Flow in a Turbulent Trailing Vortex

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The structure of turbulent line vortices is examined. A general argument is constructed to show that the vortex must develop an overshoot of circulation if it entrains fluid at a rate greater than that due to molecular diffusion. A weak hypothesis on the distribution of Reynolds stress leads to the logarithmic profile of Hoffman and Joubert and an estimate of the maximum Reynolds stress. The results of a turbulence model due to Saffman are presented and shown to be poor.

I. INTRODUCTION

The flow immediately behind an airplane wing consists of a nearly plane vortex sheet and a wake both stretching across the span of the wing. Within a short distance downstream, the vortex sheet curls up and forms a pair of nearly axisymmetric trailing vortices, concentrating a large portion of the vorticity from the vortex sheet into the cores of the vortex pair. At this early stage of vortex development, appreciable axial velocities are found in the cores of the vortices. The details of flow during this stage are still unclear at the present time. But experiments of Dosanjh *et al.*[1] show that the circulation of each vortex is about 60% of the circulation at the center of the wing producing the vortex and remains constant farther downstream.

Farther downstream, the vortices grow by turbulent diffusion. A study of this stage of vortex growth is of interest in connection with the practical problem of avoiding possible damage to aircraft that might accidentally penetrate the vortex wakes of other aircraft. During this stage, the vortices grow sufficiently slowly so that the distance between the vortices is large compared with the diameter of each. Thus, there is negligible interaction between them. Further, the axial velocities in the cores of the vortices, although large in the early parts of this stage, decay more rapidly downstream than the tangential velocities and as a good approximation (or at least a first approximation) we need study only an isolated trailing vortex with no interaction of axial motion on the tangential motion.

Lamb[2] considered the problem of the diffusion of a laminar line vortex in time. His similarity solution can be applied to a laminar trailing vortex by replacing time in the solution by z/U_0, where U_0 is the constant axial velocity and z is the axial distance. This solution is conveniently written in the similarity variable η:

$$\eta = \frac{r}{(\Gamma_0 z/U_0)^{1/2}}, \qquad (1)$$

as

$$\frac{\Gamma}{\Gamma_0} = 1 - \exp\left(\frac{-\eta^2}{4\nu/\Gamma_0}\right), \qquad (2)$$

where Γ is the circulation at any η and Γ_0 is the circulation at ∞. This solution shows a region of rigid body rotation close to $\eta = 0$ where the tangential velocity, u_ϕ, increases linearly with radius. With further increase in η, u_ϕ reaches a maximum and then decreases to zero like $1/\eta$ as $\eta \to \infty$. If we define a core circulation Γ_1 and a core radius r_1 as the circulation and the radius where the tangential velocity is a maximum, we can easily show from Eq. (2) that $\Gamma_1/\Gamma_0 = 0.716$ and r_1 is given by $\eta_1^2 = 5.04(\nu/\Gamma_0)$.

Squire[3] suggested that a turbulent trailing vortex could be described by Lamb's solution if ν in it were replaced by an eddy diffusivity ν_t as

$$\Gamma/\Gamma_0 = 1 - \exp(-\eta^2/4a), \qquad (3)$$

where $a = \nu_t/\Gamma_0$ is, on dimensional grounds, some function of the vortex Reynolds number Γ_0/ν. Many wind tunnel studies and the few flight studies to date indicate that the similarity form is reached in a short distance downstream of the wing; but the circulation profile does not conform to Eq. (3). In particular, Eq. (3) implies a value for $\Gamma_1/\Gamma_0 = 0.716$, while experiments indicate a value for the same in the range 0.4–0.6. Consequently, it is only possible to find a rough value of a. This is usually done by comparing the maximum velocity given by Eq. (3) for the observed Γ_0 with the measured value. Such a value for a provides a rough measure of vortex growth and is a convenient basis for comparing the results of various experimenters. It is given by, as is easily derived from Eq. (3),

$$a = 2.6 \times 10^{-3} (\Gamma_0 U_0 / u_m^2 z), \qquad (4)$$

where u_m is the observed value of the maximum tangential velocity at an axial distance z. Evidently, z should be measured from a virtual origin so chosen that $u_m^2 z$ is independent of z.

