Supplementary Material for "Strong planar subsystem symmetry-protected topological phases and their dual fracton orders"

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(Dated: November 23, 2019)

I. REVIEW OF 2D SPTS

In this Section, we review the group cohomological classification of SPTs in 2D, as well as some additional aspects which will prove useful for our arguments related to the SSPT. These include the interpretation of SPT phases as an anomalous action of the symmetries on the edge, and the connection to the braiding and exchange statistics of quasiparticle excitations in the dual gauge theories.

A. Group cohomological classification of 2D SPTs

In the presence of symmetry, the unique ground states of two gapped Hamiltonians belong to the the same phase if they can be transformed into each other via a symmetric local unitary (SLU) transformation.[3] That is, a finite depth quantum circuit in which each gate commutes with the symmetry operation. A state describes a nontrivial 2D SPT phase if it cannot be connected to the trivial product state via an SLU, but can be trivialized if the symmetry restriction is removed. Two dimensional bosonic SPTs with on-site symmetry G, under this phase equivalence relation, are known[2] to be classified according to the third cohomology group $H^3[G, U(1)]$. For the finite abelian group $G = \prod_i \mathbb{Z}_{N_i}$, this can be written out explicitly as

$$H^3[G,U(1)] = \prod_i \mathbb{Z}_{N_i} \prod_{i<j} \mathbb{Z}_{\gcd(N_i,N_j)} \prod_{i<j<k} \mathbb{Z}_{\gcd(N_i,N_j,N_k)} \tag{1}$$

where gcd denotes the greatest common denominator. The three factors are commonly referred to as type-I, type-II, and type-III cocycles. Type-III cocycles correspond to a gauge dual with non-abelian quasiparticle excitations; as our focus is on SSPTs with abelian fracton duals, we will discuss only on type-I and II cocycles.

1. *The Else-Nayak procedure*

Let us derive the group cohomological classification via a series of dimensional reduction procedures, introduced by Else and Nayak.[12] which will prove useful in our discussion of SSPTs. Although the original procedure observed a system with a physical edge, here we prefer to deal with a "virtual" edge, meaning: the full system has no edges, but we will consider applying the symmetry only to a finite region M of the system. At the edges of M, this symmetry will act non-trivially as if at a physical edge. The advantage of this approach is that it removes any ambiguity related to choice of how the model is defined at the physical edges (and will be useful in the case of SSPTs).

Let $|\psi\rangle$ be the unique gapped ground state of our Hamiltonian H with on-site symmetry group G, and $S(g)$ be the symmetry operation realizing the symmetry element $g \in G$. We have that $[H, S(g)] = 0$ and, without loss of generality, take the ground state to be uncharged under the symmetry $S(g)|\psi\rangle = |\psi\rangle$. Now, let $S_M(g)$ be the symmetry operation $S(g)$, but restricted to a region M. $S_M(g)$ acting on the ground state will no longer leave it invariant, but will create some excitation along the boundary of this region, ∂M. Since $|\psi\rangle$ is the unique ground state of a gapped Hamiltonian, this excitation may always be locally annihilated by some symmetric unitary transformation $U_{\partial M}(g)^\dagger$, which only has support near ∂M. That is,

$$S_M(g)|\psi\rangle = U_{\partial M}(g)|\psi\rangle \tag{2}$$

It is straightforward to show that the matrices $U_{\partial M}(g)$ form a twisted representation of G, satisfying

$$^{S_M(g_2)}U_{\partial M}(g_1)U_{\partial M}(g_2)|\psi\rangle = U_{\partial M}(g_1 g_2)|\psi\rangle \tag{3}$$

where $^{B}A \equiv BAB^\dagger$ denotes conjugation of A by B, and that they must commute with any global symmetry operation, $[U_{\partial M}(g), S(g')] = 0$.

We now perform a further restriction: from ∂M down to a segment C, $U_C(g)$. This is always possible. $U_C(g)$

need only satisfy Eq 3 up to some unitary operator $V_{\partial C}(g_1, g_2)$ at the two endpoints of C,

$$ {}^{S_M(g_2)}U_C(g_1)U_C(g_2)\left|\psi\right\rangle = V_{\partial C}(g_1, g_2)U_C(g_1 g_2)\left|\psi\right\rangle \tag{4}$$

By associativity, $V_{\partial C}$ must satisfy

$$\begin{aligned} &{}^{S_M(g_3)}V_{\partial C}(g_1, g_2)V_{\partial C}(g_1 g_2, g_3) = \\ &{}^{S_M(g_2 g_3)U_C(g_1)}V_{\partial C}(g_2, g_3)V_{\partial C}(g_1, g_2 g_3) \end{aligned} \tag{5}$$

when acting on $\left|\psi\right\rangle$. The final restriction is from ∂C, which consists of two disjoint regions a and b, down to simply a: $V_{\partial C}(g) = V_a(g)V_b(g) \to V_a(g)$. $V_a(g)$ need only satisfy Eq. 5 up to a $U(1)$ phase factor, which can be cancelled out by the contribution from $V_b(g)$.

$$\begin{aligned} &{}^{S_M(g_3)}V_a(g_1, g_2)V_a(g_1 g_2, g_3) = \\ &\omega(g_1, g_2, g_3)\,{}^{S_M(g_2 g_3)U_C(g_1)}V_a(g_2, g_3)V_a(g_1, g_2 g_3) \end{aligned} \tag{6}$$

where $\omega : G^3 \to U(1)$. This entire dimensional reduction process is shown in Figure 1.

One can further show that $\omega(g_1, g_2, g_3)$ satisfies the 3-cocycle condition[12]

$$1 = \frac{\omega(g_1, g_2, g_3)\omega(g_1, g_2 g_3, g_4)\omega(g_2, g_3, g_4)}{\omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4)} \tag{7}$$

and since $V_a(g_1, g_2)$ is only defined up to a phase factor $\beta(g_1, g_2)$, we must identify

$$\omega(g_1, g_2, g_3) \sim b(g_1, g_2, g_3)\omega(g_1, g_2, g_3) \tag{8}$$

where

$$b(g_1, g_2, g_3) = \frac{\beta(g_1, g_2)\beta(g_1 g_2, g_3)}{\beta(g_2, g_3)\beta(g_1, g_2 g_3)} \tag{9}$$

is called a coboundary. The classification of functions satisfying Eq. 7, modulo transformations Eq. 8, is exactly the definition of the third cohomology group $H^3[G, U(1)]$. The class of ω is the element of $H^3[G, U(1)]$ to which it corresponds.

2. Invariant combinations in H^3

Suppose we have followed the Else-Nayak procedure on a system and obtained the cocycle function $\omega(g_1, g_2, g_3)$. How do we identify which class in Eq. 1 it belongs to? One way to do so is to identify combinations of ω which are invariant under the transformation Eq. 8, whose value can tell us about the class.

For simplicity, we focus first on $G = (\mathbb{Z}_N)^M$. Let us first write down an explicit form[9,10] for ω,

$$\omega(g_1, g_2, g_3) = \exp\left\{ \sum_{i \le j} \frac{2\pi i p^{ij}}{N^2} g_1^i (g_2^j + g_3^j - [g_2^j + g_3^j]) \right\} \tag{10}$$

where g^i is an integer modulo N denoting the component of g in the ith $\mathbb{Z}_N$ factor, $g = (g^1, g^2, \ldots, g^M)$, $[\cdot]$ denotes the interior modulo N, and p^{ij} are integers mod N. It is straightforward to confirm that ω satisfies the 3-cocycle condition. As we will show, the different choices of p^{ij} for $i \le j$ correspond to different classes in $H^3[G, U(1)]$. From Eq. 1, $p_I^i \equiv p^{ii}$ specify the value of the type-I cocycles and $p_{II}^{ij} \equiv p^{ij}$ specify the type-II cocycles.

Define

$$\Omega(g) = \prod_{n=1}^{N} \omega(g, g^n, g) \tag{11}$$

and

$$\Omega_{II}(g, h) = \frac{\Omega(gh)}{\Omega(h)\Omega(h)} \tag{12}$$

both of which one can readily verify are invariant under transformations of the type Eq. 8. Given a choice of generators, $G = \langle a_1, \ldots, a_M \rangle$, an explicit calculation shows that

$$\Omega(a_i) = e^{\frac{2\pi i}{N} p_I^i} \tag{13}$$

and

$$\Omega_{II}(a_i, a_j) \equiv \frac{\Omega(a_i a_j)}{\Omega(a_i)\Omega(a_j)} = e^{\frac{2\pi i}{N} p_{II}^{ij}} \tag{14}$$

thus correctly identifying the value of the type-I and type-II cocycles. Thus, if we are given an unknown ω, we may simply compute $\Omega(a_i)$ and $\Omega_{II}(a_i, a_j)$ for all i and j to identify its class.

We may define the symmetric matrix $M_{ij} = p_{II}^{ij}$ and $M_{ii} = 2p_I^i$. Then, we have

$$\Omega(g) = e^{\frac{\pi i}{N} \vec{g}^T \mathbf{M} \vec{g}} \tag{15}$$

and

$$\Omega_{II}(g, h) = e^{\frac{2\pi i}{N} \vec{g}^T \mathbf{M} \vec{h}} \tag{16}$$

for arbitrary elements g and h, where $\vec{g} = (g^1, \ldots, g^M)$.

3. Group cohomology models

The group cohomology models are a powerful construction that allows us to explicitly write down models realizing SPT phases corresponding to an arbitrary cocycle[9,10]. Although these models have an elegant interpretation in terms of a path integral on arbitrary triangulations of space-time, we will use them to simply define Hamiltonian models on a square lattice.

We first define the homogenous cocycle $\nu : G^4 \to U(1)$,

$$\nu(g_1, g_2, g_3, g_4) = \omega(g_1^{-1} g_2, g_2^{-1} g_3, g_3^{-1} g_4) \tag{17}$$

which satisfies $\nu(gg_1, gg_2, gg_3, gg_4) = \nu(g_1, g_2, g_3, g_4)$. In terms of ν, the cocycle condition (Eq. 7) is

$$1 = \frac{\nu(g_1, g_2, g_3, g_4)\nu(g_1, g_2, g_4, g_5)\nu(g_2, g_3, g_4, g_5)}{\nu(g_1, g_2, g_3, g_5)\nu(g_1, g_3, g_4, g_5)} \qquad (18)$$

We will use ν to define our ground state wavefunction.

Take G-valued degrees of freedom on each site $\mathbf{r}$, $|g_\mathbf{r}\rangle$. The ground state of our model $|\psi\rangle$ is an equal amplitude sum of all possible configurations

$$|\psi\rangle = \sum_{\{g_\mathbf{r}\}} f(\{g_\mathbf{r}\})\,|\{g_\mathbf{r}\}\rangle \qquad (19)$$

where $f(\{g_\mathbf{r}\})$ is a $U(1)$ phase for each configuration. The group cohomology model is defined by the choice

$$f(\{g_\mathbf{r}\}) = \prod_\mathbf{r} \frac{\nu(g_\mathbf{r}, g_{\mathbf{r}+\mathbf{x}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}}, g^*)}{\nu(g_\mathbf{r}, g_{\mathbf{r}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}}, g^*)} \equiv \prod_\mathbf{r} f_\mathbf{r}(\{g_\mathbf{r}\}) \qquad (20)$$

where $\mathbf{x}, \mathbf{y}$ are the two unit vectors, $g^* \in G$ is an arbitrary element which we can simply take to be the identity $g^* = 1$, and we have defined a phase contribution $f_\mathbf{r}$ for each plaquette. This arises from a triangulation of each square plaquette into two triangles, each of which contribute a phase; those interested in the details of the construction are directed to Ref 9.

Performing the Else-Nayak procedure outlined in Section I A 1 on this ground state results in exactly the cocycle ω used to construct the state, up to a coboundary (Eq. 8).

To obtain a gapped local Hamiltonian realizing this state as its ground state, we simply consider a set of local ergodic transitions $\langle\{g_\mathbf{r}\} \to \{g'_\mathbf{r}\}\rangle$, multiplied by an appropriate phase factor,

$$H = -\sum_{\langle\{g'_\mathbf{r}\}\to\{g_\mathbf{r}\}\rangle} \frac{f(\{g'_\mathbf{r}\})}{f(\{g_\mathbf{r}\})}\,|\{g'_\mathbf{r}\}\rangle\langle\{g_\mathbf{r}\}| \qquad (21)$$

which by construction has $|\psi\rangle$ as its unique ground state. We can simply choose $\{g'_\mathbf{r}\}$ to differ from $\{g_\mathbf{r}\}$ by the action of a generator a_i of G on a single site $\mathbf{r}$. The Hamiltonian will then be a sum of mutually commuting terms consisting of a "flip" operator $|a_i g_\mathbf{r}\rangle\langle g_\mathbf{r}|$ on each site, multiplied by an appropriate phase factor depending on the state $\{g_\mathbf{r}\}$ near that site.

