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Kinetic energy generated by the incompressible Richtmyer–Meshkov instability in a continuously stratified fluid

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The problem of calculating the kinetic energy created by impulsive acceleration of an incompressible continuously stratified fluid is formulated. Solutions are obtained for small density perturbations and a particular profile for various Atwood numbers and length scales. The kinetic energy is reduced when the undisturbed density variation is more diffuse.

I. INTRODUCTION

When two fluids of different density separated by an initially plane interface are set in motion by an acceleration normal to the interface, small disturbances are amplified when the acceleration is directed from the light to the heavy fluid. This is the well-known Rayleigh–Taylor instability. An instability can also occur when the acceleration is impulsive, i.e., it acts for almost zero time but is so large that a finite momentum is communicated to the fluids. For example, a shock propagating normal to the interface has this effect. This instability goes under the name of the Richtmyer–Meshkov[1,2] instability. It should be noted that unlike the Rayleigh–Taylor case, it occurs independently of the direction of the acceleration. The reason is that the Richtmyer–Meshkov instability is caused by pressure gradients perpendicular to the impulsive acceleration, which arise from the action of the acceleration on the transverse density fluctuations, and the direction of the acceleration is unimportant. The amplitude of the subsequent evolution of the disturbance can of course be affected by finite accelerations or gravitational forces. It should also be noted, however, that in the absence of such forces the growth of perturbations is linear in time in contrast with the exponential growth characteristic of Rayleigh–Taylor instablity.

Experiments on shock-generated Richtmyer–Meshkov instability have recently been reported by Brouillette and Sturtevant.[3] A novel feature of this work is the use of continuous rather than piecewise constant density distributions. One of the subjects of interest is the dependence on the initial density distribution of the width of the turbulent mixing layer, which develops in the later stages.

The study of Rayleigh–Taylor instabilities in continuously stratified fluids was first undertaken by Lord Rayleigh,[4] who derived analytical solutions for exponential density profiles. For general density profiles it is necessary to solve a second-order differential equation with appropriate boundary conditions. A derivation of this result has been given by Chandrasekhar.[5] More recently, Mikaelian has obtained approximate growth rates by treating the continuous stratification as a set of fluid layers of piecewise constant densities.[6,7] This approach has also been applied to the Richtmeyer–Meshkov instability.[8]

In this paper we shall calculate the dependence of the initial kinetic energy of the disturbance on the density perturbation and original density profile. This serves two purposes. First, initial conditions are provided for a fully nonlinear simulation of the finite-amplitude stage of the instability, and second initial energy densities are found for turbulence model calculations of the development of the fully turbulent mixing layer (Barenblatt[9] and Leith[10]).

We shall assume that the fluids are incompressible. This is a reasonable approximation for shock induced impulsive acceleration when the shocks are weak as the velocities induced by the shock are subsonic. (This approximation was made, for example, by Rott[11] in a treatment of the vorticity created by diffraction of a shock by a wedge.) We can then imagine that the motion is generated by motion of containing walls with a velocity V directed parallel to the undisturbed density gradient. That is, we suppose the fluid is contained within a perfectly rigid box which at time $t=0$ is set impulsively into motion. At time $t=0^+$, the normal component of velocity on the boundaries is therefore assumed specified.

In a longitudinally infinite medium set in motion by passage of a shock as in the Brouillette and Sturtevant experiment, when the transmitted and reflected shocks have traveled a distance large compared with the width of the region of density change, the state will be like that of a shock reflected from a sharp interface. Then there is a uniform velocity far from the mixing region, which can be modeled by motion of containing walls. In the experiment, the shocks are reflected by the ends of the apparatus and return to the mixing region. This additional perturbation is not considered here, but could be modeled by the application of repeated impulsive accelerations to the evolving incompressible flow.

We take axes with the x axis parallel to the undisturbed density gradient, say vertically upward, and the y axis horizontal. The upper fluid has density ρ_1 and the lower fluid has density ρ_2. The velocity V is acting downward. We just consider two-dimensional motion (the extension of three dimensions is straightforward).