Turbulent trailing vortices have been studied experimentally. Full scale experiments to estimate the maximum velocity in the trailing vortex of a large airplane were conducted by Rose and Dee.[4] A small airplane equipped with an incidence meter (i.e., a pivoted vane) was flown through the eye of the vortex shed from the large airplane. Estimates of the maximum velocity in the vortex at various distances along the axis were made using the incidence data from the small airplane. It was found that $u_m^2 z$ was a constant indicating that the vortex was self-preserving. They found the value for a to be about 2×10^{-4}, but in their calculation it was assumed that Γ_0 was the same as the circulation at the root of the wing. If we allow for the fact that Γ_0 is only about half this value, we have a value for a about 1.0×10^{-4}. McCormick *et al.*[5] studied the flow in a vortex shed by a small airplane by using an instrument labeled the vortimeter. The vortimeter, consisting of a vertical array of horizontal cylinders, was mounted at a suitable height above the ground. By flying the airplane to one side and suitably above the instrument, it was possible to arrange the vortex to sweep across the instrument, this being visually aided by a tuft grid mounted suitably nearly. From the strain gauge data, it was possible to estimate the velocity distribution in the vortex at various distances from the airplane. Their results indicate that Γ_0 is only 0.45 of the circulation at the root of the wing. Also from their results we have calculated a value for a in the range 0.3×10^{-4} to 0.6×10^{-4}. This range for a is not far from the estimates from Rose and Dee's experiments.

Dosanjh *et al.*[1] conducted some experiments on trailing vortices in a wind tunnel. A wing of rectangular plan form was mounted normal to the side wall of a wind tunnel. Measurements of flow inclination were made in the vortex using a five hole flow direction probe. From the flow direction and total pressure data, axial and tangential velocity distributions were calculated. The results indicate that the circulation of the vortex is about 58% of the value at the root of the wing. The value of a was estimated to be about 5×10^{-3}. Other similar experiments lead to similar estimates for a. We present several of them in Table I. It is observed that the values of a in flight studies are substantially smaller than the values from wind tunnel studies. This difference should be due entirely to the difference in vortex Reynolds numbers between the two cases. Flight experiments are in the Reynolds number range of 10^6–10^7 while the wind tunnel studies are in the range 10^3–10^5. This rather large effect of Reynolds number on the growth of vortices is surprising.

Hoffman and Joubert[6] present an analysis of the turbulent trailing vortex. They derive a universal law for the distribution of circulation valid for any turbulent vortex not necessarily self-similar. In a region away from the center and not too close to the outer edge, the universal law gives a logarithmic variation of circulation which can be written as

$$\Gamma/\Gamma_1=[(1/H)\log_{10}(r/r_1)+1], \tag{5}$$

where Γ_1 and r_1 are the core circulation and core radius as already defined. $1/H$ is a universal constant whose value they find by comparison with their own experiments to be 2.14. We note that it should be equal to $\log_e 10=2.303$ for Eq. (5) to satisfy the definition of r_1 (i.e., tangential velocity should be a maximum at radius r_1). One of the ways of deriving Eq. (5), as presented by Hoffman and Joubert, assumes that the flow in the core follows a universal form independent of the conditions farther out so that Γ/Γ_1 should be a unique function of r/r_1. This assumption is verified by their experiments. It is then assumed that $d\Gamma/dr$ should be independent of r_1. Comparison of the universal profile given by Eq. (5) with experiments indicates good agreement between them throughout a vortex except for small regions near the center and the outer edge. A typical comparison is shown in Fig. 1. In Fig. 1 we have also included the curve given by Eq. (3) for constant eddy viscosity. This agrees with the universal profile for $0\leq r/r_1\leq1.2$ and there is some disagreement for larger values of r/r_1. On the other hand, we can rewrite Eq. (5) in the form (using the consistent value for H):

$$\Gamma/\Gamma_0=(\Gamma_1/\Gamma_0)[\log_e(r/r_1)+1] \tag{6}$$

which indicates that the slope of the circulation profile as a function of $\log r$ depends linearly on Γ_1/Γ_0. Since observed values of Γ_1/Γ_0 are considerably different from the constant eddy viscosity value, profiles of Γ/Γ_0 against r are considerably different for experiment and constant eddy viscosity.

As noted earlier, the vortex growth parameter a depends on the Reynolds number Γ_0/ν. Owen[7] presents an analysis of a simple model using an integral method to explain the observed variation of a with Γ_0/ν, but the model contains features which are unclear to the present writers.