4. Gauge duality

The group cohomological classification of an SPT has an elegant interpretation in terms of braiding statistics of its gauge dual.[11] We will briefly outline the gauging process (as applied to the group cohomology models), and discuss the relevant statistical processes.

Consider the group cohomology SPT model on a square lattice given by Eq. 21. To gauge the global symmetry, we define gauge degrees of freedom $g_{\mathbf{r},\mathbf{r}'} = g^{-1}_{\mathbf{r}',\mathbf{r}}$ for each nearest neighbor pair $(\mathbf{r}, \mathbf{r}')$, and enforce a Gauss's law constraint at each vertex $\mathbf{r}$ which involves the matter degree of freedom $g_\mathbf{r}$ and the adjacent gauge degrees of freedom $g_{\mathbf{r},\mathbf{r}'}$. Then, we minimally couple the symmetric Hamiltonian to the gauge degrees of freedom by replacing the operators $g_{\mathbf{r}'}g^{-1}_\mathbf{r}$ with $g_{\mathbf{r}'}g_{\mathbf{r},\mathbf{r}'}g^{-1}_\mathbf{r}$ throughout. In addition, we energetically enforce the zero-flux constraint $g_{\mathbf{r}_1\mathbf{r}_2}g_{\mathbf{r}_2\mathbf{r}_3}g_{\mathbf{r}_3\mathbf{r}_4}g_{\mathbf{r}_4\mathbf{r}_1} = 1$ for the square plaquette with corners $\mathbf{r}_{1\ldots4}$ (labeled going clockwise or counterclockwise), by adding an appropriate projection term to the Hamiltonian.

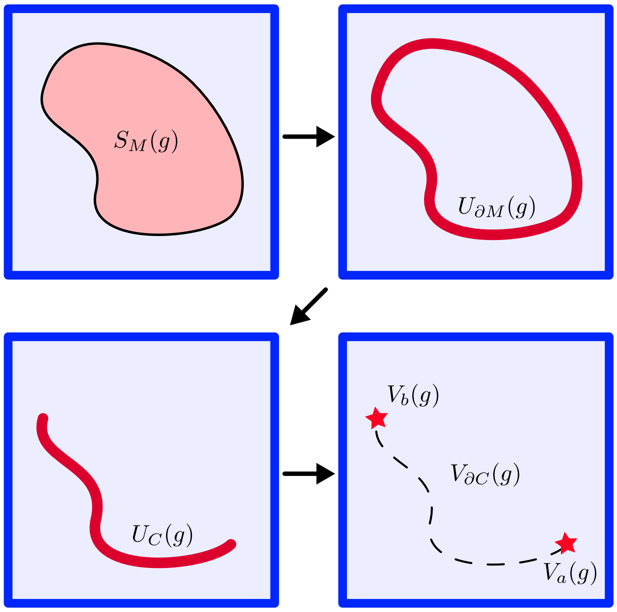

FIG. 1. The dimensional reduction procedure in the Else-Nayak procedure. We start with a truncated global symmetry operator, $S_M(g)$. This acts on the ground state as a unitary $U_{\partial M}(g)$ along the edge of M. We further restrict this unitary down to a line segment C, $U_C(g)$. Restricted to C, $U_C(g)$ behaves as a representation of G only up to unitaries $V_{\partial C}(g)$ at its endpoints. Finally, we restrict to a single endpoint $V_a(g)$, where associativity of the representation is only satisfied up to a phase $\omega(g_1, g_2, g_3)$, defining our 3-cocycle.

The resulting model describes a topologically ordered phase, with characteristic properties such as a topological ground state degeneracy on a torus and quasiparticle excitations with anyonic braiding statistics. There are two types of excitations: gauge charge, denoted by e_g, and gauge flux, denoted by m_g, for each $g \in G$. The former are created by gauged versions of operators of the form

$$Z_g^\dagger(\mathbf{r}_1)Z_g(\mathbf{r}_2) = \sum_{\{g_\mathbf{r}\}} e^{\frac{2\pi i}{N}(g^i_{\mathbf{r}_2} - g^i_{\mathbf{r}_1})}\,|\{g_\mathbf{r}\}\rangle\langle\{g_\mathbf{r}\}| \qquad (22)$$

which creates a charge-anticharge pair, e_g and e_g^{-1}, at positions $\mathbf{r}_2$ and $\mathbf{r}_1$. To create gauge flux excitations, instead consider the gauged version of the operator

$$L(g)\,|\psi\rangle \equiv U^\dagger_{\partial M}(g)S_M(g)\,|\psi\rangle = |\psi\rangle \qquad (23)$$

where $S_M(g)$ is a symmetry operator restricted to a region M and $U_{\partial M}(g)$ is the action on the boundary ∂M, as in the dimensional reduction procedure of Section I A 1. The gauged version of $S_M(g)$ only flips $g_{\mathbf{r},\mathbf{r}'}$ near at the boundary, and so the gauged $L(g)$ operator has support only on ∂M. Now, if we further restrict $L(g) \to L_C(g)$ to an open segment C, $L_C(g)$ creates two quasiparticle excitations at the two endpoints, which we identify as the gauge flux-antiflux pair m_g and m_g^{-1}. Note that there is an ambiguity in defining the gauged version of $L(g)$, which may result in a different definition of the gauge flux excitation, $m_g \sim m_g e_{g'}$. Thus, gauge fluxes are only well defined modulo attachment of charges.

The group cohomological classification of the ungauged SPT manifests in the self and mutual statistics of gauge fluxes in the gauged theory. Let a_i be the generator of the ith factor of $\mathbb{Z}_N$ in G, and e_i and m_i be its gauge charge and flux excitations. For two identical excitations, we can define an exchange phase via a process in which their two positions are exchanged. For two different excitations, we may instead define the full braiding phase, which is accumulated when one particle encircles another. In the gauge theory, e_i all have trivial exchange and only braid non-trivially with its own gauge flux m_i. Meanwhile, the gauge flux m_i has an exchange statistic $e^{\frac{2\pi i p_I^i}{N^2}}$ with itself, and a mutual braid $e^{\frac{2\pi i p_{II}^{ij}}{N^2}}$ with m_j. Notice that the exchange and mutual braid of m_i is only well defined modulo $e^{\frac{2\pi i}{N}}$, since m_i is only well defined modulo charge attachment. For a general gauge flux m_g, its exchange phase is given by an Nth root of $\Omega(g)$, which can be straightforwardly calculated from the $\mathbf{M}$ matrix (Eq. 15).

For those familiar with the K matrix formulation of Abelian topological orders, we note that the K matrix characterizing the gauged theory is given by

$$\mathbf{K} = \begin{bmatrix} -\mathbf{M} & N\mathbf{1} \\ N\mathbf{1} & 0 \end{bmatrix} \tag{24}$$

such that

$$\mathbf{K}^{-1} = \begin{bmatrix} 0 & \frac{1}{N}\mathbf{1} \\ \frac{1}{N}\mathbf{1} & \frac{1}{N^2}\mathbf{M} \end{bmatrix} \tag{25}$$

where the indices represent quasiparticle excitations ordered as $\{e_1, e_2, \dots, m_1, m_2, \dots\}$. The exchange statistic of a quasiparticle $\vec{l}$ written in this basis is given by $e^{\pi i \vec{l}^T \mathbf{K}^{-1} \vec{l}}$, and the mutual braiding statistic between $\vec{l}_1$ and $\vec{l}_2$ is $e^{2\pi i \vec{l}_1^T \mathbf{K}^{-1} \vec{l}_2}$.

II. THE SYMMETRY ACTION IN SSPTS

Let us briefly discuss how symmetries may act anomalously on the edges in an SSPT. Again, let us consider only virtual edges, as in our earlier 2D discussion.

We first discuss the case for 3-foliated phases. Let us take a cubic subregion M, and consider applying symmetry operations restricted to this subregion. An xy planar symmetry restricted to this region, $S_M^{xy}(z; g)$, will act on the ground state as some unitary operation along the boundary near the plane z,

$$S_M^{(xy)}(z; g) |\psi\rangle = U_{\partial M}(z; g) |\psi\rangle \tag{26}$$

exactly as for the 2D SPT earlier. The same is true for yz or zx planar symmetries.

Now, let us instead consider applying the global symmetry restricted to this subsystem, which we call simply $S_M(g)$. Along the xy faces of the cube M, $S_M(g)$ looks like a symmetry operation $S_M^{xy}(z; g)$, and similarly along the yz and zx faces. Thus, $S_M(g)$ acting on the ground state cannot act non-trivially along the edges. The only place where $S_M(g)$ does not look like a symmetry operator is along the hinges of M, which we denote by h_M. Thus,

$$S_M(g) |\psi\rangle = U_{h_M}(g) |\psi\rangle \tag{27}$$

acts as some unitary operator along the hinges of M. For 2-foliated phases, we may apply the same argument, except that only two of the planar symmetries exist. Suppose we only have xy and yz planar symmetries. Then, $S_M(g)$ acting on the ground state may act non-trivially along the hinges and the zx face of M. This is shown in Fig. 2 for the 3 and 2-foliated cases.

Knowing the way the symmetry acts along the edges of M is sufficient to obtain the $H^3[G^L, U(1)]$ classification of the phase. For example, one can readily apply the Else-Nayak procedure detailed earlier in order to extract the cocycle function ω. However, there is more information contained in $U_{h_M}(g)$ that is missed in this process. We know that $U_{h_M}(g)$ must commute with all untruncated symmetries, such as $S^{(xy)}(z; g)$, when acting on the ground state. That is, $U_{h_M}(g)$ has to be overall charge neutral under all symmetries. Consider the symmetry $S^{(xy)}(z; g')$ which intersects with four hinges in $U_{h_M}(g)$, as in Fig. 2. The four intersection locations are spatially separated, thus one can sensibly define a charge on each of the four hinges, which do not have to be trivial. That is, $S^{xy}(z; g')$ commuting with the first hinge may result in a phase $e^{i\phi_1}$, the second $e^{i\phi_2}$, and so forth, which is fine as long as $e^{i(\phi_1+\phi_2+\phi_3+\phi_4)} = 1$. These charges pinned to the hinges cannot be removed by a symmetric local unitary transformation, and are therefore a sign of a non-trivial phase. Such charges arise due to the existence of 2D linear SSPTs: 2D phases with SSPT order protected by line-like subsystem symmetries.[4] In the following section we will construct an explicit example of this.

III. NON-TRIVIAL SSPT PHASES WITH TRIVIAL H^3

In this section, we highlight a mechanism by which an SSPT phase may be non-trivial, despite appearing trivial in our $H^3[G^L, U(1)]$ picture along all planar directions (but still a weak phase overall). Let us begin with an example: the so-called semionic X-cube model (See Eq. 14

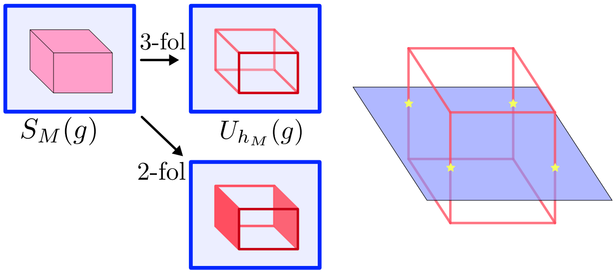

FIG. 2. (Left) The action of a symmetry restricted to a large cube, $S_M(g)$, acts on the ground state as a unitary supported along the hinges $U_{h_M}(g)$ in a 3-foliated model. In the 2-foliated model with only xy and yz planar symmetries, it may act non-trivially along the zx face of M as well. (Right) A phase may be non-trivial if the hinges of the cube operator contain non-trivial charge under a planar symmetry. This type of non-triviality does not show up in its H^3 classification, but can be generated by stacking 2D linear SSPTs.

and Fig. 9 in Ref. 1, also Ref. 6). This is a $G = \mathbb{Z}_2$ fracton gauge theory in which lineons satisfying a triple fusion rule have a -1 braiding statistic with one another. There is a qubit degree of freedom on each site, which are acted on by Pauli matrices X and Z. The ungauged model is given by the Hamiltonian

$$H_{\text{sem}} = -\sum_{\mathbf{r}} X_{\mathbf{r}} C_{\mathbf{r}} C_{\mathbf{r}-\mathbf{x}-\mathbf{y}-\mathbf{z}} \tag{28}$$

where

$$C_{\mathbf{r}} = \prod_{a=0}^{1} \prod_{b=0}^{1} \prod_{c=0}^{1} Z_{\mathbf{r}+a\mathbf{x}+b\mathbf{y}+c\mathbf{z}} \tag{29}$$

is a product of 8 Zs on the corners of a cube.