At the initial time $t=0^-$, the density is taken to be

$$\rho_0=\bar{\rho}(x)+\epsilon\rho'(x,y), \tag{1}$$

where $\bar{\rho}\to\rho_{1,2}$ and $\rho'\to 0$ as $x\to\pm\infty$. The perturbation is assumed to introduce no mass, so that $\iint \rho'\,dx\,dy=0$. The initial velocity field at $t=0^+$ after the impulsive acceleration satisfies the boundary conditions $(u,v)\to(-V,0)$ as

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$x \to \pm \infty$ and $v = 0$ on vertical side walls.

The imposition of the velocity produces an impulsive pressure $P(x,y)\delta(t)$, where $\delta(t)$ is the delta function. The initial motion is a balance between acceleration and pressure gradient (viscous and convective inertial terms are negligible). Hence the initial velocity field is

$$u_0(x,y) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}H(t), \quad v_0(x,y) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}H(t), \tag{2}$$

where $H(t)$ is the Heaviside function

The impulsive pressure is determine by the requirement that the velocity field is incompressible, i.e.,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

which give the equation to determine the impulsive pressure

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial P}{\partial y}\right) = 0, \tag{4}$$

together with the boundary conditions

$$\frac{\partial P}{\partial x} \to \rho_1 V, \quad \text{as } x \to +\infty,$$
$$\frac{\partial P}{\partial x} \to \rho_2 V, \quad \text{as } x \to -\infty. \tag{5}$$

At a vertical wall $y = \text{const}$, we have simply $\partial P/\partial y = 0$.

Writing

$$P = V\int_0^x \bar{\rho}\, dx + \epsilon p', \tag{6}$$

we find that the perturbation impulsive pressure satisfies exactly

$$\epsilon\left[\frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial y}\right)\right] = -V\frac{\partial}{\partial x}\left(\frac{\bar{\rho}}{\rho_0}\right). \tag{7}$$

Note that the right-hand side of (7) is in fact $O(\epsilon)$ through the use of (1). This equation is to be solved, subject to the requirement that $\partial p'/\partial x$ vanishes as $x \to \pm \infty$ and $\partial p'/\partial y = 0$ on side walls. Note that we do not and in general cannot require that p' vanishes as $x \to \pm \infty$.

The vorticity generated by the acceleration is

$$\omega = \frac{\epsilon}{\rho_0^2}\left(\frac{\partial \rho_0}{\partial x}\frac{\partial p'}{\partial y} - \frac{\partial \rho'}{\partial y}\frac{\partial p'}{\partial x}\right). \tag{8}$$

Note that this perturbation vorticity is zero, and the flow is uninteresting unless the density perturbation depends upon y.

The initial kinetic energy (KE) (per unit span), relative to the frame moving with velocity V, is

$$\text{KE} = \frac{1}{2}\iint \rho_0[(u+V)^2 + v^2]dx\, dy, \tag{9}$$

where

$$u + V = V\left(1 - \frac{\bar{\rho}}{\rho_0}\right) - \frac{\epsilon}{\rho_0}\frac{\partial p'}{\partial x}, \quad v = -\frac{\epsilon}{\rho_0}\frac{\partial p'}{\partial y}. \tag{10}$$

II. SMALL DISTURBANCES

We now suppose that ϵ is small and consider finding p' and the velocities to lowest order. Suppose

$$\rho' = (1/L)g(x/L)\cos ky, \tag{11}$$

where L is a characteristic length scale of the undisturbed density profile $\bar{\rho}$, and $2\pi/k$ is the channel width. We normalize ϵ so that $\int g(\xi)d\xi = 1$, where $\xi = x/L$. This makes the integrated (with respect to x) density perturbation of order ϵ. Note that ϵ has dimensions density times length. Then

$$p' = Vf(x/L)\cos ky, \tag{12}$$

where $f(\xi)$ satisfies the equation

$$\frac{d}{d\xi}\left(\frac{1}{\bar{\rho}}\frac{df}{d\xi}\right) - \frac{k^2L^2}{\bar{\rho}}f = \frac{d}{d\xi}\left(\frac{g}{\bar{\rho}}\right), \tag{13}$$

with $f \to 0$ as $\xi \to \pm \infty$.