An interesting secondary effect of trailing vortex flow is the generation of an axial pressure gradient. Low pressure is produced near the axis of the vortex due to centrifugal acceleration of the fluid in it. As the tangential velocity in the vortex decays with axial distance, suction near the axis of the vortex is reduced and thus we have a positive axial pressure gradient in the core of the vortex. This pressure gradient induces an axial velocity defect very much resembling an ordinary wake. Batchelor[8] studied the development of axial velocity in a laminar trailing vortex assuming that the perturbation of axial velocity is small compared with the free stream velocity. This assumption is believed to be satisfied in the later stages of vortex growth. In the early stages of vortex growth, the axial velocity defect is not small compared with the free-stream velocity and may significantly affect the development of the vortex itself.

TABLE I. Growth parameters for trailing vortices.

	Γ_0/ν	Γ_1/Γ_0	a	$b=r_1(U_0/\Gamma_0 z)^{1/2}$
Rose and Dee	$\sim10^7$	0.4	2×10^{-4}	1.8×10^{-2}
Rose and Dee (corrected)	$\sim10^7$	0.4	1×10^{-4}	1.3×10^{-2}
Dosanjh *et al.*	2×10^3	0.6	5×10^{-3}	1.3×10^{-1}
Newman	2×10^4	0.5	2×10^{-3}	7×10^{-2}
McCormick *et al.*	$\sim10^6$	0.37	5×10^{-5}	8.2×10^{-3}
Saffman's model	∞	1.2	7.6×10^{-3}	3.1×10^{-1}

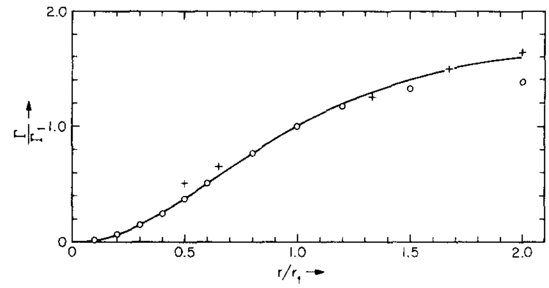

FIG. 1. Comparison of circulation profiles. Solid line corresponds to the universal profile of Hoffman and Joubert. ○-points corresponding to constant eddy viscosity; +-typical measurement by McCormick *et al.* (Ref. 5).

In what follows we study the diffusion of a turbulent trailing vortex assuming that the axial velocity defect is small so that it does not affect the tangential motion. We shall show, from very general considerations, that the circulation distribution in a turbulent vortex is very likely to acquire a maximum value which is greater than Γ_0. That is, we expect a strange situation to develop in which the circulation rises above Γ_0 and then falls back to Γ_0 as $r\rightarrow\infty$. However, a profile with regions of negative circulation gradient is unstable in the Rayleigh sense, which leads us to believe that the overshoot will be small and followed by a long tail in which the circulation gradient is small and negative. Experiments to to date do not indicate the presence of any significant overshoot in the circulation profile implying, perhaps, that the height of the overshoot is about the same as the accuracy of the measurements. But, another possibility is that the instability associated with a negative circulation gradient produces an axial periodic structure with qualitative features like Taylor vortices in cylindrical Couette flow. (Casual observations of contrails sometimes show this type of structure. One can also speculate that there may be a relation to some types of vortex breakdown.)

Saffman[9] proposed a model for inhomogeneous turbulent flows. The model was applied to some simple flows by Govindaraju[10] with encouraging results. We shall present the results from a similarity solution using this model, but the solution so obtained indicates a rather large overshoot in the circulation profile. Further, the rate growth of the vortex seems to be overestimated by this model. The reason for the failure of the model is still unclear.

It is also possible to study the development of axial velocity in a trailing vortex using Saffman's model assuming the axial velocity defect to be small. The solution exhibits the same structure as the laminar vortex studied by Batchelor.[8] We do not present the details of such an analysis as the results are not sufficiently interesting.