We immediately notice that this model has a trivial $H^3[G^L, U(1)]$, which one can confirm by simply noting that a planar symmetry actually acts trivially along the edges (except at a corner), and so cannot produce anything non-trivial under the Else-Nayak procedure. This is due to the presence of higher symmetry: this model actually is symmetric under *line-like* symmetries. For example, a product of Xs along the x direction,

$$S^x(y, z; g) = \prod_{x} X_{\mathbf{r}=(x,y,z)} \tag{30}$$

commutes with H_{sem}. Acting on the ground state, the global symmetry operator truncated to a cube $S_M(g)$ acts by creating charges localized at its corners (as it must be for a 3D system with line-like subsystem symmetries). It is this pattern of charges which leads to the non-trivial lineon braiding phase.

Notice that there is no contradiction between the system having a trivial $H^3[G^L, U(1)]$ classification and lineons having a -1 braiding statistic. This is due to the fact that the fundamental braiding process which $H^3[G^L, U(1)]$ cares about is between lineon dipoles. Braiding two lineons in a plane $z_0 - 1/2$ is like braiding a stack of lineon dipoles on planes $z < z_0$ with another stack $z \geq z_0$. However, the braiding phase in a $\mathbb{Z}_2$ theory is only defined modulo ± 1, and so a braiding phase of -1 is the same as trivial from this perspective.

It is straightforward to show that the model described by H_{sem} is weak. One can write the Hamiltonian as

$$H_{\text{sem}} = -\sum_{\mathbf{r}} U X_{\mathbf{r}} U^{\dagger} \tag{31}$$

where U is a local unitary circuit consisting of CZ gates. The ground state is then

$$|\psi\rangle = U |\psi_0\rangle \tag{32}$$

where $|\psi_0\rangle$ is the trivial paramagnetic phase. It is possible to write U as

$$U = \prod_{z} U_z \tag{33}$$

where U_z acts only between layers z and $z+1$, and U_z commutes with all planar symmetries (and each other) This is exactly the form of a planar-symmetric local unitary circuit (just a higher dimensional version of the linearly-symmetric local unitary circuit defined in Ref 4). Thus, $U_z |\psi_0\rangle$, where $|\psi_0\rangle$ is the trivial paramagnetic state $X_{\mathbf{r}} |\psi_0\rangle = |\psi_0\rangle$, describes a 2D phase on planes z, $z+1$ (which is actually a 2D linear SSPT), and the ground state of H_{sem} is simply the stack of these.

One of the consequences of the fact that H_{sem} describes a weak phase is that there is no obstruction to constructing an SSPT phase which is described by H_{sem} for $z \ll 0$ and is completely trivial H_{triv} for $z \gg 0$. For example, we can define the Hamiltonian

$$\begin{aligned} H_{\text{half}} = &- \sum_{\mathbf{r}:r_z<0} X_{\mathbf{r}} C_{\mathbf{r}} C_{\mathbf{r}-\mathbf{x}-\mathbf{y}-\mathbf{z}} - \sum_{\mathbf{r}:r_z>0} X_{\mathbf{r}} \\ &- \sum_{\mathbf{r}:r_z=0} X_{\mathbf{r}} Z_{\mathbf{r}} Z_{\mathbf{r}+\mathbf{x}} Z_{\mathbf{r}+\mathbf{y}} Z_{\mathbf{r}+\mathbf{x}+\mathbf{y}} C_{\mathbf{r}-\mathbf{x}-\mathbf{y}-\mathbf{z}} \end{aligned} \tag{34}$$

which is composed of commuting terms. If we look at the action of $S_M(g)$ where M is a large cube crossing $z = 0$, one finds that there is a single Z pinned to each hinge at $z = 0$, exactly as discussed in Section II. Indeed, by stacking 2D SSPTs, it is possible to realize phases with various choices of allowable charges pinned on each hinge.

Finally, we note that we do not have a proof that *all* phases with a trivial $H^3[G^L, U(1)]$ classification are weak. There may also exist other mechanisms by which a phase may be non-trivial. However, we are not aware of any counterexamples.

IV. MOBILITY OF SINGLE LINEONS

Here, we show that in the fracton dual of a 3-foliated phase, a "single lineon" need not actually be mobile along lines. In a slight abuse of nomenclature, we will still call this excitation the lineon, even though it may not be mobile along a line.

First, let us identify what is commonly referred to as the lineon. Again, consider the action of the global symmetry truncated to a cube, $S_M(g)$, which acts on the ground state as some unitary along the hinges h_M,

$$S_M(g)\,|\psi\rangle = U_{h_M}(g)\,|\psi\rangle \tag{35}$$

Then, the operator $L_M(g) \equiv S_M(g)U^{\dagger}_{h_M}(g)$ acts trivially on the ground state. The gauged version of the operator $L_M(g)$ will define our lineon.

Let us review how the generalized gauging process works for a 3-foliated planar symmetric model (see Refs. 13–15 for details). For each xy plaquette, we introduce the plaquette gauge degree of freedom $\tilde{g}^{(xy)}_{\mathbf{r}}$ which resides on the plaquette centered at $\mathbf{r}+\frac{1}{2}\mathbf{x}+\frac{1}{2}\mathbf{y}$, and similarly for yz and zx plaquettes. A Gauss's law constraint is subsequently enforced at each vertex $\mathbf{r}$ which involves the matter degree of freedom $g_{\mathbf{r}}$ and the 12 adjacent gauge degrees of freedom. We then minimally couple the subsystem symmetric Hamiltonian to the gauge degrees of freedom by replacing the symmetric coupling terms $g_{\mathbf{r}}g^{-1}_{\mathbf{r+x}}g^{-1}_{\mathbf{r+y}}g_{\mathbf{r+x+y}}$ with the term $\tilde{g}^{(xy)}_{\mathbf{r}}g_{\mathbf{r}}g^{-1}_{\mathbf{r+x}}g^{-1}_{\mathbf{r+y}}g_{\mathbf{r+x+y}}$ throughout. Finally, we energetically enforce the zero-flux constraints

$$1 = \tilde{g}^{(zx)}_{\mathbf{r}}\tilde{g}^{(yz)}_{\mathbf{r+x}}\left(\tilde{g}^{(zx)}_{\mathbf{r+y}}\tilde{g}^{(yz)}_{\mathbf{r}}\right)^{-1} \tag{36}$$

$$1 = \tilde{g}^{(xy)}_{\mathbf{r}}\tilde{g}^{(zx)}_{\mathbf{r+y}}\left(\tilde{g}^{(xy)}_{\mathbf{r+z}}\tilde{g}^{(zx)}_{\mathbf{r}}\right)^{-1} \tag{37}$$

$$1 = \tilde{g}^{(yz)}_{\mathbf{r}}\tilde{g}^{(xy)}_{\mathbf{r+z}}\left(\tilde{g}^{(yz)}_{\mathbf{r+x}}\tilde{g}^{(xy)}_{\mathbf{r}}\right)^{-1} \tag{38}$$

by adding terms to the Hamiltonian which projects onto this subspace.

In terms of these plaquette variables, the symmetry $S_M(g)$ acts only on the hinges of the cube M. If the gauged version of U_{h_M} can be written such that it only acts along the hinges as well, then the gauged version of $L_M(g) = S_M(g)U^{\dagger}_{h_M}$ also only acts along the hinges. In this case, if we truncate $L_M(g)$, we obtain an operator which creates a g lineon excitation at each of its truncated hinges. A single lineon is guaranteed to be mobile along a line, simply due to the fact that the operator $L_M(g)$ is line-like along the hinges. While this has been true in virtually all previously studied models, it is not true generally.

Indeed, consider H_{half} from Section III. In that model, $U_{h_M}(g)$ has a single charge $Z_{\mathbf{r}}$ pinned at each place where h_M crossed $z=0$. However, there is no way to gauge $U_{h_M}(g)$ in such a way as to keep the support of the operator only along the hinges. This means that one cannot construct a lineon which crosses the $z=0$ plane alone.

From the perspective of the fracton order, we may consider a z-moving lineon at $z>0$. Now, suppose we naively move this lineon down to $z<0$, crossing the $z=0$ plane. What one will find is that, upon crossing, there is a single fracton charge excitation stuck at the $z=0$ plane. As it is a single fracton, which is immobile, one cannot simply move it along with the lineon (which would simply amount to a redefinition of the lineon for $z<0$ vs $z>0$). Thus, a z-moving lineon cannot cross the $z=0$ plane without paying an energy penalty in the form of a fracton stuck at $z=0$.

Now, instead of having simply a single plane at $z=0$ where charges are pinned, we can imagine constructing a model in which charges are pinned on every plane, or every other plane, for example. In this case, a single lineon moving would create fracton excitations as it moved along, which are unable to be annihilated or moved along with the lineon as a redefinition. A single lineon therefore cannot be moved along a line without creating additional excitations. However, a pair of lineon anti-lineon on adjacent planes (the gauge flux) is always guaranteed to be a planon.

V. PROOF OF CONSTRAINTS

Here, we prove the two constraints mentioned in the main text for 2 or 3-foliated models. Let us label by g_z the group element g in the zth factor of G^L, and $g_{\text{glo}} = \prod_z g_z$ a global symmetry. Again, we view the SSPT as a quasi-2D system with the large symmetry group G^L. We will denote the representation of the symmetry g acting on the zth plane by simply $S(g_z)$, rather than $S^{xy}(z;g)$ as in the main text. We take the system to also be symmetric under yz-planar symmetries.

Consider a square region M of this quasi-2D system (M would contain all sites x,y,z with $x_1 < x < x_2$, $y_1 < y < y_2$, and all z, for some choice of $x_{1,2}, y_{1,2}$). The key fact is that the global symmetry truncated to M, $S_M(g_{\text{glo}})$, acts trivially along the yz face of M (simply due to the fact that it acts identically to yz-planar symmetries). Thus, $U_{\partial M}(g_{\text{glo}})$ acts trivially along the yz face. Now, we may perform the Else-Nayak procedure, further choosing a restriction to an open segment C which ends along the yz face. Going through the procedure with a trivial $U_{\partial M}(g_{\text{glo}})$, we can always get $\omega(g_{\text{glo}}, h_{\text{glo}}, k_{\text{glo}}) = 1$ for arbitrary $g,h,k \in G$ (up to a coboundary).

This leads to our second constraint. Calculating the invariant $\Omega(g) = \prod_{n=1}^{N}\omega(g,g^n,g)$ for any global symmetry results in a trivial type-I cocycle with itself, $\Omega(g_{\text{glo}}) = 1$. In terms of the $\mathbf{M}$ matrix,

$$\Omega(g_{\text{glo}}) = e^{\pi i \vec{g}^{T}_{\text{glo}}\mathbf{M}\vec{g}_{\text{glo}}/N} = 1 \tag{39}$$

for each generator of G is exactly the global constraint from the main text.

Next, consider the type-II cocycle between a global symmetry h_{glo} and g_z. This is given by the ratio

$$\Omega_{II}(g_z, h_{\text{glo}}) = \Omega(g_z h_{\text{glo}})/(\Omega(g_z)\Omega(h_{\text{glo}})). \tag{40}$$

We can calculate $\Omega(h_{\text{glo}}g_z)$ using the Else-Nayak procedure, which we wish to show is simply equal to $\Omega(g_z)$.

