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The touching pair of equal and opposite uniform vortices

P. G. Saffman and S. Tanveer
Department of Applied Mathematics, California Institute of Technology, Pasadena, California 91125

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The shape and speed of a pair of touching finite area vortices are calculated and an error in previous work corrected.

Pierrehumbert[1] presents steady-state solutions for two-dimensional finite area vortex pairs of equal and opposite uniform vorticity in an inviscid incompressible fluid. He claims that as the gap between the vortices decreases, the shapes approach a limit in which the pair touch along the axis of symmetry with a cusp at each end. The purpose of this letter is to point out that Pierrehumbert's analysis is incomplete and that there exist solutions without cusps, the vortex boundary being perpendicular to the axis as sketched in Fig. 1, although the curvature is infinite. We have recalculated the shapes and find, however, that apart from the behavior near the axis, our shape and his are in good agreement (see Fig. 2) and the calculated speeds also agree to two significant figures.

The flow can be reduced to rest by superposing a velocity U equal and opposite to the speed of the pair. The axis of symmetry is a streamline and it is sufficient to consider the flow in the upper half-plane. We shall also assume fore and aft symmetry. (The existence of nonsymmetrical solutions is an open question.) To consider the shape near the end A of the axis of symmetry, take polar coordinates r, θ centered on A. The stream function ψ satisfies $\nabla^2\psi = 0$ in region I and $\nabla^2\psi = -\omega$ in region II. Take the local expansions in the form

$$\omega^{-1}\psi_{\rm I} = (2\pi)^{-1}(r^2 \ln r \sin 2\theta + \theta r^2 \cos 2\theta) + \alpha r^2 \sin 2\theta + \beta_1\left(\frac{r^2 \ln r \sin 2\theta - \theta r^2 \cos 2\theta}{(\ln r)^2 + \theta^2}\right) + O(r^2/(\ln r)^2), \quad (1)$$

$$\omega^{-1}\psi_{\rm II} = (2\pi)^{-1}(r^2 \ln r \sin 2\theta + \theta r^2 \cos 2\theta) - \tfrac{1}{2} r^2 \sin^2\theta - \tfrac{1}{2} r^2 \cos 2\theta + \alpha r^2 \sin 2\theta + \beta_2\left(\frac{r^2 \ln r \sin 2\theta - \theta r^2 \cos 2\theta}{(\ln r)^2 + \theta^2}\right) + O(r^2/(\ln r)^2). \quad (2)$$

It is easily verified that $\psi_{\rm I}$ and $\psi_{\rm II}$ satisfy the differential equations. Further $\psi_{\rm I} = 0$, $\psi_{\rm II} = 0$ on $\theta = 0, \pi$, respectively. The velocity has to be continuous on the boundary of the vortex, i.e., $\psi_{\rm I} = \psi_{\rm II} = 0$ and $(\partial\psi_{\rm I}/\partial\theta) = (\partial\psi_{\rm II}/\partial\theta)$ when $\theta = \Theta(r)$ say (continuity of pressure is then automatically satisfied). A little algebra shows that these equations are satisfied to $O(r^2/\ln r)$ accuracy if the boundary is taken to be

$$\Theta(r) = \pi/2 - \pi/(4 \ln r) + \gamma/(\ln r)^2 + \ldots, \quad (3)$$

provided $\beta_2 - \beta_1 = \pi/8$ and $\gamma = \pi/8 + \frac{1}{2}\alpha\pi^2$. The assumption that the leading order term for $\psi_{\rm I}$ and $\psi_{\rm II}$ is $r^2 \ln r \sin 2\theta$ can be checked by local expansion around A of $w(z)$ given in Eq. (5), provided the slope of the boundary at A is nonzero. Pierrehumbert's argument for the nonexistence of solutions with nonzero slope overlooked the possibility of logarithmic terms.