II. ANALYSIS

A. The Overshoot of Circulation

We can easily derive the basic equation governing the growth of an axisymmetric turbulent vortex starting from the Navier–Stokes equations. Under the boundary-layer approximation ($\partial/\partial r \gg \partial/\partial z$) and the assumption of small axial velocity defect, this equation can be written as

$$U_0 \frac{\partial u_\phi}{\partial z} = \frac{\nu}{r^2}\frac{\partial}{\partial r}\left[r^3 \frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right] - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\overline{u_\phi' u_r'}\right). \qquad (7)$$

In the above U_0 is the free stream velocity. u_ϕ, the tangential velocity, should be proportional to r as $r\rightarrow 0$ and should go like $\Gamma_0/2\pi r$ for large r so that the circulation tends to Γ_0 as $r\rightarrow\infty$. The distribution of the Reynolds stress $-\overline{u_\phi' u_r'}$ is unknown, but it should be bounded for $r\rightarrow 0$ and should decay to zero as $r\rightarrow\infty$. The flow for large r approaches potential flow since vorticity fluctuations decay. Thus, it is reasonable to assume, in fact, that the turbulent fluctuations u_ϕ', u_r' each go to zero faster than $1/r$ for large r so that $-\overline{u_\phi' u_r'}$ tends to zero faster than $1/r^2$. A sufficient condition is that the vorticity fluctuations should be $O(r^{-3})$. It is reasonable to assert that the vorticity fluctuations are exponentially small as $r\rightarrow\infty$.

Equation (7) can be written in terms of the mean circulation $\Gamma = 2\pi r u_\phi$ as (since Γ_0 is a constant):

$$U_0 \frac{\partial}{\partial z}(\Gamma-\Gamma_0) = \frac{\nu}{r}\frac{\partial}{\partial r}\left[r^3 \frac{\partial}{\partial r}\left(\frac{\Gamma}{r^2}\right)\right] - \frac{1}{2\pi r}\frac{\partial}{\partial r}\left(r^2 u_\phi' r_r'\right). \qquad (8)$$

Multiplying both sides of Eq. (8) by r and integrating throughout with respect to r, we have:

$$U_0 \frac{\partial}{\partial z}\int_0^\infty (\Gamma-\Gamma_0) r\,dr = \nu r^3 \frac{\partial}{\partial r}\left(\frac{\Gamma}{r^2}\right)\Bigg|_0^\infty - \frac{r^2}{2\pi}\overline{u_\phi' u_r'}\Bigg|_0^\infty. \qquad (9)$$

The second term on the right-hand side vanishes because $\overline{u_\phi' u_r'}$ goes to zero faster than $1/r^2$. From Eq. (8) it is easily shown that $\Gamma-\Gamma_0$ goes to zero faster than $1/r^2$ if $\overline{u_\phi' u_r'}$ goes to zero faster than $1/r^2$ for large r. Thus, the integral on the left-hand side of Eq. (9) exists and in the first term on the right-hand side we replace Γ by Γ_0 as $r\rightarrow\infty$ when Eq. (9) becomes

$$\frac{\partial}{\partial z}\int_0^\infty \frac{\Gamma_0-\Gamma}{\Gamma_0} r\,dr = \frac{2\nu}{U_0}. \qquad (10)$$

It is easily verified that Lamb's solution Eq. (2) satisfies this identity. Integrating Eq. (10), we have

$$\int_0^\infty \frac{\Gamma_0-\Gamma}{\Gamma_0} r\,dr = A + \frac{2\nu(z-z_0)}{U_0}, \qquad (11)$$

where A is some constant of dimensions $(\text{length})^2$ determined at the station $z=z_0$. This integral can be given a physical interpretation in terms of the angular momentum defect. The integral is also equal to one-half of the radius of gyration squared of the mean vorticity distribution.

From Eq. (11) we can deduce that a growing turbulent vortex will acquire a profile in which Γ is somewhere greater than Γ_0. We take $r_1(z)$ as a characteristic radius of the vortex, where r_1 has the definition given in Sec. I, and introduce a new variable

$$\xi=r/r_1(z), \tag{12}$$

then

$$J(z)=\int_0^\infty \frac{\Gamma_0-\Gamma}{\Gamma_0}\,\xi\,d\xi=\frac{A}{r_1^2}+\frac{2\nu(z-z_0)}{U_0r_1^2}. \tag{13}$$

By definition,

$$\frac{\partial}{\partial\xi}\left(\frac{\Gamma}{\Gamma_0}\right)=\frac{\Gamma}{\Gamma_0}, \qquad \text{when } \xi=1. \tag{14}$$