First, note that if we have $S_{M'}(g)$ defined on some larger M', which has a boundary action $U_{\partial M'}(g)$, we may always use this to construct an edge action for $S_M(g)$ as

$$\begin{aligned} S_M(g)\,|\psi\rangle &= S_M(g)S^\dagger_{M'}(g)S_{M'}(g)\,|\psi\rangle \\ &= S_M(g)S^\dagger_{M'}(g)U_{\partial M'}(g)\,|\psi\rangle \\ &\equiv U_{\partial M}(g)\,|\psi\rangle \end{aligned} \tag{41}$$

which acts simply as $S_M(g)S^\dagger_{M'}(g)$ near ∂M, and has deferred all the non-triviality over to $\partial M'$. Now, we may use this construction for $U_{\partial M}(g_z)$ in the Else-Nayak procedure, which, along the yz face, is equivalent to $U_{\partial M}(g_z h_{\rm glo})$ (since $U_{\partial M}(g_z) = 1$ is trivial along this edge). The procedure then continues, and since $U_{\partial M}(g_z h_{\rm glo})$ is (by construction) invariant under conjugation by $S_M(h_{\rm glo})$, the process proceeds exactly the same regardless of whether we had chosen $g_z h_{\rm glo}$ or just g_z. We can therefore always choose to have

$$\omega(g_z h_{\rm glo}, (g_z h_{\rm glo})^n, g_z h_{\rm glo}) = \omega(g_z, (g_z)^n, g_z) \tag{42}$$

so that $\Omega(g_z h_{\rm glo}) = \Omega(g_z)$, and therefore $\Omega_{II}(g_z, h_{\rm glo}) = 1$. In terms of the $\mathbf{M}$ matrix,

$$\Omega_{II}(g_z, h_{\rm glo}) = e^{2\pi i \vec{h}_z^T \mathbf{M} \vec{g}_{\rm glo}/N} = 1 \tag{43}$$

for h, g, being generators of G, is exactly the local constraint in the main text.

VI. VARIOUS PROOFS FOR INVARIANTS F_1 AND F_2

A. Independence of direction for F_1 and F_2, and triviality of F_2 in 3-foliated model

In this section, we prove the claims in the main text that 1) the invariants F_1 and F_2 must be the same regardless of which direction of planar symmetry we look at, and 2) that F_2 must be trivial in a 3-foliated model.

We first introduce some ideas for a regular 2D SPT. First, let us make the simplifying assumption that $U_{\partial M}(g)$ is a purely diagonal operator. This is always possible to do in our class of models, where $|\psi\rangle$ is an equal amplitude sum

$$|\psi\rangle = \sum_{\{g_{\mathbf{r}}\}} f(\{g_{\mathbf{r}}\})\,|\{g_{\mathbf{r}}\}\rangle \tag{44}$$

since if $S_M(g)$ sends $\{g_{\mathbf{r}}\} \to \{g'_{\mathbf{r}}\}$, then we may simply choose

$$U_{\partial M}(g) = \sum_{\{g_{\mathbf{r}}\}} \frac{f(\{g_{\mathbf{r}}\})}{f(\{g'_{\mathbf{r}}\})}\,|\{g'_{\mathbf{r}}\}\rangle\,\langle\{g'_{\mathbf{r}}\}| \tag{45}$$

which one can verify satisfies $U^\dagger_{\partial M}(g)S_M(g)\,|\psi\rangle = |\psi\rangle$ and will be only supported along ∂M as $|\psi\rangle$ is symmetric. Note that although we have made this assumption, the spirit of our argument should remain the same even without it. In the Else-Nayak procedure, this means that $U_C(g)$ and $V_{\partial C}(g_1, g_2)$ can also be chosen to be purely diagonal, and Eq. 6 reads

$$\begin{aligned} {}^{S_M(g_3)}V_a(g_1,g_2)V_a(g_1g_2,g_3) = \\ \omega(g_1,g_2,g_3)V_a(g_2,g_3)V_a(g_1,g_2g_3) \end{aligned} \tag{46}$$

To measure $\Omega(g)$, consider the product

$$Q_a(g) = \prod_{n=1}^{N} V_a(g, g^n) \tag{47}$$

which one can show using Eq. 46 satisfies

$${}^{S_M(g)}Q_a(g) = \Omega(g)Q_a(g) \tag{48}$$

That is, the charge of $Q_a(g)$ under $S_M(g)$ is exactly the type-I invariant $\Omega(g)$. This procedure has the nice interpretation in the gauged language as measuring (half) the charge of N gauge fluxes m_g.

Next, consider a measurement of $\Omega_{II}(g,h)$. One way to do so is by noting that we may use a region M_1 for $S_M(g) = S_{M_1}(g)$, but instead consider a much larger region M_2 which fully contains M_1 for the symmetry $S_{M_2}(h)$, and also define $S_M(g^n h^n) = S_{M_1}(g^n)S_{M_2}(h^n)$ (formally, we would absorb some of the symmetry into $U_{\partial M}(h)$, like in Eq 41). Then, using the fact that $U_{\partial M}(g)$ commutes with all full symmetries $S(h)$, and $S_{M_2}(h) \approx S(h)$ when acting on $U_{\partial M_1}(g)$ since M_2 is much larger than M_1, we have

$$U_{\partial M}(gh) = U_{\partial M_1}(g)U_{\partial M_2}(h) \tag{49}$$

Next, one can always choose the truncation to a segment $U_{\partial M_1}(g) \to U_{C_1}(g)$ in a way that $U_{C_1}(g)$ also commutes with all full symmetries $S(h)$, in which case

$$V_{\partial C}(gh, g^n h^n) = V_{\partial C_1}(g, g^n)V_{\partial C_2}(h, h^n) \tag{50}$$

as well. Using this choice, we have that $Q_a(gh) = Q_{a_1}(g)Q_{a_2}(h)$.

From this, one can readily compute the type-II cocycle

$$\Omega_{II}(g,h) = \frac{{}^{S(h)}Q_a(g)}{Q_a(g)} \tag{51}$$

And by symmetry,

$$\Omega_{II}(g,h) = \frac{{}^{S(g)}Q_a(h)}{Q_a(h)} \tag{52}$$

(note that these expressions are unambiguous since both numerator and denominator are diagonal). These have the nice interpretation in the gauged picture of measuring the number of charges e_h obtained as a fusion result of N gauge fluxes m_g, or vice versa.

Notice that while $Q_a(g)$ carries a charge under $S_M(g)$ and $S(h)$, if we consider the contribution from the other endpoint of ∂C, $Q_b(g)$, then one must have that

$$\begin{aligned} {}^{S_M(g)}(Q_a(g)Q_b(g)) &= Q_a(g)Q_b(g) \\ {}^{S(h)}(Q_a(g)Q_b(g)) &= Q_a(g)Q_b(g) \end{aligned} \tag{53}$$

the phase factors cancel out from the two endpoints. This is simply due to the fact that the phase ω only appears when isolating $V_{\partial C}(g)$ to a single endpoint.

Now, let us begin talking about the SSPT. Consider applying a symmetry $S_M(g)$ to a cubic region M, which acts non-trivially as $U_{h_M}(g)$ along the hinges. Then, consider a symmetric truncation of $U_{h_M}(g) \to U_C(g)$ which leads to $V_a(g_1, g_2)$ in the Else-Nayak procedure, and consider $Q_a(x, z; g)$ on an upper hinge (see Fig. 3), where we are now explicitly labeling the x and z coordinate of the hinge. Notice that if we had instead chosen to consider $V'_a(g_1, g_2)$ defined from the bottom hinge, we would end up with the conjugate $Q^*_a(x, z; g)$ instead (as shown in Fig. 3), which follows from the fact that the bottom hinge of $S_M(g)$ is related by a symmetry action to the top hinge of $S_M(g^{-1})$. Knowing $Q_a(x, z; g)$ is sufficient to calculate the invariants F_1 and F_2.

Consider calculating F_1, using $H^3[G^L, U(1)]$ obtained from xy planar symmetries. Let us choose g to be the generator of $G = \mathbb{Z}_{2N}$. Then, the invariant F_1 corresponds to

$$e^{\pi i F_1} = \Omega_{II}(g_<, g_\geq)^N \tag{54}$$

where

$$g_< = \prod_{z=z_0}^{z_1-1} g_z \tag{55}$$

$$g_\geq = \prod_{z=z_1}^{z_2} g_z \tag{56}$$

for some arbitrary z_1, with $z_0 \ll z_1 \ll z_2$. Let us take the region M to be some region $x < x_1$, such that the relevant edge is at x-coordinate x_1. Then, applying Eq 48

$$\begin{aligned} \Omega(g_<) &= \frac{{}^{S_M(g_<)}Q^*_a(x_1, z_0; g)\, {}^{S_M(g_<)}Q_a(x_1, z_1; g)}{Q^*_a(x_1, z_0; g)Q_a(x_1, z_1; g)} \\ \Omega(g_\geq) &= \frac{{}^{S_M(g_\geq)}Q^*_a(x_1, z_1; g)\, {}^{S_M(g_\geq)}Q_a(x_1, z_2; g)}{Q^*_a(x_1, z_1; g)Q_a(x_1, z_2; g)} \\ \Omega(g_< g_\geq) &= \frac{{}^{S_M(g_< g_\geq)}Q^*_a(x_1, z_0; g)\, {}^{S_M(g_< g_\geq)}Q_a(x_1, z_2; g)}{Q^*_a(x_1, z_0; g)Q_a(x_1, z_2; g)} \end{aligned} \tag{57}$$

For convenience, let us divide $Q_a(x_1, z_1; g)$ into four quadrants, as shown in Fig. 4, and denote its charge in each quadrant as $Q_{\text{◰}}$,$Q_{\text{◳}}$,$Q_{\text{◲}}$, and $Q_{\text{◱}}$. For example,

$$Q_{\text{◱}} = \frac{{}^{S_M(g_<)}Q_a(x_1, z_1; g)}{Q_a(x_1, z_1; g)} \tag{58}$$

Using this, we can express using Eqs 57

$$\Omega_{II}(g_<, g_\geq) = Q_{\text{◰}}/Q_{\text{◱}} \tag{59}$$

Alternatively, we could have used Eq. 51 and Eq. 52 to obtain

$$\Omega_{II}(g_<, g_\geq) = Q_{\text{⬒}} \tag{60}$$

and

$$\Omega_{II}(g_<, g_\geq) = 1/Q_{\text{⬓}} \tag{61}$$

where $Q_{\text{⬒}} \equiv Q_{\text{◰}}Q_{\text{◳}}$, and similarly for others.

Eq. 59, 60, and 61 imply that the charge distribution in Q must satisfy

$$1 = Q_{\text{◰}}/Q_{\text{◲}} = Q_{\text{◳}}/Q_{\text{◱}} \tag{62}$$

Thus, there are two degrees of freedom for the charge distribution in Q, which we may call q_1 and q_2,

$$\begin{aligned} e^{2\pi i q_1/(2N)} &= Q_{\text{◰}} = 1/Q_{\text{◲}} \\ e^{2\pi i q_2/(2N)} &= Q_{\text{◳}} = 1/Q_{\text{◱}} \end{aligned} \tag{63}$$

in which case $\Omega_{II}(g_<, g_\geq) = e^{2\pi i(q_1+q_2)/(2N)}$. The invariant F_1 is then $F_1 = q_1 + q_2 \mod 2$.

Now, suppose we calculate the same quantity except using yz planar symmetries instead. We may perform the calculation using the same hinge $Q_a(x_1, z_1; g)$, as shown in Fig. 3. In this case, one finds that

$$\Omega^{(yz)}_{II}(g^{(yz)}_<, g^{(yz)}_\geq) = Q_{\text{◨}} = e^{2\pi i(q_2-q_1)/(2N)} \tag{64}$$

where we have explicitly labeled everything with yz to avoid confusion ($g^{(yz)}_<$ is the product of $g^{(yz)}_x$ for $x < x_1$, for example). In this case, one has $F^{(yz)}_1 = q_2 - q_1 \mod 2$. However, $q_2 - q_1 = q_2 + q_1 \mod 2$, and so $F^{(yz)}_1 = F_1$ is independent of whether we had chosen the xy or yz plane. In the 3-foliated case, we may use the same argument along a different hinge to show that $F^{(zx)}_1$ is also given by the same quantity.