A. Limiting cases

There are two limiting cases. First we consider $kL \to \infty$. (The width of the layer is large compared with the horizontal scale of the density perturbation.) Then

$$f \approx -\frac{\bar{\rho}}{k^2L^2}\frac{d}{d\xi}\left(\frac{g}{\bar{\rho}}\right), \tag{14}$$

provided that g or $\bar{\rho}$ do not have jumps or cusps. Substitution into the expression for the kinetic energy gives the leading term for the kinetic energy per unit span and per unit width, denoted by KE′,

$$\text{KE}' = \frac{\epsilon^2 V^2}{4L}\int_{-\infty}^{\infty}\frac{g^2}{\bar{\rho}}\,d\xi. \tag{15}$$

The second limiting case is $kL \to 0$. This corresponds to a sharp interface that is displaced into the surface $x = \eta \cos ky$. The equivalent density perturbation is a delta function, $g = \delta(x/L), \epsilon = (\rho_2 - \rho_1)\eta$. The results in this limit can be obtained from the differential equation by employing the method of matched asymptotic expansions (see the Appendix). To determine the leading term, one can work out directly the disturbance for a perturbed sharp interface in the straightforward way used for Rayleigh–Taylor instability. The result is

$$p' = [\rho_1 V/(\rho_1 + \rho_2)]e^{-kx}\cos ky, \quad \text{for } x > 0,$$
$$p' = [\rho_2 V/(\rho_1 + \rho_2)]e^{kx}\cos ky, \quad \text{for } x < 0. \tag{16}$$

The kinetic energy is

$$\text{KE}' = \epsilon^2 V^2 k/4(\rho_1 + \rho_2). \tag{17}$$

Comparing the two expressions (15) and (17), we see that less kinetic energy is generated when the width of the undisturbed density distribution is increased. Thus to limit the amount of kinetic energy in the Richtmyer–Meshkov instability, the jump in density should be spread out.

B. Streamfunction

An alternative way of carrying out the calculation is to employ the streamfunction ψ, which exists by virtue of the incompressibility equation. We write

$$\psi = -Vy + \epsilon\psi', \quad u' = \frac{\partial \psi'}{\partial x}, \quad v' = -\frac{\partial \psi'}{\partial y}, \tag{18}$$

where

$$u_0 = -V + \epsilon u', \quad v_0 = \epsilon v'. \tag{19}$$

Then

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TABLE I. Values of $E(\lambda, A)$ for selected values of A and λ.

A \ λ	0.01	0.1	0.2	0.5	1	1.5	2	2.5	3	4	5
0	0.495 03	0.453 83	0.414 33	0.324 83	0.233 70	0.179 80	0.144 934	0.120 826	0.103 304 9	7.973 6 − 2	6.4735 − 2
0.1	0.495 07	0.454 08	0.414 72	0.325 33	0.234 140	0.180 163	0.145 225	0.121 071 1	0.103 514	7.989 7 − 2	6.4866 − 2
0.2	0.495 16	0.454 84	0.415 898	0.326 86	0.235 48	0.181 249	0.146 115	0.121 816	0.104 152	8.039 0 − 2	6.5266 − 2
0.4	0.495 57	0.458 021	0.420 847	0.333 36	0.241 222	0.185 904	0.149 930	0.125 014	0.106 889	8.250 11 − 2	6.6977 − 2
0.6	0.496 30	0.463 881	0.430 123	0.345 88	0.252 448	0.195 045	0.157 430	0.131 301	0.112 271	8.665 11 − 2	7.0340 − 2
0.8	0.497 50	0.473 84	0.446 390	0.369 039	0.273 833	0.212 598	0.171 861	0.143 402	0.122 629	9.463 23 − 2	7.6805 − 2
0.9	0.498 41	0.481 698	0.459 727	0.389 367	0.293 378	0.228 805	0.185 215	0.154 600	0.132 209	0.102 00 − 2	8.2773 − 2