If the profile were given by Eq. (3), $J(z)=0.4$. If the profile were of constant angular velocity up to $r=r_1$ (i.e., rigid body rotation for $r\leq r_1$) and potential flow for $r>r_1$, $J(z)=0.25$. Thus, the initial value of $J(z)$ can be expected to lie between 0.25 and 0.4. But as z grows larger, r_1 increases and

$$J(z)\rightarrow(2\nu/U_0)\lim(z/r_1^2).$$

Now, as defined in Table I, $r_1=b(\Gamma_0z/U_0)^{1/2}$. Hence as z increases, the growth of the vortex reduces $J(z)$ until it is of order $2\nu/\Gamma_0b^2$. Using Table I, we have estimates of this quantity ranging from about 1.2×10^{-3} for Rose and Dee's data to about 6×10^{-2} for the data of Dosanjh *et al.* In any event, provided the vortex grows at a rate faster than the spread due to molecular diffusion alone, the quantity $J(z)$ decreases from a value greater than 0.25 to a value small compared with one. For the limiting case of infinite Reynolds number, the limiting value of $J(z)$ is zero.

From the definition of $J(z)$ and the constraint Eq. (14), plus the further requirement that $\Gamma\propto\xi^2$ as $\xi\rightarrow0$, it is clear that a reduction in the value of $J(z)$ to something close to zero can only be accomplished by letting $\Gamma_0-\Gamma$ have negative values, unless pathological behavior is allowed. In fact, if we impose the plausible condition that u_ϕ should have only one maximum, it can be proved rigorously that $\Gamma_0-\Gamma$ must be negative for some range in $\xi>1$; for the minimum value of $J(z)$ is $\frac{1}{6}$ when $\Gamma<\Gamma_0$ everywhere and u_ϕ has only one maximum. (See Fig. 2.)

We have thus shown that depending upon the initial distribution of tangential velocity and the Reynolds number of the vortex, the mean circulation in the outer part of the vortex must be greater than Γ_0 when the vortex grows sufficiently (i.e., when the vortex grows to roughly twice the initial size and beyond). It is to be stressed that this result is independent of any hypothesis about the Reynolds stress and any numerical calculations involving any closure approximation should demonstrate an overshoot of circulation, provided the closure approximation is self-consistent with the conservation of angular momentum. (The overshoot has been found independently by Donaldson and Sullivan,[11] who studied the turbulent vortex numerically with a particular closure approximation.)

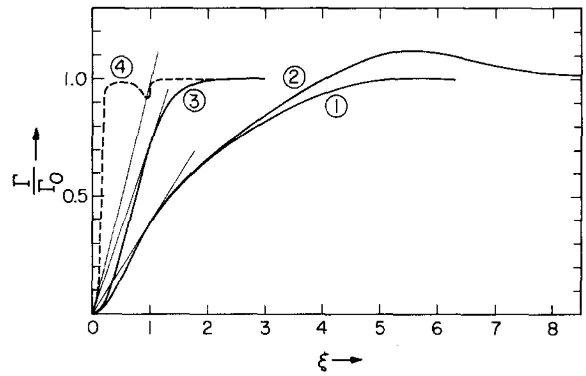

FIG. 2. Sketch of Γ/Γ_0 against ξ. Tangent at $\xi=1$ must pass through origin and the curve must touch tangent from below. $J(z)$ is the area between curve and $\Gamma/\Gamma_0=1$, weighted with ξ. For a single maximum of u_ϕ, the curve must lie below tangent from origin. For a profile with only one maximum for u_ϕ, minimum of J occurs when $\Gamma=\Gamma_0\xi$ for $\xi<1$ and $\Gamma=\Gamma_0$ for $\xi>1$. ① Profile roughly as measured by McCormick. ② Profile for which $J\simeq0$. ③ The profile corresponding to constant eddy viscosity. ④ "Pathological" shape for which J is small and $\Gamma<\Gamma_0$ everywhere.