Next, consider the quantity F_2. Take $G = \mathbb{Z}_N \times \mathbb{Z}_N$, and choose g and h to be the two generators of G. Then, we wish to compute

$$e^{2\pi i F_2/N} = \Omega_{II}(g_<, h_\geq)/\Omega_{II}(h_<, g_\geq) \tag{65}$$

using the same set-up as before. Let us define

$$Q^{h,g}_{\text{◱}} = \frac{{}^{S_M(h_<)}Q_a(x_1, z_1; g)}{Q_a(x_1, z_1; g)} \tag{66}$$

to be the h charge in the ◱ quadrant of $Q_a(x_1, z_1, g)$, and similarly for the other quadrants. Then, using Eq. 51 and Eq. 52,

$$\Omega_{II}(g_<, h_\geq) = Q^{h,g}_{\text{⬒}} \tag{67}$$

$$\Omega_{II}(h_<, g_\geq) = 1/Q^{h,g}_{\text{⬓}} \tag{68}$$

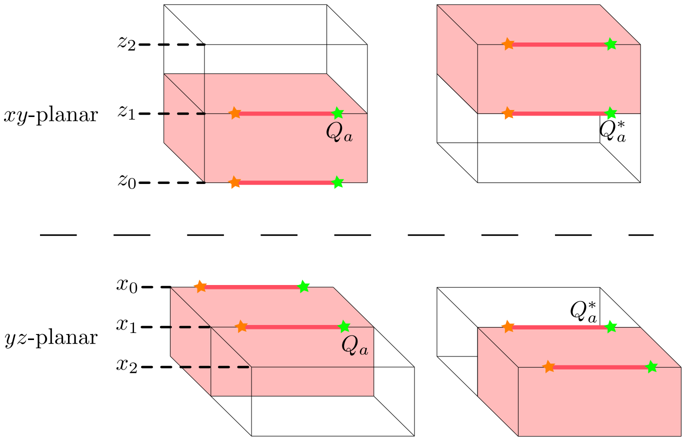

FIG. 3. A truncated symmetry $S_M(g)$ (shown in pink) acts non-trivially along its hinges. Truncating this to a segment as in the Else-Nayak procedure allows us to identify the diagonal operator Q_a. (top) The operator Q_a is used to compute invariants F_1 and F_2 for xy-planar symmetries. (bottom) The same operator Q_a is also used to compute invariants $F_1^{(yz)}$ and $F_2^{(yz)}$ using yz planar symmetries.

such that

$$e^{2\pi i F_2/N} = Q^{h,g}_{\boxplus} \tag{69}$$

is simply the total h charge of $Q_a(x_1, z_1, g)$.

Clearly, if we were to perform this calculation for the yz plane using this same hinge (x_1, z_1), we would find exactly the same result, $e^{2\pi i F_2^{(yz)}/N} = Q^{h,g}_{\boxplus}$. (depending on choice of convention: we could have had $F_2^{(yz)} = -F_2$ instead). Thus, F_2 is independent of whether we measure using the xy or yz planes.

Now, suppose our model is 3-foliated. We have shown that if we consider every endpoint (not just Q_a), the total charge must be zero under any untruncated symmetry (Eq. 53). However, in a 3-foliated model, we may choose a symmetry operator which acts like a global symmetry near Q_a, but does not act on the other endpoints at all (see Fig. 4). This means that Q_a must have trivial total charge under any global symmetry. Thus, $e^{2\pi i F_2/N} = Q^{h,g}_{\boxplus} = 1$ must be trivial.

On the gauged side this has a natural interpretation: for the gauged 3-foliated model, N lineons at (x_1, z_1) (which are mobile in the y direction) cannot carry any charge under $S^{(zx)}(y; h)$, otherwise they could not have been mobile in the y direction in the first place.

B. Completeness and basis change

Here, we first go through how to obtain $\mathbf{M}$ for a 3D SSPT after stacking by a 2D SPT, as described by the main text. Then, we prove that the invariants F_1 and F_2 are a complete classification of all matrices $\mathbf{M}$ modulo this stacking. We prove this by showing that all possible $\mathbf{M}$ may be brought into a canonical form $\mathbf{M}^{\text{canon}}$, determined solely by F_1 and F_2, via stacking 2D SPTs.

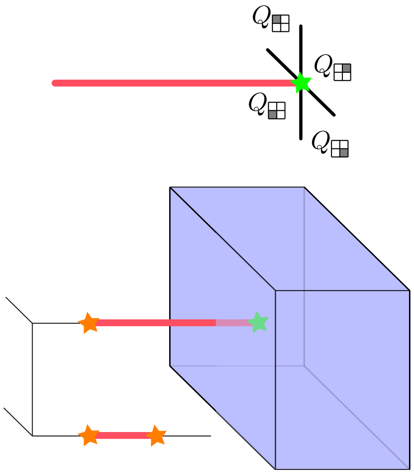

FIG. 4. (top) The charge of the operator Q_a is divided into contributions from four quadrants as shown. (bottom) In a 3-foliated model, a product of zx-planar symmetries (blue) act as a global symmetry near Q_a (the green star), while not acting on any of the other truncation points (orange stars). This implies that the total charge of Q_a must be zero.

1. *Basis change*

First, let us go over the details of the basis change. For this section, we will work with general $G = \prod_{\alpha=1}^{M} \mathbb{Z}_{N_\alpha}$. In this case, $\mathbf{M}$ is an $ML \times ML$ integer matrix, where the off-diagonal elements $M_{(\alpha,z),(\beta,z')}$ are defined modulo $\gcd(N_\alpha, N_\beta) \equiv N_{\alpha\beta}$, and the diagonal elements $M_{(\alpha,z),(\alpha,z)}$ are even integers modulo $2N_\alpha$.

Recall that we wish to stack a 2D SPT with symmetry group $G_{2D} = G^K$, and that we do so by identifying each factor of G in G_{2D} with a plane z_k in the SSPT. That is, let $k = 1, \ldots, K$ label the factors of G in G_{2D}, which we associate with the plane z_k, and $\widetilde{M}^{2D}_{(\alpha,k),(\beta,k')}$ the $KM \times KM$ matrix characterizing the 2D SPT. To ensure locality, all $\{z_k\}$ must reside within some finite $\mathcal{O}(1)$ interval. Then, define $\mathbf{M}^{2D}$ to be the matrix with the same dimensions as $\mathbf{M}$, whose elements are obtained directly from $\widetilde{\mathbf{M}}^{2D}$,

$$M^{2D}_{(\alpha,z_k),(\beta,z_{k'})} = \widetilde{M}^{2D}_{(\alpha,k),(\beta,k')} \tag{70}$$

and all other elements with $z \notin \{z_k\}$ zero. If we were stacking on to a 1-foliated model described by $\mathbf{M}$, we would simply modify $\mathbf{M} \to \mathbf{M} + \mathbf{M}^{2D}$. However, when stacking to a 2- or 3-foliated model, we instead define the 2D SPT in terms of $d_{\mathbf{r}}$ degrees of freedom. Thus, one instead has $\mathbf{M} \to \mathbf{M} + \mathbf{W}^T \mathbf{M}^{2D} \mathbf{W}$, where

$$W_{(\alpha,z),(\beta,z')} = \delta_{\alpha\beta}(\delta_{z+1,z'} - \delta_{z,z'}) \tag{71}$$

For example, suppose $G = \mathbb{Z}_N$ and we have a single type-I cocycle on plane z_1,

$$\mathbf{M}^{2D} = \begin{array}{c} \begin{matrix} z_1 & z_1+1 \end{matrix} \\ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \tag{72}$$

where we show only the $\{z_1, z_1+1\}$ submatrix. Then, within this submatrix,

$$\begin{aligned} \mathbf{W}^T\mathbf{M}^{2D}\mathbf{W} &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{aligned} \tag{73}$$

and all other elements outside of this submatrix are 0. This therefore results in two type-I cocycles valued 1 (recall that the diagonal elements are $M_{ii} = 2p_I^i$) on planes z_1 and z_1+1, and a type-II cocycle valued -2 between the two planes. This is one of the examples shown in Fig. 1 of the main text.

2. *Completeness*

Next, let us show completeness of the invariants F_1 and F_2. In a general group $G = \prod_{\alpha=1}^{M} \mathbb{Z}_{N_\alpha}$, we may define F_1^α for each even N_α, and $F_2^{\alpha\beta}$ for each $N_{\alpha\beta} \neq 1$. These are defined in reference to some plane z_0,

$$F_1^\alpha \equiv \sum_{z<z_0} \sum_{z' \geq z_0} M_{(\alpha,z)(\alpha,z')} \mod 2 \tag{74}$$

and

$$F_2^{\alpha\beta} \equiv \sum_{z<z_0} \sum_{z' \geq z_0} \left(M_{(\alpha,z),(\beta,z')} - M_{(\beta,z),(\alpha,z')} \right) \mod N_{\alpha\beta} \tag{75}$$

and are independent of the precise choice z_0. Let us define equivalence classes of $\mathbf{M}$, where $\mathbf{M}_1$ and $\mathbf{M}_2$ belong to the same equivalence class if $\mathbf{M}_1 = \mathbf{M}_2 + \mathbf{W}^T\mathbf{M}^{2D}\mathbf{W}$ for some $\mathbf{M}^{2D}$. Then, we claim that F_1^α and $F_2^{\alpha\beta}$ (for $\alpha < \beta$) are a complete set of invariants, and are sufficient to characterize all equivalence classes of $\mathbf{M}$.

Our strategy is as follows: assume we are given a general $\mathbf{M}$, with some longest range coupling d_{max}, defined as the maximum $|z - z'|$ where $M_{(\alpha,z),(\beta,z')}$ is non-zero. We then show that by stacking $\mathbf{M}^{2D}$ we can reduce $\mathbf{M}$ down to one in which $d_{max} = 1$, such that there is only couplings between planes z and $z \pm 1$. Then, we finally reduce $\mathbf{M}$ down to some canonical $\mathbf{M}^{\text{canon}}$, which only depends on F_1^α and $F_2^{\alpha\beta}$. Thus, any two $\mathbf{M}$ with the same F_1^α and $F_2^{\alpha\beta}$ can be related to one another by stacking various $\mathbf{M}^{2D}$, and are therefore a complete set of invariants.

First, suppose we have some matrix $\mathbf{M}$ with some longest range coupling $d_{max} > 1$ (which is always $\mathcal{O}(1)$ due to locality). This means there is some element $M_{(\alpha_1,z_1),(\alpha_2,z_2)} \neq 0$ where $|z_2 - z_1| = d_{max}$. By symmetry of $\mathbf{M}$, we may consider $z_2 > z_1$ without loss of generality.

Suppose $\alpha_1 = \alpha_2$, then take

$$\mathbf{M}^{2D}_{\alpha_1,\alpha_2} = \begin{array}{c} \begin{matrix} z_1 & z_1+1 & z_2-1 & z_2 \end{matrix} \\ \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{76}$$

where $\mathbf{M}^{2D}_{\alpha_1\alpha_2}$ is viewed as a matrix indexed by z, with fixed α_1, α_2, and we only show the relevant non-zero submatrix. We then have

$$(\mathbf{W}^T\mathbf{M}^{2D}\mathbf{W})_{\alpha_1,\alpha_2} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ \hline 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \tag{77}$$

which has a single -1 as its $((\alpha_1,z_1),(\alpha_2,z_2))$th element (along with its symmetric partner), and all other elements are of $|z - z'| < d_{max}$. Thus, we may take

$$\mathbf{M}' = \mathbf{M} + M_{(\alpha_1,z_1),(\alpha_2,z_2)}\mathbf{W}^T\mathbf{M}^{2D}\mathbf{W} \tag{78}$$

which now has $M'_{(\alpha_1,z_1),(\alpha_2,z_2)} = 0$. Note that although in writing the submatrix we have assumed $d_{max} = z_2 - z_1 > 2$, this also works for $d_{max} = 2$.

If $\alpha_1 \neq \alpha_2$, then we may instead use

$$\mathbf{M}^{2D}_{\alpha_1,\alpha_2} = \begin{array}{c} \begin{matrix} z_1 & z_1+1 & z_2-1 & z_2 \end{matrix} \\ \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{79}$$

such that

$$(\mathbf{W}^T\mathbf{M}^{2D}\mathbf{W})_{\alpha_1,\alpha_2} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{80}$$

again only has a -1 as its $((\alpha_1,z_1),(\alpha_2,z_2))$th element, and all other elements have range smaller than d_{max}.

We may repeat this on all non-zero elements of $\mathbf{M}$ with distance d_{max}, after which we end up with some matrix with $d'_{max} < d_{max}$. We can repeat this process until we have $d_{max} = 1$, meaning $\mathbf{M}_{\alpha_1\alpha_2}$ is a tridiagonal matrix.