$$\frac{\partial\psi'}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \frac{V\rho'}{\rho_0}, \quad \frac{\partial\psi'}{\partial x} = \frac{1}{\rho_0}\frac{\partial p'}{\partial y}. \tag{20}$$

Eliminating p' gives the equation for ψ',

$$\rho_0\nabla^2\psi' + \frac{\partial\psi'}{\partial x}\frac{\partial\rho_0}{\partial x} + \frac{\partial\psi'}{\partial y}\frac{\partial\rho_0}{\partial y} = V\frac{\partial\rho'}{\partial y}. \tag{21}$$

In terms of the streamfunction, the kinetic energy is easily shown to be

$$\mathrm{KE}' = -V\epsilon^2\int\int\psi'\frac{\partial\rho'}{\partial y}\,dx\,dy. \tag{22}$$

This expression is exact.

If we write

$$\psi' = Vh(x/L)\sin ky, \quad \text{where } \psi'\to 0, \text{ as } x\to\pm\infty, \tag{23}$$

then to lowest order in ϵ, h satisfies the equation

$$\frac{d^2h}{d\xi^2} - k^2L^2h + \frac{1}{\bar\rho}\frac{d\bar\rho}{d\xi}\frac{dh}{d\xi} = -\frac{kLg}{\bar\rho}, \tag{24}$$

with boundary condition $h\to 0$, as $\xi\to\pm\infty$. The kinetic energy produced by the disturbance is

$$\mathrm{KE}' = \frac{\epsilon^2}{4}kV^2\int hg\,d\xi. \tag{25}$$

In the limit $kL\to\infty$, $h\sim g/kL\bar\rho$ and the kinetic energy is the same as that obtained before in Eq. (15).

III. A PARTICULAR PROFILE

We now consider a definite case. We take

$$\bar\rho = 1 + A\tanh(x/L), \quad g = \tfrac{1}{2}\operatorname{sech}^2(x/L). \tag{26}$$

The quantity A is the Atwood number. The initial density is

$$\rho_0 = \bar\rho + (\epsilon/2L)\operatorname{sech}^2(x/L)\cos ky \tag{27}$$

produced by a deformation of the lines of constant density by a vertical distance $\eta = \epsilon/2A\cos ky$.

We can write (25) in the form

$$\mathrm{KE}' = [\epsilon^2kV^2/2(\rho_1+\rho_2)]E(\lambda,A), \tag{28}$$

where $\lambda = kL$. Comparison with the limits (15) and (17), using the results $\int_{-\infty}^{\infty}\operatorname{sech}^4\xi\,d\xi = \frac{4}{3}$, shows that

$$E\sim\tfrac{1}{2} \quad\text{or}\quad E\sim 1/3\lambda, \tag{29}$$

as $\lambda\to 0$ or $\lambda\to\infty$, respectively.

Results of numerical solution of the equation, using a two-point boundary value ordinary differential equation code, are shown in Table I and Fig. 1.

It is perhaps more appropriate to express the kinetic energy in terms of the crest to trough displacement $\bar\eta$ of the constant density lines, which produce the density perturbation,

$$\bar\eta = 2\epsilon/(\rho_1+\rho_2)A = 2\epsilon/(\rho_1-\rho_2). \tag{30}$$

Then the expression (28) for the kinetic energy is

$$\mathrm{KE}' = [kV^2\bar\eta^2(\rho_1-\rho_2)^2/8(\rho_1+\rho_2)]E(\lambda,A). \tag{31}$$

For given k, $\bar\eta$, V, and A, the energy is least when L is largest.