B. Similarity Solutions

We consider a similarity solution for which

$$\Gamma=\Gamma_0f\left(\frac{r}{\delta},\frac{\nu}{\Gamma_0}\right); \qquad \overline{u_\phi'u_r'}=\left(\frac{\Gamma_0}{\delta}\right)^2g\left(\frac{r}{\delta},\frac{\nu}{\Gamma_0}\right), \tag{15}$$

where $\delta=\delta(z)$ is a characteristic radius of the vortex. By substitution into Eq. (8), it is found that a necessary condition for the existence of similarity solutions is that $\delta\propto(\Gamma_0z/U_0)^{1/2}$. For definiteness, we choose $r_1(z)$ as the characteristic radius. Then,

$$r_1=b(\Gamma_0z/U_0)^{1/2}, \tag{16}$$

where b is a constant to be determined. Substitution into Eq. (8) gives

$$\tfrac{1}{2}(b^2\xi^2)\frac{df}{d\xi}=(2\pi)^{-1}\frac{d}{d\xi}(\xi^2g)-\frac{\nu}{\Gamma_0}\frac{d}{d\xi}\left[\xi^3\frac{d}{d\xi}\left(\frac{f}{\xi^2}\right)\right], \tag{17}$$

where the explicit dependence of f and g on the Reynolds number Γ_0/ν is omitted. In addition, $f(\infty)=1$, $f\propto\xi^2$ as $\xi\rightarrow0$ and

$$f'(1)=f(1) \tag{18}$$

by the definition of r_1. Integrating (17), we obtain

$$\int_0^\infty \xi(1-f)\, d\xi = \frac{2\nu}{\Gamma_0 b^2} \tag{19}$$

which could have been obtained directly from Eq. (13). We repeat that if $2\nu/\Gamma_0 b^2 < \frac{1}{6}$ and the velocity distribution has only one maximum (or equivalently no minimum), it follows rigorously from Eq. (19) that $f>1$ for some values of ξ greater than 1.

Of course, Eq. (17) is one equation for two unknown functions, and cannot be solved in detail without a closure approximation relating g and f. (One such approach is described in Sec. IIC.) Nevertheless, some interesting results can be derived from a fairly weak hypothesis about $g(\xi)$. We know that $g(\xi)=o(\xi^{-2})$ as $\xi \to \infty$, and also $g(\xi) \propto \xi^2$ as $\xi \to 0$ because the velocity field is regular at the origin. It follows from Eq. (17) that g is positive near $\xi=0$, corresponding to an outward flux of angular momentum and it is reasonable to believe that g will be positive everywhere. Then, as ξ increases from 0 to ∞, $g(\xi)$ starts like ξ^2, reaches a maximum at some value of ξ, and then drops back to zero.

We denote this maximum value of $g(\xi)$ by g_m and suppose it occurs at $\xi=\xi_m$. We now make the hypothesis that when the Reynolds number is large, the distribution of g in the neighborhood of its maximum is flat, i.e., we suppose that $d^2g/d\xi^2 \ll g_m$ for ξ around ξ_m. This hypothesis is equivalent to supposing that the structure of the turbulence depends weakly on the distance from the origin in the vicinity of ξ_m. It resembles the hypothesis of a constant stress layer in wall turbulence. Then, we can approximate Eq. (17) by

$$\tfrac{1}{2}(b^2\xi^2)\frac{df}{d\xi} = (2\pi)^{-1}\frac{d}{d\xi}(g_m\xi^2) \tag{20}$$

which integrates to give

$$f(\xi) = (g_m/\pi b^2)\log_e\xi + f(1), \tag{21}$$

provided $\xi=1$ is in the region of constant Reynolds stress. Now, $f(1)=\Gamma_1/\Gamma_0$, and it follows from Eq. (18) that

$$g_m/\pi b^2 = \Gamma_1/\Gamma_0 \tag{22}$$

or in dimensional terms

$$\max(\overline{u_\phi' u_r'}) = \tfrac{1}{2}(\pi U_0 \Gamma_1). \tag{23}$$

Sufficiently accurate measurements of the maximum Reynolds stress to check this result do not seem to be available. In addition, we can, of course, write Eq. (21) as

$$\Gamma/\Gamma_1 = \log(r/r_1) + 1$$

which is the logarithmic profile of Hoffman and Joubert, but it should be stressed that the derivation given here is fundamentally different from theirs. (Their Reynolds stress is, in fact, of the opposite sign to that of our argument.)