Let us now define a canonical form $\mathbf{M}^{\text{canon}}$, for a given set of F_1^α and $F_2^{\alpha\beta}$, by

$$\mathbf{M}^{\text{canon}}_{\alpha\alpha} = \begin{bmatrix} \ddots & \ddots & & & & \\ & -2F_1^\alpha & F_1^\alpha & 0 & 0 & \\ \ddots & F_1^\alpha & -2F_1^\alpha & F_1^\alpha & 0 & \\ & 0 & F_1^\alpha & -2F_1^\alpha & F_1^\alpha & \\ & 0 & 0 & F_1^\alpha & -2F_1^\alpha & \ddots \\ & & & & \ddots & \ddots \end{bmatrix} \tag{81}$$

and

$$\mathbf{M}^{\text{canon}}_{\alpha\beta} = \begin{bmatrix} \ddots & \ddots & & & & \\ & -F_2^{\alpha\beta} & F_2^{\alpha\beta} & 0 & 0 & \\ & 0 & -F_2^{\alpha\beta} & F_2^{\alpha\beta} & 0 & \\ & 0 & 0 & -F_2^{\alpha\beta} & F_2^{\alpha\beta} & \\ & 0 & 0 & 0 & -F_2^{\alpha\beta} & \ddots \\ & & & & & \ddots \end{bmatrix} \tag{82}$$

for $\alpha < \beta$. For $\beta < \alpha$, we simply have $\mathbf{M}^{\text{canon}}_{\alpha\beta} = (\mathbf{M}^{\text{canon}}_{\beta\alpha})^T$. We have also simply set $F_1^\alpha = 0$ for any odd N_α, and $F_2^{\alpha\beta} = 0$ for any $N_{\alpha\beta} = 1$. The strong examples shown in Fig. 1 of the main text are both already in canonical form. We will now show that our tridiagonal $\mathbf{M}$ can always be brought into its canonical form.

First, for each α, examine the symmetric matrix $\mathbf{M}_{\alpha\alpha}$. Consider each 2×2 block coupling z_1 and $z_1 + 1$. We may stack with

$$\mathbf{M}^{2D}_{\alpha,\alpha} = \begin{array}{c} \begin{array}{cc} z_1 & z_1+1 \end{array} \\ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \tag{83}$$

which realizes

$$(\mathbf{W}^T \mathbf{M}^{2D} \mathbf{W})_{\alpha,\alpha} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \tag{84}$$

which we can add to $\mathbf{M}$ to modify the offdiagonal element to be 0 or 1 depending on its parity (if N_α even) or 0 (if N_α odd). We may do this for all the offdiagonal elements, bringing them all to F_1^α. The diagonal elements are automatically constrained by the local constraint (Eq 43) to be $-2F_1^\alpha$.

Next, we may do a similar thing to $\mathbf{M}_{\alpha,\beta}$ for each $\alpha < \beta$. In this case, we stack

$$\mathbf{M}^{2D}_{\alpha,\beta} = \begin{array}{c} \begin{array}{cc} z_1 & z_1+1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \tag{85}$$

$$(\mathbf{W}^T \mathbf{M}^{2D} \mathbf{W})_{\alpha,\beta} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{86}$$

which we can use to eliminate all the lower-diagonal elements $M_{(\alpha,z_1+1),(\beta,z_1)}$. Then, the upper-diagonal elements are $M_{(\alpha,z_1),(\beta,z_1+1)} = F_2^{\alpha\beta}$ and the diagonal elements are all automatically fixed by the local constraint to be $-F_2^{\alpha\beta}$. We have therefore brought an arbitrary initial matrix $\mathbf{M}$, via moves of the form $\mathbf{W}^T\mathbf{M}^{2D}\mathbf{W}$ (stacking 2D SPTs), to a canonical form which only depends on F_1^α and $F_2^{\alpha\beta}$. From this, we conclude that F_1^α and $F_2^{\alpha\beta}$ are a complete set of invariants for $\mathbf{M}$.

VII. STRONG MODELS

In this section, we introduce two exactly solvable models of strong planar SSPT phases. The first is the 3-foliated Type 1 strong phase with $G = \mathbb{Z}_2$, which we write down in the form of a Hamiltonian. The fracton dual is a novel fracton model which we explicitly write down. The second is the 2-foliated Type 1 and Type 2 strong phase with $G = \mathbb{Z}_N \times \mathbb{Z}_N$, for which we write down the ground state wavefunction $|\psi\rangle$. We may consider the 2-foliated model as part of a model with two sets of 2-foliated symmetries, in which case the fracton dual is again a novel model with unusual braiding statistics between fractons. Alternatively, we may examine the fracton dual of a single 2-foliated model by itself, which results in a 2-foliated fracton phase, with non-trivial braiding statistics between gauge fluxes. To obtain models for strong phases for more general groups G, one may simply identify $\mathbb{Z}_2$ or $\mathbb{Z}_N \times \mathbb{Z}_N$ subgroups of G, and define the model in terms of those degrees of freedom.

A. 3-foliated Type 1 strong model

The $G = \mathbb{Z}_2$ strong 3-foliated model is defined on the square lattice with qubit degrees of freedom on each site. Define the Pauli matrices Z and X,

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{87}$$

as well as the $S = \sqrt{Z}$ matrix and the controlled-Z (CZ) matrix

$$S = i^{(1-Z)/2} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{88}$$

$$CZ_{12} = (-1)^{(1-Z_1)(1-Z_2)/4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{89}$$

The Hamiltonian will be written as a sum of terms of the form

$$H = -\sum_{\mathbf{r}} X_{\mathbf{r}} F_{\mathbf{r}}(\{Z_p\}) \equiv -\sum_{\mathbf{r}} B_{\mathbf{r}} \tag{90}$$

where Z_p are products of Z on the four corners of a plaquette p, and $F_{\mathbf{r}}(\{Z_p\})$ is some function of these variables near the site $\mathbf{r}$. The planar symmetries will act as products of Xs along xy, yz, or zx planes. As $F_{\mathbf{r}}(\{Z_p\})$ only depends on the combinations Z_p which commutes with all planar symmetries, this Hamiltonian is explicitly symmetry respecting.

The function $F_{\mathbf{r}}(\{Z_p\})$ consists of 6 Z_p, 12 S_p, and 12 $CZ_{p_1p_2}$ operators on various plaquettes, and an overall factor of i. Fig. 5 shows the model on the dual lattice, where plaquettes are represented by bonds, and the site $\mathbf{r}$ is mapped on to the red cube. Careful calculation will show that $[B_{\mathbf{r}}, B_{\mathbf{r}'}] = 0$ and $B_{\mathbf{r}}^2 = 1$. This Hamiltonian is is therefore simply a commuting projector Hamiltonian, and as every term is independent (only $B_{\mathbf{r}}$ can act as $X_{\mathbf{r}}$) and there are the same number of terms as sites, H has a unique gapped group state $|\psi\rangle$ and describes a valid

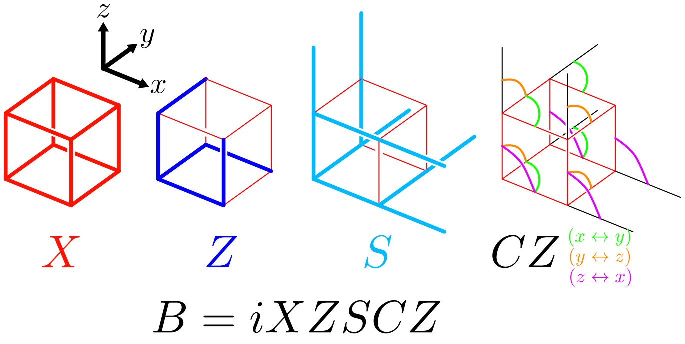

FIG. 5. The operator B_c for each cube in the gauged Type 1 strong model. The action of the Z, S, and CZ operators precede the action of the X. The CZ operators are always between two bonds oriented in different directions, and are denoted by a line connecting the two bonds. For ease of viewing, CZ operators between bonds of various pairs of orientations are shown in a different color. The model is symmetric under three-fold rotation about the (111) axis.

SSPT. We found it simplest to write a small computer script to confirm these commutation relations (and to compute the wireframe operator later), rather than doing so by hand.

The wireframe operator (Fig. 6) obtained as a product of $B_{\mathbf{r}}$ over a large cube, when ungauged, gives the action of the symmetry on the hinges of the cube. One may confirm using the Else-Nayak procedure that this model has $\mathbf{M}$ matrix

$$\begin{aligned} M_{z,z} &= 2 \\ M_{z,z+1} = M_{z,z-1} &= 1 \end{aligned} \tag{91}$$

and all other elements zero. This therefore realizes the Type 1 strong phase shown in Fig. 1 for $G = \mathbb{Z}_2$. Computing the charges Q defined in Section VI, one finds $Q_{\text{◰}} = Q_{\text{◲}} = -1$ and $Q_{\text{◱}} = Q_{\text{◳}} = 1$.

The fracton dual of this model is defined on the square lattice with qubit degrees of freedom on the bonds. The Hamiltonian is given by

$$H = -\sum_v (A_v^{xy} + A_v^{yz} + A_v^{zx}) - \sum_c B_c \tag{92}$$

where c represents cubes, B_c is the operator shown in Fig. 5, v represents vertices, and $A_v^{\mu\nu}$ is the product of Zs along the four bonds touching v in the $\mu\nu$ plane (the usual cross term from the X-cube model). B_c consists of Xs along the cube (the cube term from the X-cube model) but with an additional phase factor depending on the Z state around it in the form of S, Z, and CZ operators. Note that while the ungauged operator $B_{\mathbf{r}}$ squares to 1, the gauged operator B_c does not square to 1, it instead squares to a product of A_v operators.

This model has the same fracton charge excitations as the usual X-cube model. However, the lineon excitations are modified. To find out what they are, consider the product of B_c over a large cube, $\prod_c B_c$, shown in Fig. 6. This results in an operator with support only along the hinges of the cube. This operator, when truncated, is the operator which creates lineon excitations at its ends.

From this, the crossing (braiding) statistic of two lineon can be readily extracted. Reading off of Fig. 6, a pair of x-moving lineons on line (y_1, z_1) is constructed by the operator

$$L_x \equiv \prod_{x=x_0}^{x_2} X_{x,y_1,z_1}^{(x)} S_{x,y_1,z_1}^{(z)} CZ_{x,y_1,z_1}^{(x\leftrightarrow y)} \tag{93}$$

where $X_{x,y,z}^{(x)}$ is an X on the bond originating from the vertex at (x, y, z) going in the positive x direction, and similarly for $S_{x,y,z}^{(z)}$, and $CZ_{x,y,z}^{(x\leftrightarrow y)}$ is a CZ between $Z_{x,y,z}^{(x)}$ and $Z_{x,y,z}^{(y)}$. L_x creates two lineons at x_0 and x_2. Meanwhile, a pair of y-moving lineons is constructed by

$$L_y \equiv \prod_{y=y_0}^{y_2} X_{x_1,y,z_1}^{(y)} S_{x_1,y,z_1}^{(x)} CZ_{x_1,y,z_1}^{(y\leftrightarrow z)} \tag{94}$$

which creates two lineons at y_0 and y_2. Note that depending on which hinge of the wireframe we obtain L_x and L_y from, there may be additional Z operators, which correspond to a choice of lineon or antilineon (and will affect the braiding phase by a ±1). It can be readily verified that when these two operators cross (i.e. $y_0 < y_1 < y_2$ and $x_0 < x_1 < x_2$), they only commute up to a factor of i,

$$L_y L_x = i L_y L_x \tag{95}$$

using the relations $XSX = iZS$ and $X_1 CZ_{12} X_1 = Z_2 CZ_{12}$. Thus, the braiding phase of any two lineons in this model is $\pm i$.

B. 2-foliated strong model

In this section, we describe a 2-foliated model which realizes both Type 1 and/or Type 2 strong phases.

1. *A group cohomology model on the square lattice*

First, let us explicitly construct a group cohomology model on the square lattice, for $G = \mathbb{Z}_N^M$. Recall that the ground state of such models are an equal amplitude sum of all configurations

$$|\psi\rangle = \sum_{\{g_{\mathbf{r}}\}} f(\{g_{\mathbf{r}}\}) \,|\{g_{\mathbf{r}}\}\rangle \tag{96}$$

where $f(\{g_{\mathbf{r}}\})$ is a pure phase up to an overall normalization, which we ignore.