The initial conditions calculated here produced by an incompressible acceleration can either be used as initial values for an incompressible stratified Euler code to study the latter evolution (this is currently under study) or as providing initial data for a turbulence model or subgrid model calculation. Milinazzo and Saffman[12] proposed a turbulence model for incompressible stratified flow that worked well for the effects of density difference on the turbulent mixing layer and it might be worth studying in the present context.

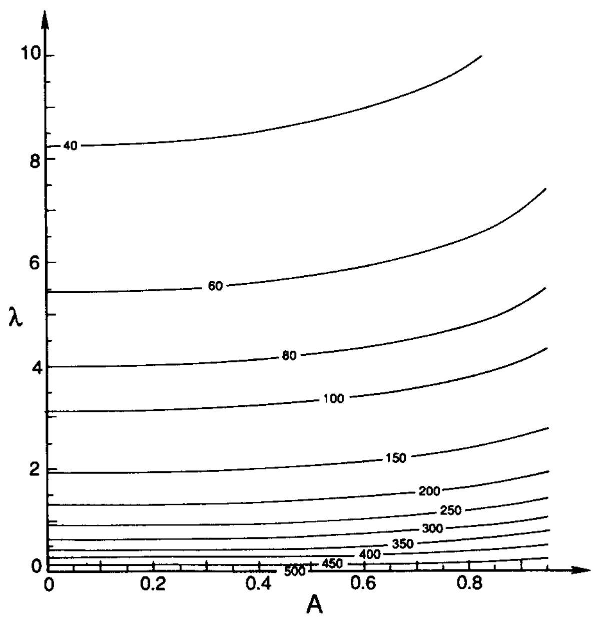

FIG. 1. Contours of constant $E(\lambda,A)$ scaled by 1000 for selected values of A and λ.

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APPENDIX: BOUNDARY LAYER ANALYSIS

Here we consider the behavior of the scaled pressure $f(x)$ in the limit as $\lambda \to 0$. Using matched asymptotic expansions we confirm the results in (16) and (17). We will also compute the correction to the leading-order term for the normalized energy $E(\lambda,A)$ given by (29).

The task at hand is to analyze the solution of (13):

$$\frac{d}{d\xi}\left(\frac{1}{\bar{\rho}}\frac{df}{d\xi}\right) - \lambda^2 \frac{f}{\bar{\rho}} = \frac{d}{d\xi}\left(\frac{g}{\bar{\rho}}\right), \tag{A1}$$

where $\lambda = kL$ and $\xi = x/L$. In order to generate an outer solution we make the substitution $x = \xi L$ and consider the limit as $x \to \infty$ with k held fixed. We assume here that the density $\bar{\rho}(x/L)$ and the perturbation $g(x/L)$ approach their respective limiting values at an exponential rate as $|x| \to \infty$. Thus our derivation will only be valid for a stratification of the form (26). The modification of this derivation to allow more general forms of the stratification is straightforward. The outer solution of (A1) to $O(\lambda^2)$ is given by

$$f_O(x) \sim \begin{cases} (c_{R0} + c_{R1}\lambda + c_{R2}\lambda^2)\exp(-kx), & x>0, \\ (c_{L0} + c_{L1}\lambda + c_{L2}\lambda^2)\exp(+kx), & x>0. \end{cases} \tag{A2}$$

The inner solution is obtained by solving (A1) in a perturbation series in λ again to $O(\lambda^2)$. The result is

$$\begin{aligned} f_I(\xi) \sim & \int_0^\xi g(\xi_1)d\xi_1 + c_{I00} + \lambda\left(c_{I10} + c_{I11}\int_0^\xi \bar{\rho}(\xi_1)d\xi_1\right) + \lambda^2\left(c_{I20} + c_{I21}\int_0^\xi \bar{\rho}(\xi_1)d\xi_1 \right. \\ & + c_{I00}\int_0^\xi \bar{\rho}(\xi_2)d\xi_2 \int_0^{\xi_2} \frac{1}{\bar{\rho}(\xi_1)}d\xi_1 \\ & \left. + \int_0^\xi \bar{\rho}(\xi_3)d\xi_3 \int_0^{\xi_3} \frac{1}{\bar{\rho}(\xi_2)}d\xi_2 \int_0^{\xi_2} g(\xi_1)d\xi_1\right). \end{aligned} \tag{A3}$$

