton solutions of the form

$$\psi(z, t) = b_0\,\mathrm{sech}\left[\frac{k^2b_0}{(1-3\sin^2\alpha)^{1/2}}\left(z - \frac{\omega_0}{2k_0}\cos\alpha\; t\right)\right] \times\exp[-(i/4)\,\omega_0k_0^2b_0^2t] \tag{5}$$

(see Fig. 1). For $\alpha > 35.26°$, no steady solutions which decay as $|z| \to \infty$ exist.

It has been shown by Vakhitov and Kolokolov[2] that the envelope soliton solution (5) is stable to one-dimensional perturbations independent of the coordinate normal to $z$. We discuss its stability to infinitesimal two-dimensional perturbations. We define

$$T = -\omega_0 t\,,\quad X = 2k_0\left(x - \frac{\omega_0}{2k_0}t\right),\quad Y = 2k_0y\,,\quad A = \frac{k_0\psi}{\sqrt{2}} \tag{6}$$

and in terms of $A$, $X$, $Y$, and $T$, rewrite (1) as

$$i\frac{\partial A}{\partial T} + \frac{1}{2}\frac{\partial^2 A}{\partial X^2} - \frac{\partial^2 A}{\partial Y^2} + |A|^2A = 0\,. \tag{7}$$

Without loss of generality, we consider the case of $\alpha = 0$, since all other values of $\alpha < 35.26°$ can be recovered by a simple geometrical scaling. The soliton solution is

$$A^{(0)}(X, T) = \phi^{(0)}(X)\exp(i\gamma^2T)\,, \tag{8}$$

where

$$\phi^{(0)}(X) = \sqrt{2}\gamma\,\mathrm{sech}(\sqrt{2}\gamma X)\,. \tag{9}$$

We examine two-dimensional perturbations of the form

$$A^{(1)}(X, Y, T) = \phi^{(1)}(X)\exp(iKY - i\Omega T)\,,$$

where $A^{(1)}$ is small compared with $A^{(0)}$.

Zakharov and Rubenchik[3] showed that the real and imaginary parts of $\phi^{(1)}$, denoted by $u$ and $v$, respectively, satisfy the following fourth order, non-self-adjoint eigenvalue problem

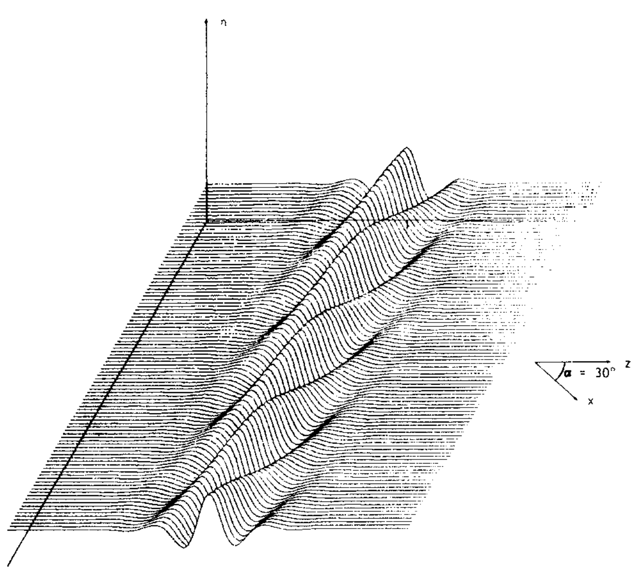


FIG. 1. Free surface corresponding to a plane oblique envelope soliton making an angle of 30° with the carrier wave vector.

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$$(L_0 - K^2)u = \Omega v \ ,$$
$$(L_1 - K^2)v = \Omega u \ , \quad -\infty < X < \infty \ , \tag{10}$$

where

$$L_0 = -\frac{1}{2}\frac{d^2}{dX^2} + \gamma^2 - \phi^{(0)2} \ ,$$
$$L_1 = -\frac{1}{2}\frac{d^2}{dX^2} + \gamma^2 - 3\phi^{(0)2} \ , \tag{11}$$

and the eigenfunctions vanish as $|X| \to \infty$.

They examined the system analytically for $K$ and $\Omega$ small, and demonstrated by ingenious arguments the existence of two branches of eigenvalues starting from $K = \Omega = 0$. These branches correspond to even and odd modes. The even mode is stable with $\Omega^2 > 0$; the odd mode is unstable with $\Omega^2 < 0$. They developed $\Omega^2$ as a series in $K^2$ and found the coefficients of $K^2$ and $K^4$. Extrapolation based on the leading terms suggested the existence of a most unstable transverse perturbation wavenumber $K_{max}$ and a cutoff wavenumber $K_c$.

We have extended their results numerically by computing solutions of (10) for arbitrary values of $K$ and have also searched for other possible branches with $\Omega^2$ real.

By examining the system (10) for large $X$, we can analytically demonstrate the existence of a continuous spectrum (i.e., improper eigenfunctions) for

$$\Omega^2 > 0 \ , \qquad K^2 > \gamma^2 \ ,$$
$$\Omega^2 > (\gamma^2 - K^2)^2 \ , \ K^2 < \gamma^2 \ .$$

In the remaining part of the $(K^2, \Omega^2)$ plane, we obtain two branches corresponding to those found by Zakharov and Rubenchik, with properties in qualitative agreement with their predictions. We find that for the odd unstable mode

$$K^2_{max} = 0.65\ \gamma^2 \ , \ |\mathrm{Im}\Omega|_{max} = 0.66\ \gamma^2 \ ,$$

and

$$K_c^2 = 1.18\ \gamma^2 \ .$$

The even stable mode bends back and apparently connects with the continuous spectrum at $K = 0$. The results are shown in Fig. 2.

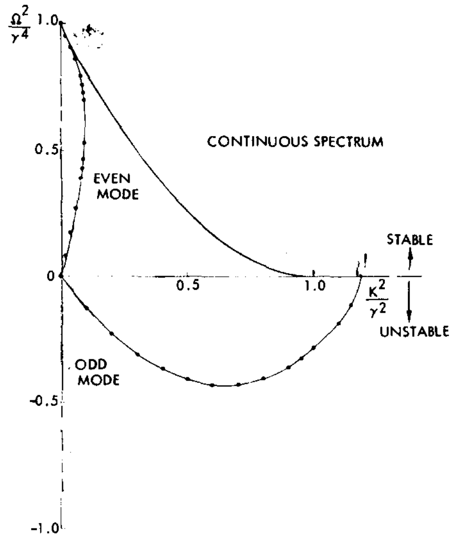


FIG. 2. Stability diagram for a plane soliton subject to infinitesimal two-dimensional perturbations. The stability curves are obtained by joining the numerical results shown by the dots.

Vakhitov and Kolokolov[2] proved that for $K^2 = 0$, $\Omega^2$ must be real and nonnegative and their results suggested the possible existence of discrete eigenvalues for $K^2 = 0$, different from $\Omega = 0$, and branches emanating from these points. Our numerical results indicate that no such discrete eigenvalues exist. The possibility, however, of complex eigenvalues for $K^2 > 0$ cannot be ruled out at present.


This work was sponsored by the Applied Physics Laboratory of The Johns Hopkins University under Contract APL/JHU No. 600613.



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