C. Analysis Based on Saffman's Model

Saffman[9] proposed a model for inhomogeneous turbulence. The model describes turbulence in terms of two scalar densities, the "energy density" e and the "vorticity density" ω governed by nonlinear diffusion equations:

$$\frac{\partial\omega^2}{\partial t} + u_i\frac{\partial\omega^2}{\partial x_i} = \alpha\omega^2\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2\right]^{1/2} - \beta\omega^3 + \frac{\partial}{\partial x_i}\left(\sigma E\frac{\partial\omega^2}{\partial x_i}\right),$$

$$\frac{\partial e}{\partial t} + u_i\frac{\partial e}{\partial x_i} = \alpha^* e(2s_{ij}^2)^{1/2} - e\omega + \frac{\partial}{\partial x_i}\left(\sigma^* E\frac{\partial e}{\partial x_i}\right), \tag{24}$$

with

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad E = \frac{e}{\omega}. \tag{25}$$

In the above u_i are the components of the mean velocity vector, α, α^*, β, σ, and σ^* are universal constants. For the solution of any flow problem, we have, in addition to the model equations above, the equations of conservation of mass and momentum as

$$\frac{\partial u_i}{\partial x_i} = 0,$$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\rho^{-1}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(2Es_{ij}). \tag{26}$$

It will be observed that the Reynolds stress tensor $2Es_{ij}$ is described by a scalar eddy diffusivity E, related to the turbulent densities e and ω by Eq. (25). Also, the vorticity equation contains $(\partial u_i/\partial x_j)^2$ and is thus sensitive to rigid body rotation while the energy equation contains the strain s_{ij} which is not. This is in accordance with the intuitive idea that energy production should not depend on rotation while vorticity should, being related to angular momentum.

The constants α, α^*, β, σ, and σ^* are determined once and for all by comparing the solutions obtained by using the model equations with the known properties of some simple turbulent flows. From such considerations Saffman found that

$$\alpha^* = 0.3, \quad \sigma = \sigma^* = 0.5, \quad \tfrac{5}{3} \le \beta \le 2, \quad \tfrac{1}{2}\alpha^* \le \alpha \le \alpha^*/\sqrt{2}. \tag{27}$$

Solutions of the model equations are not very sensitive to the values of the parameters in the range given by Eq. (27). One can conveniently choose the values of the parameters in the middle of the range.

A notable feature of the above model is the presence of sharp interfaces dividing turbulent and nonturbulent parts of the fluid. Such interfaces occur in all free turbulent flows including the turbulent vortex of interest here. Full analytical solutions of the model equations are not possible because of their complexity and numerical solutions cannot be satisfactorily extended all the way to such interfaces. We get round this difficulty by using analytical (series) solutions near the interfaces and extend them into the turbulent fluid using a numerical solution.

For the flow in a turbulent trailing vortex, we can look for a similarity-type solution of the form

$$u_\phi=(\Gamma_0 U_0/z)^{1/2}F(\xi),$$
$$e=(\Gamma_0 U_0/z)J(\xi), \tag{28}$$
$$\omega=(U_0/z)K(\xi),$$

where ξ is the similarity variable defined in Sec. IIB. Use of Eq. (28) in Eqs. (24) and (26) results in a set of ordinary differential equations for F, J, and K. Details of the solution, not presented here, may be found in Govindaraju[10] where the development of axial velocity in such a flow is also studied. Results of interest in the present context are presented in Fig. 3 and Table I.

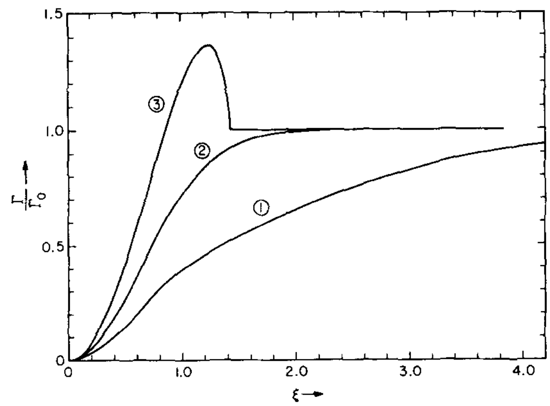

FIG. 3. Comparison of circulation profiles. ① Profile typical of McCormick's measurements; ② Profile for constant eddy viscosity; ③ From Saffman's model.