Next we examine the behavior of the inner solution $f_I(\xi)$ as $|\xi| \to \infty$ and the behavior of the outer solution $f_O(\lambda\xi)$ as $\lambda \to 0$ such that $\lambda\xi \to 0$. This determines the matching region. As $\xi \to \pm\infty$ the inner solution has the asymptotic expansion

$$\begin{aligned} f_I(\xi) \sim & Z_\pm + c_{I00} + \lambda[c_{I10} + c_{I11}(A_\pm\xi + B_\pm)] + \lambda^2 \\ & \times [(C_\pm\xi^2 + D_\pm\xi + E_\pm) + c_{I21}(A_\pm\xi \\ & + B_\pm) + c_{I22}(F_\pm\xi^2 + G_\pm\xi + H_\pm) + c_{I20}], \end{aligned} \tag{A4}$$

where

$$Z_\pm = \pm\int_0^\infty g(\pm\xi_1)d\xi_1,$$

$$A_\pm = \lim_{\xi\to\infty}\frac{1}{\xi}\int_0^\xi \bar{\rho}(\pm\xi_1)d\xi_1,$$

$$B_\pm = \lim_{\xi\to\infty}\left(\int_0^\xi \bar{\rho}(\pm\xi_1)d\xi_1 - A_\pm\xi\right),$$

$$C_\pm = Z_\pm/2,$$

$$D_\pm = \lim_{\xi\to\infty}\left(\frac{1}{\xi}\int_0^\xi \bar{\rho}(\pm\xi_3)\int_0^{\xi_3}\frac{1}{\bar{\rho}(\pm\xi_2)}d\xi_2 \times \int_0^{\xi_2} g(\pm\xi_1)d\xi_1 - C_\pm\xi\right),$$

$$E_\pm = \lim_{\xi\to\infty}\left(\int_0^\xi \bar{\rho}(\pm\xi_3)\int_0^{\xi_3}\frac{1}{\bar{\rho}(\pm\xi_2)}d\xi_2 \times \int_0^{\xi_2} g(\pm\xi_1)d\xi_1 - C_\pm\xi^2 - D_\pm\xi\right),$$

$$F_\pm = \tfrac{1}{2},$$

$$G_\pm = \lim_{\xi\to\infty}\left(\frac{1}{\xi}\int_0^\xi \bar{\rho}(\pm\xi_2) \times \int_0^{\xi_2}\frac{1}{\bar{\rho}(\pm\xi_1)}d\xi_1 - F_\pm\xi\right),$$

$$H_\pm = \lim_{\xi\to\infty}\left(\int_0^\xi \bar{\rho}(\pm\xi_2)\int_0^{\xi_2}\frac{1}{\bar{\rho}(\pm\xi_1)}d\xi_1 - F_\pm\xi^2 - G_\pm\xi\right).$$

In the matching region the outer solution has the asymptotic expansion

$$f_O(x) \sim \begin{cases} c_{R0} + \lambda(c_{R1} - c_{R0}\xi) + \lambda^2\left(c_{R2} - c_{R1}\xi + \frac{c_{R0}}{2}\xi^2\right), & x>0, \\ c_{L0} + \lambda(c_{L1} + c_{L0}\xi) + \lambda^2\left(c_{L2} + c_{L1}\xi + \frac{c_{L0}}{2}\xi^2\right), & x<0. \end{cases} \tag{A5}$$