From Eq 20, $f(\{g_{\mathbf{r}}\})$ is a product of terms $f_{\mathbf{r}}$ coming from each square plaquette at $\mathbf{r}$, given by

$$f_{\mathbf{r}}(\{g_{\mathbf{r}}\}) = \frac{\nu(g_{\mathbf{r}}, g_{\mathbf{r}+\mathbf{x}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}}, 1)}{\nu(g_{\mathbf{r}}, g_{\mathbf{r}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}}, 1)} \tag{97}$$

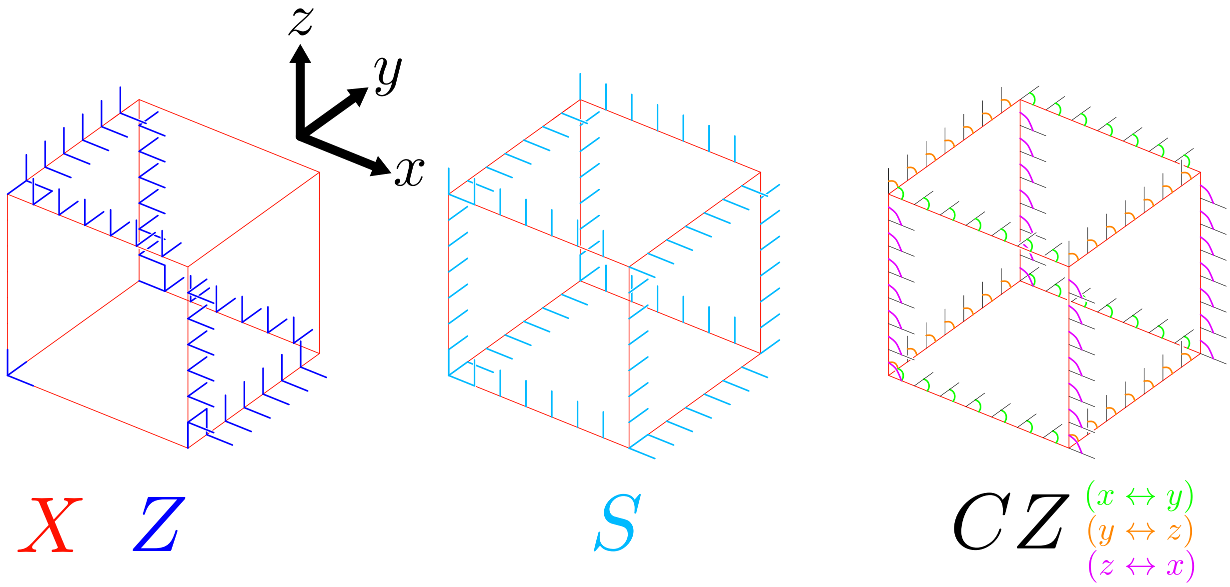

FIG. 6. Taking a product of the cubic terms B_c (Fig. 5) results in a wireframe operator with support along the hinges of the cube. This wireframe operator is shown here for a $7 \times 7 \times 7$ cube, where the action of the Z, S, and CZ operators precede the X (which acts along the red cube).

Alternatively, we may choose to defined the same wavefunction using instead

$$f'_{\mathbf{r}}(\{g_{\mathbf{r}}\}) = \frac{\nu(1, g_{\mathbf{r}}, g_{\mathbf{r+y}}, g_{\mathbf{r+x+y}})}{\nu(1, g_{\mathbf{r}}, g_{\mathbf{r+x}}, g_{\mathbf{r+x+y}})} \tag{98}$$

which one can verify using the cocycle condition (Eq 18) differs from $f_{\mathbf{r}}$ only by terms along the edges of the plaquette which are cancelled out by the same terms from neighboring plaquettes. Plugging the explicit form for the cocycles, we get

$$f^{(2D)}_{\mathbf{r}}(\{g_{\mathbf{r}}\}) = \exp\left\{\sum_{i\leq j} \frac{2\pi i p^{ij}}{N^2} g^i_{\mathbf{r}} \left([g^j_{\mathbf{r+y}} - g^j_{\mathbf{r}}] + [g^j_{\mathbf{r+x+y}} - g^j_{\mathbf{r+y}}] - [g^j_{\mathbf{r+x}} - g^j_{\mathbf{r}}] - [g^j_{\mathbf{r+x+y}} - g^j_{\mathbf{r+x}}]\right)\right\} \tag{99}$$

which we have called $f^{(2D)}_{\mathbf{r}}$.

2. *The strong SSPT*

Let us take $G = \mathbb{Z}_N \times \mathbb{Z}_N$, with subsystem symmetries along xy and yz planes. The ground state of our strong SSPT is again described by a function $f(\{g_{\mathbf{r}}\})$, which can be written as a product of $f_{\mathbf{r}}(\{g_{\mathbf{r}}\})$, which are now associated with the cube at $\mathbf{r}$. The function $f_{\mathbf{r}}$ is given by

$$\begin{aligned} f^{(SSPT)}_{\mathbf{r}}(\{g_{\mathbf{r}}\}) = \exp\Bigg\{\sum_{\alpha\leq\beta} \frac{2\pi i q^{\alpha\beta}}{N^2} \Bigg(& (g^\alpha_{\mathbf{r+z}} - g^\alpha_{\mathbf{r}})([g^\beta_{\mathbf{r+y+z}} - g^\beta_{\mathbf{r+z}}] + [g^\beta_{\mathbf{r+x+y+z}} - g^\beta_{\mathbf{r+y+z}}] \\ & - [g^\beta_{\mathbf{r+x+z}} - g^\beta_{\mathbf{r+z}}] - [g^\beta_{\mathbf{r+x+y+z}} - g^\beta_{\mathbf{r+x+z}}]) \\ & -(g^\alpha_{\mathbf{r+y}} - g^\alpha_{\mathbf{r}})([g^\beta_{\mathbf{r+x+y+z}} - g^\beta_{\mathbf{r+y+z}}] - [g^\beta_{\mathbf{r+x+y}} - g^\beta_{\mathbf{r+y}}] \\ & + [g^\beta_{\mathbf{r+y+z}} - g^\beta_{\mathbf{r+x+y+z}} + g^\beta_{\mathbf{r+x+y}} - g^\beta_{\mathbf{r+y}}]) \Bigg)\Bigg\} \end{aligned} \tag{100}$$

for $q^{\alpha\beta}$ integers mod N. Here, $\alpha, \beta \in \{1, 2\}$ for each factor of $\mathbb{Z}_N$ in G.

We claim that $f_{\mathbf{r}}^{(SSPT)}$ describes an SSPT phase which is Type 1 strong if q^{11} or q^{22} are odd (and N is even), and Type 2 strong if $q^{12} \neq 0$.

First, let us examine the state as a quasi-2D SPT along the xy plane, with a G^L symmetry group. Let us label each generator of G^L by (α, z), for $\alpha \in \{1, 2\}$ and $z \in [1, L]$. The second term in the exponent (the term multiplying $(g^i_{\mathbf{r}+\mathbf{y}} - g^i_{\mathbf{r}})$) is completely invariant under an xy planar symmetry. This second term therefore cannot affect the xy cocycle class, as it can be removed by a symmetric local unitary transformation respecting all xy planar symmetries (but will break the yz planar symmetries). Thus, the xy cocycle class is determined simply by the first term. However, this term is exactly of the form $f_{\mathbf{r}}^{(2D)}(\{g_{\mathbf{r}}\})$ for G^L, with the mapping

$$\begin{aligned} p^{(\alpha,z),(\beta,z)} &= q^{\alpha\beta} \\ p^{(\alpha,z),(\beta,z+1)} &= -q^{\alpha\beta} \end{aligned} \tag{101}$$

and other elements zero. In terms of the $\mathbf{M}$ matrix,

$$\begin{aligned} M_{(\alpha,z),(\beta,z)} &= (1 + \delta_{\alpha\beta}) q^{\alpha\beta} \\ M_{(\alpha,z),(\beta,z+1)} &= -q^{\alpha\beta} \end{aligned} \tag{102}$$

and all other elements (except those related by symmetry) are zero.

The F_1 invariants are therefore simply q^{11} and q^{22} modulo 2, and the F_2 invariant is $-q^{12}$. By the proof from our previous section, the invariants will also the same for the yz symmetries.

But before we can conclude that we have constructed a strong phase, we must show that this state is symmetric under yz symmetries. The purpose of the second term in $f_{\mathbf{r}}^{(SSPT)}$ is to ensure that this is the case. Let us examine how $f_{\mathbf{r}}(\{g_{\mathbf{r}}\})$ transforms under a yz planar symmetry which sends $\{g_{\mathbf{r}}\} \to \{g^{(yz)} g_{\mathbf{r}}\}$, or, on the relevant degrees of freedom,

$$\begin{aligned} &(g_{\mathbf{r}}, g_{\mathbf{r}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{y}+\mathbf{z}}) \to (g g_{\mathbf{r}}, g g_{\mathbf{r}+\mathbf{y}}, g g_{\mathbf{r}+\mathbf{z}}, g g_{\mathbf{r}+\mathbf{y}+\mathbf{z}}) \\ &(g_{\mathbf{r}+\mathbf{x}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}+\mathbf{z}}) \quad \text{unchanged} \end{aligned} \tag{103}$$

A calculation shows that

$$\begin{aligned} \frac{f_{\mathbf{r}}(\{g^{(yz)} g_{\mathbf{r}}\})}{f_{\mathbf{r}}(\{g_{\mathbf{r}}\})} = \exp\Bigg\{ \sum_{\alpha \le \beta} \frac{2\pi i q^{\alpha\beta}}{N^2} \Bigg(&(g^\alpha_{\mathbf{r}+\mathbf{z}} - g^\alpha_{\mathbf{r}})([g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{y}+\mathbf{z}} - g^\beta_{\mathbf{r}+\mathbf{y}+\mathbf{z}} + g^\beta] - [g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{y}+\mathbf{z}} - g^\beta_{\mathbf{r}+\mathbf{y}+\mathbf{z}}] \\ &\qquad - [g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{z}} - g^\beta_{\mathbf{r}+\mathbf{z}} + g^\beta] + [g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{z}} - g^\beta_{\mathbf{r}+\mathbf{z}}]) \\ &-(g^\alpha_{\mathbf{r}+\mathbf{y}} - g^\alpha_{\mathbf{r}})([g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{y}+\mathbf{z}} - g^\beta_{\mathbf{r}+\mathbf{y}+\mathbf{z}} + g^\beta] - [g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{y}+\mathbf{z}} - g^\beta_{\mathbf{r}+\mathbf{y}+\mathbf{z}}] \\ &\qquad - [g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{y}} - g^\beta_{\mathbf{r}+\mathbf{y}} + g^\beta] + [g^\beta_{\mathbf{r}+\mathbf{x}+\mathbf{y}} - g^\beta_{\mathbf{r}+\mathbf{y}}]) \Bigg) \Bigg\} \end{aligned} \tag{104}$$

which simplifies to

$$\frac{f_{\mathbf{r}}(\{g^{(yz)} g_{\mathbf{r}}\})}{f_{\mathbf{r}}(\{g_{\mathbf{r}}\})} = \frac{P(g_{\mathbf{r}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{y}+\mathbf{z}}, g)}{P(g_{\mathbf{r}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{y}}, g)} \frac{P(g_{\mathbf{r}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{z}}, g)}{P(g_{\mathbf{r}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{y}+\mathbf{z}}, g)} \frac{P(g_{\mathbf{r}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{y}}, g_{\mathbf{r}+\mathbf{y}}, g)}{P(g_{\mathbf{r}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{x}+\mathbf{z}}, g_{\mathbf{r}+\mathbf{z}}, g)} \tag{105}$$

where

$$P(g_1, g_2, g_3, g) = \exp\left\{ \sum_{\alpha \le \beta} \frac{2\pi i q^{\alpha\beta}}{N^2} \left(g_1^\alpha([g_2^\beta - g_3^\beta + g^\beta] - [g_2^\beta - g_3^\beta]) \right) \right\} \tag{106}$$

If one considers the contribution from neighboring cubes, one finds that the factors of $P(\dots)$ exactly cancel out between neighboring cubes. Repeating this calculation for a yz-planar symmetry which transforms the other four sites in Eq 103, one finds the same result. Thus, the wavefunction is indeed symmetric under yz planar symmetries and describes a strong SSPT phase for a 2-foliated model. If one wished, one could confirm that the matrix $\mathbf{M}^{(yz)}$ obtained from yz planar is also strong with the same F_1 and F_2 invariants, by following the Else-Nayak procedure. Obtaining a gapped local Hamiltonian corresponding to this ground state is straightforward, and is done in the same way as for the standard group cohomology models, Eq 21.