III. DISCUSSION

A notable feature of all the experimental results reviewed is the absence of an overshoot in the circulation profile. This is very surprising as we have shown in Sec. IIB that an overshoot in the circulation profile is extremely likely. Thus, we are led to believe that the measurements are of insufficient accuracy or are not sufficiently extended in the radial direction or are both. As an example we may consider McCormick's profile of Fig. 3 from which we can compute the integral $J(z)$ of Eq. (13) to be about 2 while we have shown in Sec. IIA that for any reasonable initial distribution of circulation, $J(z)$ is less than 0.4 and decreases downstream. Thus, we have strong indication that the measurement of circulation by McCormick *et al.* is of insufficient accuracy. This conclusion is somewhat strengthened by a significant amount of scatter found in McCormick's data. More accurate studies of circulation distribution in vortices are needed to verify this conclusion.

In Sec. IIC we presented the results of an analysis using Saffman's turbulence model. Figure 3 compares the circulation profile obtained this way with profiles corresponding to constant eddy viscosity and experiments of McCormick. We note rather large differences. The model leads to a circulation profile with a large overshoot (about 40% over the value at ∞). Although the presence of overshoot is clearly correct, the magnitude of the overshoot is disturbing. Also reference to values of a and b in Table I shows that the rate of growth of the vortex is severely overestimated by the model. This failure of the model to adequately describe vortex flow is perhaps associated with the stability of flow due to strong rotation in this type of flow. Some attempts to modify the model to get a more satisfying description of vortex flow (without affecting the results for other simple flows at the same time) have not been very successful to date.

It should be pointed out that the presence of a sharp interface between turbulent and nonturbulent parts of the flow is not a serious fault of the model. The boundary between turbulent and nonturbulent regions of any real flow is, in fact, sharp, but unsteady on a time scale large compared with that of the turbulent fluctuations. Thus, if the solution of the model equations were unsteady, the sharp edge between the turbulent and nonturbulent regions would be smoothed out when the flow is averaged over a time scale large compared with that of the unsteadiness of the flow. Further, such an averaging would considerably reduce the overshoot in the circulation profile. This combined with the fact that the steady circulation profile is strongly unstable in the Rayleigh sense leads us to believe that the model equations permit unsteady solutions with more desirable averaged properties. Finding such solutions, if they exist, does not seem to be feasible because of the complexity of the problem.

Now, some comments about the experimental studies of the trailing vortices are in order. As the Table I indicates, wind tunnel studies of the trailing vortices are for a Reynolds number in the range 10^3–10^5 while the flight studies are in the range 10^6–10^7. From the table it is seen that the vortex growth rate is significantly different for flight tests and the wind tunnel experiments. This difference, barring experimental errors, can only be attributed to the effect of Reynolds number. The data about vortex growth at the higher Reynolds numbers come from the only two flight studies to date. The flight studies are difficult and the accuracy of this data is not known. Wind tunnel studies to date do not cover this range. Thus, it seems desirable to conduct accurate studies of the vortex in a wind tunnel in the range of Reynolds numbers 10^5–10^7.

Such a study of a vortex can be conveniently carried out using a wing spanning the test section of a wind tunnel such that one-half of the wing has an angle of attack equal and opposite the other. The vortices shed by the parts of the wing merge to form a single vortex from the center of the wing. We can easily derive an expression for the vortex Reynolds number and the outer radius of the vortex r_0 (defined as the radius at which $\Gamma/\Gamma_0=0.95$) as

$$\Gamma_0/\nu=C_LU_0c/2\nu, \qquad r_0=k(\tfrac{1}{2}C_Lcz)^{1/2}, \tag{29}$$

where k is a number whose value is somewhat uncertain. It is likely to be between 0.1 and 0.3, the former derived from full flight data and the latter from wind tunnel data. In the above C_L is the lift coefficient of each part of wing of chord c placed in a stream of speed U_0. Calculations based on Eq. (29) indicate that it is not feasible to obtain Reynolds numbers in the range 10^6–10^7 using reasonable dimensions of equipment. It looks feasible to obtain a Reynolds number of 10^5 which can be obtained using a wing of chord 15 cm in a stream of 20 m/sec. The resulting vortex is about 30 cm radius about 10 m downstream of the wing. An accurate study of even such a vortex is likely to provide valuable data about the vortices.

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