Matching powers of λ and ξ as $\xi \to +\infty$ we obtain the linear equations:

$$O(\lambda^0\xi^0):\quad Z_+ + c_{I00} = c_{R0},$$

$$O(\lambda\xi^0):\quad c_{I10} + c_{I11}B_+ = c_{R1},$$

$$O(\lambda\xi):\quad c_{I11}A_+ = -c_{R0},$$

$$O(\lambda^2\xi^0):\quad E_+ + c_{I21}B_+ + c_{I00}H_+ + c_{I20} = c_{R2},$$

$$O(\lambda^0\xi):\quad D_+ + c_{I21}A_+ + c_{I00}G_+ = -c_{R1},$$

$$O(\lambda^2\xi^2):\quad C_+ + 2F_+c_{I00} = c_{R0}. \tag{A6}$$

Matching powers of λ and ξ as $\xi \to -\infty$ we obtain

$$O(\lambda^0\xi^0):\quad Z_- + c_{I00} = c_{L0},$$

$$O(\lambda\xi^0):\quad c_{I10} + c_{I11}B_- = c_{L1},$$

$$O(\lambda\xi):\quad c_{I11}A_- = c_{L0},$$

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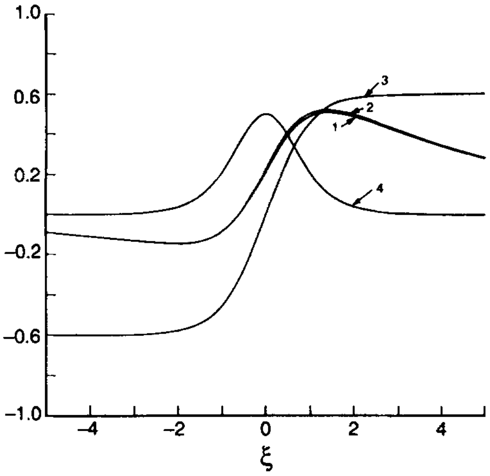

FIG. 2. A comparison of the numerical solution of (A1) with the boundary layer approximation of (A11) for $A = 0.6$ and $\lambda = 0.1$. (i) the numerical solution for $f(\xi)$; (ii) the boundary layer approximation of $f(\xi)$; (iii) the density $\bar{\rho} - 1$; and (iv) the density perturbation $g(\xi)$.

$$O(\lambda^2\xi^0):\quad E_- + c_{I21}B_- + c_{I00}H_- + c_{I20} = c_{L2},$$
$$O(\lambda^0\xi):\quad D_- + c_{I21}A_- + c_{I00}G_- = c_{L1},$$
$$O(\lambda^2\xi^2):\quad 2C_- + 2F_-c_{I00} = c_{L0}. \qquad \text{(A7)}$$

Equations (A6) and (A7) comprise 12 equations for 11 unknowns. However, on closer inspection it is found that the first and last equations of (A6) and (A7) are dependent and this leaves 10 equations for 11 unknowns. Thus a complete solution can only be determined to $O(\lambda)$. In order to obtain a complete solution to $O(\lambda^2)$ it is necessary to continue to $O(\lambda^3)$.

For the choice of $\bar{\rho}$ and g given in (26) it is possible to solve for the boundary layer solution in closed form. We have

$$Z_\pm = \pm \tfrac{1}{2},$$
$$A_\pm = 1 \pm A,$$
$$B_\pm = -A \log 2,$$
$$C_\pm = \pm \tfrac{1}{4},$$
$$D_\pm = \mp [1/2(A \mp 1)]\log[(1 \pm A)/2],$$
$$G_\pm = \pm [A/(A \mp 1)]\log[(1 \pm A)/2].$$

Solving (A6) and (A7) we have

$$c_{I00} = A/2,$$
$$c_{I10} = \tfrac{1}{4}(A^2 - 1)\log[(1-A)/(1+A)] - A\log 2/2,$$
$$c_{I11} = \tfrac{1}{2},$$
$$c_{R0} = (1+A)/2, \qquad \text{(A8)}$$
$$c_{R1} = \tfrac{1}{4}(A^2-1)\log[(1-A)/(1+A)],$$
$$c_{L0} = -(1-A)/2,$$
$$c_{L1} = \tfrac{1}{4}(A^2-1)\log[(1-A)/(1+A)].$$