The equation to be solved for the determination of the vortex boundary is

$$\mathrm{Im}[w(z)] = 0, \quad (4)$$

where

$$w(z) = -Uz - \frac{\omega}{4\pi}\int_C [(z - z')\ln(z - z') + (z + z')\ln(z + z')]\,d\overline{z}' + [(z - \overline{z}')\ln(z - \overline{z}') + (z + \overline{z}')\ln(z + \overline{z}')]\,dz', \quad (5)$$

and C is the anticlockwise contour in the first quadrant of Fig. 1. Deem and Zabusky[2] first used a similar formula to determine vortex boundaries. Equation (4) is a nonlinear in-

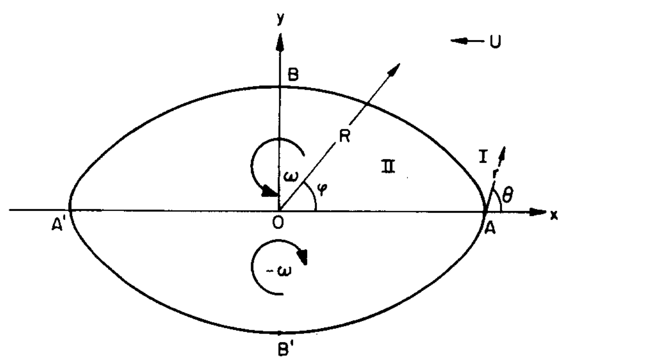

FIG. 1. Sketch of flow geometry and coordinate system for touching vortex pair.

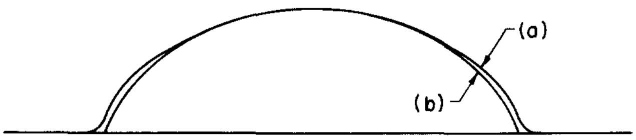

FIG. 2. Comparison of Pierrehumbert's[1] solution (a) and present solution (b) for vortices of the same width.

tegral equation for the vortex boundary. The only length scale in the problem is (U/ω). Given U and ω, the flow and vortex boundary are completely determined and in particular the area S of each vortex is given by

$$S = kU^2/\omega^2, \tag{6}$$

where k is a constant to be determined.

We introduce polar coordinates R, φ centered on 0 (Fig. 1). We constructed numerical solutions by writing R as:

$$R(\bar{\varphi}) = \sum_{n=0}^{N-1} R_n \cos(2n\bar{\varphi}), \tag{7}$$

where $\bar{\varphi}$ is related to the actual angle φ by

$$\varphi = \bar{\varphi} - 0.5\sin(2\bar{\varphi}). \tag{8}$$

Equation (4) was then satisfied at the points $\bar{\varphi}_j = j\pi/2N$; $j = 1,2,...,N$. The stretching Eq. (8) concentrated the collocation points close to the axis. Notice also that $d\varphi/d\bar{\varphi} = 0$ at $\varphi = 0$ and therefore a Fourier cosine series in $\bar{\varphi}$ does not preclude cases with cusps on the axis. Approximating the integrals in Eq. (5) by sums then reduces Eq. (4) to set of N nonlinear equations in the N unknowns $R_0, R_1,...,R_{N-1}$. Since, irrespective of the vortex boundary choice, the imaginary part of $w(z)$ given by Eq. (5) is close to zero near the axis, we chose

$$\mathrm{Im}(f_j w\{R(\bar{\varphi}_j)\exp[i\varphi(\bar{\varphi}_j)]\}) = 0 \tag{9}$$

for $j = 1,...,N$ as our modified nonlinear system, where the numbers f_j were made appropriately big for points close to the axis. In this way, sensitivity of the left-hand side of Eq. (9) to the boundary curve near the axis was ensured. The system was solved by Newton iteration. Here $N = 20$ was found sufficient to give $R(\bar{\varphi})$ to 6 significant figures. Our numerical results support the analytical result that $\Theta \sim \pi/2 - \pi/(4 \ln r)$ near A. For $r = 1.3\times 10^{-2}$ in units of U/ω, the 2 term approximation gives $\Theta = 1.689$ and numerical calculation yielded $\Theta = 1.697$. Comparison over a range of r between 10^{-3} to 10^{-2} revealed about the same kind of agreement. For smaller r, the agreement was a little worse for the quantity $(\theta - \pi/2)$ because of limitations of numerical accuracy. The value of k in Eq. (6) was found to be 37.11. Pierrehumbert's value is 37, inferred from his numbers.

Saffman and Szeto[3] argued that this solution with touching vortices may be isolated and not the limit of a vortex pair. This question still remains open.

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