VIII. RELATION TO p-STRING CONDENSATION

In this section, we will discuss the gauge duality between weak 3-foliated SSPTs and twisted X-cube models constructed by stacking 1-foliated gauge theories and inducing a p-string condensation transition.[6,7] This procedure is a straightforward generalization of the coupled layers construction of the X-cube model and its twisted variants. We will demonstrate the correspondence by showing that our zero-correlation length Hamiltonian models for weak SSPTs are dual to the effective Hamiltonians that emerge from strongly coupling stacked 1-foliated gauge theories.

First, let us briefly view the coupled layers construction of the X-cube model. The starting point is 3 intersecting stacks of 2D toric code layers, oriented along xy, yz, and zx planes respectively. The toric code layers contain qubits on the edges of square lattices, subject to the Hamiltonian

$$H_{TC} = -\sum_v A_v - \sum_p B_p \tag{107}$$

where A_v is the tensor product of Pauli Z operators over the four edges adjacent to vertex v, and B_p is the product of Pauli X operators over the four edges around plaquette p. They are arranged such that the degrees of freedom coincide on the edges of a cubic lattice, such that each edge contains two qubits from different stacks. Then, the stacks are coupled together via the term ZZ acting on the two qubits on a given edge. In the strong-coupling limit, the two qubit degrees of freedom become one effective qubit degree of freedom, and the X-cube Hamiltonian

$$H_{XC} = -\sum_v (A_v^{xy} + A_v^{yz} + A_v^{zx}) - \sum_c B_c \tag{108}$$

emerges as the effective Hamiltonian governing these degrees of freedom.[6,7] Here $A_v^{\mu\nu}$ is the tensor product of Pauli Z operators over the edges emanating from vertex v in the $\mu\nu$ plane, and B_c is the product of X operators over the 12 edges of the elementary cube c. The transition between decoupled stacks and emergent X-cube order is driven by a mechanism which has been named p-string condensation.[6] The essential idea is that the ZZ coupling acts on the ground state by creating a small loop composed of 4 charge excitations. In the strong coupling limit, loops of gauge charges of all sizes, called p-strings, proliferate throughout the system and form a condensate in the ground state.

This construction can be generalized by replacing each stack of 2D toric codes by an arbitrary 1-foliated gauge theory, viewed as a quasi-2D topological order that compactifies to a G^L twisted gauge theory (with G an arbitrary abelian group). An example of such a 1-foliated gauge theory was constructed in Ref. 8. Exactly solvable Hamiltonians for 1-foliated gauge theories can be constructed as generalized string-net models. For simplicity let us consider the case $G = \mathbb{Z}_2$, with planar symmetries oriented along the xy plane; the generalization to arbitrary abelian gauge group is straightforward. The degrees of freedom are qubits attached to the x- and y-oriented links of a cubic lattice, and the Hamiltonian for an arbitrary 1-foliated gauge theory takes the general form

$$H_{\text{1-fol}} = -\sum_v A_v - \sum_p \tilde{B}_p + \text{h.c.} \tag{109}$$

Here, A_v is simply the vertex constraint enforcing the string-net branching rules, which acts as the product of Pauli Z operators on the four links around vertex v. $\tilde{B}_p$ is a plaquette term associated to the xy oriented plaquette p, and has the form

$$\tilde{B}_p = X_1 X_2 X_3 X_4 \phi_p(\{Z\}) P_p \tag{110}$$

where edges 1, 2, 3, and 4 bound plaquette p, $\phi_p(\{Z\})$ is a particular phase-valued function of the Pauli Z variables in the vicinity of plaquette p, and P_p is a projector onto the subspace satisfying the vertex constraints in the vicinity of p. The function ϕ_p takes the same form for all plaquettes in the same plane z_0, but may vary between planes if the model is not translation invariant in the z direction. The particular form of ϕ_p depends on the element of the cohomology group $H^3[\mathbb{Z}_2^L, U(1)]$ represented by the model. It has been shown that representative models of all cohomology classes admit Hamiltonian descriptions of this form.[16]

We now consider coupling together three mutually perpendicular 1-foliated $\mathbb{Z}_2^L$ gauge theories oriented along the xy, yz and zx planes, via a ZZ coupling between the two qubits on each link of the cubic lattice. In the strong-coupling limit, the two qubit degrees of freedom merge into one effective qubit, and the following effective Hamiltonian emerges:

$$H_{\text{3-fol}} = -\sum_v (A_v^{xy} + A_v^{yz} + A_v^{zx}) - \sum_c \tilde{B}_c \tag{111}$$

This Hamiltonian is identical to H_{XC}, except that the cube operator takes the form

$$\tilde{B}_c = \left(\prod_{e\in c} X_e\right) \phi_c(\{Z\}) P_c \tag{112}$$

where $\phi_c(\{Z\})$ is a phase-valued function of the Pauli Z variables in the vicinity of cube c, and P_c is a projector onto the subspace satisfing all vertex contraints in the vicinity of c. The function $\phi_c(\{Z\})$ is defined in terms of the phase functions $\phi_p(\{Z\})$ of the 1-foliated gauge theories as follows:

$$\phi_c(\{Z\}) = \prod_{p\in c} \phi_p(\{Z\}), \tag{113}$$

where the $\phi_p(\{Z\})$ are now interpreted to act on the merged effective qubits.

If we ungauge $H_{3\text{-fol}}$ by regarding the fractonic excitations of the cube terms $\tilde{B}_c$ as gauge charge, the resulting model contains one spin per site $\mathbf{r}$ of the dual cubic lattice. The Hamiltonian takes the form

$$H^{\text{un}}_{3\text{-fol}} = -\sum_{\mathbf{r}} X_{\mathbf{r}} \phi_{\mathbf{r}}(\{Z\}) \tag{114}$$

where $\phi_{\mathbf{r}}(\{Z\})$ is the ungauged version of $\phi_c(\{Z\})$ for site $\mathbf{r}$ dual to cube c, which is a function of plaquette variables $ZZZZ$ (tensor product over the four edges in a plaquette) hence manifestly symmetric. Because $\phi_c(\{Z\})$ has the form 113, it follows that $\phi_{\mathbf{r}}(\{Z\})$ has the form

$$\begin{aligned}\phi_{\mathbf{r}}(\{Z\}) = {} & \phi_{\mathbf{r}+\mathbf{x}/2}(\{d_{\mathbf{r}+\mathbf{x}/2}\})\phi_{\mathbf{r}-\mathbf{x}/2}(\{d_{\mathbf{r}-\mathbf{x}/2}\}) \\ & \times \phi_{\mathbf{r}+\mathbf{y}/2}(\{d_{\mathbf{r}+\mathbf{y}/2}\})\phi_{\mathbf{r}-\mathbf{y}/2}(\{d_{\mathbf{r}-\mathbf{y}/2}\}) \\ & \times \phi_{\mathbf{r}+\mathbf{z}/2}(\{d_{\mathbf{r}+\mathbf{z}/2}\})\phi_{\mathbf{r}-\mathbf{z}/2}(\{d_{\mathbf{r}-\mathbf{z}/2}\})\end{aligned} \tag{115}$$

where each function $\phi_{\mathbf{r}+\sigma/2}(d_{\mathbf{r}+\sigma/2})$ is dual to one of the plaquette phase functions $\phi_p(\{Z\})$, and the variables $d_{\mathbf{r}+\sigma/2}$, similar to the variables $d_{\mathbf{r}}$ in the main text, are defined as $d_{\mathbf{r}+\sigma/2} = Z_{\mathbf{r}}Z_{\mathbf{r}+\sigma}$ for $\sigma = \mathbf{x}, \mathbf{y}, \mathbf{z}$ the three unit vectors. Hence, it is straightforwardly verified that $H^{\text{un}}_{3\text{-fol}}$ is the result of stacking three 1-foliated SSPTs with respective Hamiltonians

$$\begin{aligned}H^{xy}_{1\text{-fol}} &= -\sum_{\mathbf{r}} X_{\mathbf{r}}\phi_{\mathbf{r}+\mathbf{z}/2}(\{d_{\mathbf{r}+\mathbf{z}/2}\})\phi_{\mathbf{r}-\mathbf{z}/2}(\{d_{\mathbf{r}-\mathbf{z}/2}\}) \\ H^{yz}_{1\text{-fol}} &= -\sum_{\mathbf{r}} X_{\mathbf{r}}\phi_{\mathbf{r}+\mathbf{x}/2}(\{d_{\mathbf{r}+\mathbf{x}/2}\})\phi_{\mathbf{r}-\mathbf{x}/2}(\{d_{\mathbf{r}-\mathbf{x}/2}\}) \\ H^{zx}_{1\text{-fol}} &= -\sum_{\mathbf{r}} X_{\mathbf{r}}\phi_{\mathbf{r}+\mathbf{y}/2}(\{d_{\mathbf{r}+\mathbf{y}/2}\})\phi_{\mathbf{r}-\mathbf{y}/2}(\{d_{\mathbf{r}-\mathbf{y}/2}\})\end{aligned} \tag{116}$$

Therefore, we have demonstrated that the Hamiltonian $H_{3\text{-fol}}$, obtained by strongly coupling three mutually perpendicular 1-foliated gauge theories, is dual to a weak SSPT Hamiltonian $H^{\text{un}}_{3\text{-fol}}$. Moreover, any weak 3-foliated SSPT can be constructed in this way, since $H^{\text{un}}_{3\text{-fol}}$ describes a stacking of three 1-foliated SSPTs which can in principle by arbitrary. While our discussion has focused on the $G = \mathbb{Z}_2$ case for simplicity, it can be straightforwardly generalized to arbitrary abelian group G.

A. Condensation transitions

As alluded to in the main text, and discussed in Ref. 8, the procedure of stacking a 2D SPT onto a 3-foliated 3D SSPT is dual to the procedure of adding a 2D twisted gauge theory to a 3D twisted X-cube model, and condensing composite planon excitations composed of fracton dipole and 2D gauge charge pairs. This planon condensation process has the effect of confining the lineon dipoles and 2D gauge fluxes that braid non-trivially with these fracton dipoles and 2D gauge charges respectively, leaving deconfined only the composites of lineon dipoles and 2D gauge fluxes, which become the lineon dipoles of the condensed phase. The result is that the statistics of these lineon dipoles are now modified by the addition of the 2D gauge flux statistics.

Let us consider a simple example. Consider stacking an xy oriented 2D $\mathbb{Z}_2$ SPT, with $\mathbf{M}$ matrix a single entry-matrix equal to 2, between layers $z = 0$ and $z = 1$ of a trivial 3D 3-foliated $\mathbb{Z}_2$ planar SSPT, dual to a copy of the X-cube model. This procedure is dual to adding a 2D double semion layer, whose gauge flux has semionic exchange statistics, and condensing the planon composed of the 2D gauge charge plus the fracton dipole centered around $z = 1/2$. This condensation has the effect of confining both the 2D gauge flux and the lineon dipoles centered around $z = 0$ and $z = 1$, and leaving deconfined the composite of the $z = 0$ lineon dipole and the 2D gauge flux, and the composite of the $z = 1$ lineon dipole and the 2D gauge flux. Both of these composites therefore obtain semionic exchange statistics. This procedure corresponds to the addition of a single self-loop to a 3-foliated SSPT in our graphical notation (see Fig. 1).

This dual picture interpretation of the stacking construction of weak 3-foliated SSPTs sheds light on the correspondence with p-string condensation. The key point is that the p-string condensation procedure resulting in untwisted and twisted X-cube models, and the planon condensation procedure outlined above, commute with one another because they both involve condensation of pure gauge charge. Therefore, one can construct the dual phases of weak 3-foliated SSPTs by 1) starting with three decoupled stacks of 2D toric code layers, 2) adding 2D twisted gauge theory layers and identifying the added gauge symmetries with existing gauge symmetries by condensing pairs of gauge charges, and 3) driving a p-string condensation transition. Since all 1-foliated SSPTs are weak, and thus can be constructed by stacking 2D SPTs, step 2 allows for the creation of arbitrary 1-foliated gauge theories. Therefore, any fracton model which is obtained by performing p-string condensation on intersecting 1-foliated gauge theories, is dual to a weak 3-foliated SSPT.

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