This allows us to determine the outer solution correct to $O(\lambda)$ from (A2). The inner solution is given by

$$f_I(\xi) \sim \tfrac{1}{2}[\tanh(\xi) - 1] + c_{I00} + \lambda\{c_{I10} + c_{I11}[\xi + A\log(\cosh\xi)]\}. \qquad \text{(A9)}$$

In the matching region the solution has the form

$$f_M(\xi) \sim \begin{cases} c_{R0} + \lambda(c_{R1} - c_{R0}\xi), & \xi > 0, \\ c_{L0} + \lambda(c_{L1} + c_{L0}\xi), & \xi < 0. \end{cases} \qquad \text{(A10)}$$

Thus a solution correct to $O(\lambda)$, which is uniformly valid over the interval $-\infty < \xi < \infty$, is given by

$$f_U(\xi) = f_O(L\xi) + f_I(\xi) - f_M(\xi). \qquad \text{(A11)}$$

In Fig. 2, we compare the asymptotic solution (A11) with a numerical solution of (13) for $\lambda = 0.1$ and $A = 0.6$. Note that as $\lambda \to 0$ the outer solution agrees with (16).

Finally we turn to the calculation of the normalized energy $E(\lambda,A)$. This is given by

$$E(\lambda,A) = \frac{1}{\lambda}\int_{-\infty}^{+\infty} d\xi \frac{1}{\bar{\rho}(\xi)}\left[\left(g - \frac{df}{d\xi}\right)^2 + \lambda^2 f^2\right]. \qquad \text{(A12)}$$

In order to compute the asymptotic expansion of E, we break up the region of integration as follows:

$$\int_{-\infty}^{+\infty} = \int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty},$$

where δ is in the matching region of the inner and outer solutions given by (A2) and (A3). Over the interval $-\delta < \xi < \delta$ we substitute into (A12) the inner solution. Similarly the appropriate outer solution is used for $|\xi| > \delta$. We then compute the asymptotic expansion of each integral in the limit $\delta \to \infty$, $\lambda \to 0$ such that $\lambda\delta \to 0$. The final result must be independent of δ. We omit the details and indicate the result:

$$E(\lambda,A) \sim \tfrac{1}{2} + \lambda\,[(A^2-1)/4A\,]\log[(1+A)/(1-A)], \qquad \text{(A13)}$$

in agreement with (29) in the limit $\lambda \to 0$.

[1] R. D. Richtmyer, Comm. Pure Appl. Math. **8**, 297 (1960).
[2] E. E. Meshkov, NASA Technical Translation TT F13 074, 1970.
[3] M. Brouillette and B. Sturtevant, in *Advances in Fluid Turbulence* (Elsevier, New York, in press).
[4] Lord Rayleigh, *Scientific Papers* (Dover, New York, 1900), Vol. 2.
[5] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford U.P., London, 1968).
[6] K. O. Mikaelian, Phys. Rev. A **26**, 2140 (1982).
[7] K. O. Mikaelian, Lawrence Livermore Laboratory Preprint UCRL-100097, 1988.
[8] K. O. Mikaelian, Phys. Rev. A **31**, 410 (1986).
[9] G. I. Barenblatt, in *Nonlinear Dynamics and Turbulence*, edited by G. I. Barenblatt, G. Iooss, and D. D. Joseph (Pitman, New York, 1983), p. 48.
[10] C. E. Leith, Lawrence Livermore Laboratory Report UCRL-96036, 1986.
[11] N. Rott, J. Fluid Mech. **1**, 111 (1956).
[12] F. Milinazzo and P. G. Saffman, Stud. Appl. Math. **55**, 45 (1976).

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