Conspiracy Relations in Vector-Meson Production*

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We list the conspiracy relations for reactions of type $\Pi+N\to V+N$ and $\Pi+N\to V+\Delta$, emphasizing that there are relations connecting states with opposite parity at $t=0$ even when all four masses are unequal. We show in detail that if the relations are satisfied by vanishing of the couplings of individual trajectories at $t=0$, all helicity-flip amplitudes are suppressed at forward angles; whereas if the relations are satisfied by conspiracy among various trajectories, the flip-flip amplitudes which conserve angular momentum in forward scattering can contribute with full strength.

I. INTRODUCTION

RECENTLY, the existence of "conspiracy relations" among scattering amplitudes at $t=0$ has been recognized.[1,2] In the present paper, we discuss various aspects of these relations for reactions of type $\Pi+N\to V+N$ and $\Pi+N\to V+\Delta$.

We begin in Sec. II by listing the relations at $t=0$ for the above reactions. Our method of derivation is due to Cohen-Tannoudji *et al.*,[3,4] and is described in Appendix A with special emphasis on the relations for $\Pi+N\to V+\Delta$.[5]

If only Regge poles contribute to the scattering amplitudes, the relations at $t=0$ can be satisfied either by zeros in the couplings of individual trajectories, or by conspiracy among different trajectories. These same two alternatives are open to contributions from other singularities in the angular-momentum plane, such as cuts or fixed poles. The two possibilities can be experimentally distinguished by the property that in the first case, all helicity-flip amplitudes are suppressed in the forward direction, whereas in the second case, only those amplitudes which fail to conserve angular momentum along the forward direction are suppressed. This property, which we have alluded to in previous papers,[6,7] is proved for $\Pi+N\to V+N$ and $\Pi+N\to V+\Delta$ in Sec. III. The proof makes use of the special form the crossing matrix takes for forward scattering; this form is given in Appendix B.

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[2] M. Gell-Mann and E. Leader, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, Calif., 1967).
[3] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Saclay Report, 1967 (unpublished).
[4] A discussion is also given by H. Hogaasen and P. Salin, Nucl. Phys. **B2**, 615 (1967). Equation (2.3) was obtained from the related reaction $\gamma p\to\pi^+n$ by M. Halpern, Phys. Rev. **160**, 1441 (1967).
[5] E. Leader, Phys. Rev. **166**, 1599 (1968).
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[7] S. Frautschi and Lorella Jones, Phys. Rev. **163**, 1820 (1967).

II. CONSPIRACY RELATIONS IN $\Pi N\to VN$ AND $\Pi N\to V\Delta$

We shall use the customary notation $f_{cd;ab}{}^t$ to denote a helicity amplitude for the t-channel reaction $a+b\to c+d$. The conspiracy relations for unequal-mass reactions involve the reduced amplitudes

$$\bar{f}_{cd;ab}{}^t=f_{cd;ab}{}^t(\sin\tfrac{1}{2}\theta_t)^{-|c-d-a+b|}(\cos\tfrac{1}{2}\theta_t)^{-|c-d+a-b|}\quad(2.1)$$

formed into "parity-conserving" combinations ($\bar{f}_{cd;ab}{}^t\pm\bar{f}_{-c-d;ab}{}^t$). Each combination can be written

$$\bar{f}_{cd;ab}{}^t\pm\bar{f}_{-c-d;ab}{}^t=K_{cd;ab}{}^{\pm}(t)\tilde{f}_{cd;ab}{}^{t(\pm)}(t,s)\,,\quad(2.2)$$

where K is a known factor containing the kinematic singularities and zeros listed by Wang,[8] and $\tilde{f}$ contains the dynamics. The conspiracy relations provide additional kinematic zeros at $t=0$ in certain linear combinations of the parity-conserving amplitudes.

Using the methods of Appendix A, we find two conspiracy relations at $t=0$ for $\Pi N\to VN$[4]:

$$(\sqrt{t})[(\bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{1}{2}-\frac{1}{2}}{}^t)+i(\bar{f}_{10;\frac{1}{2}\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{1}{2}\frac{1}{2}}{}^t)]\sim t\,,\quad(2.3)$$

$$(\sqrt{t})[f_{00;\frac{1}{2}-\frac{1}{2}}{}^t+2if_{00;\frac{1}{2}\frac{1}{2}}{}^t]\sim t\,.\quad(2.4)$$

For all four parity-conserving amplitudes involved here, the Wang kinematical factor[9] K allows a $t^{-1/2}$ behavior at $t=0$ (Table I). The additional kinematic conditions imposed by Eqs. (2.3) and (2.4) can be satisfied in either of two ways: (a) Each of the two amplitudes involved in an equation can have an additional factor of t from $\tilde{f}$; in this case each term in the equation would vanish separately like t (no conspiracy); or (b) each amplitude can retain the singular $t^{-1/2}$ behavior at $t=0$; in this case the two terms in the equation must approach the same constant (conspiracy). For example, one conspiracy solution behaves like

$$\begin{aligned}\bar{f}_{10;\frac{1}{2}\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{1}{2}\frac{1}{2}}{}^t&=c_4/\sqrt{t}\,,\\ \bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{1}{2}-\frac{1}{2}}{}^t&=-ic_4/\sqrt{t}\end{aligned}\quad(2.5)$$

near $t=0$.

[8] L. L. Wang, Phys. Rev. **142**, 1187 (1966).
[9] The kinematic factors $K(t)$ used throughout this paper are those found by Wang (Ref. 8) from the crossing matrix. They do not include the additional factors which can be found from factorization in some cases. This is a departure from the usage in previous papers by the present authors.

TABLE I. Amplitudes for $\Pi N \to VN$.

Amplitudes	μ	λ	Dominant parity	Kinematic factor	Extra factor in no-conspiracy case
$f_{00;\frac{1}{2}\frac{1}{2}}{}^t$	0	0	$(-1)^{J+1}$	$1/\sqrt{t}$	t
$\bar{f}_{00;\frac{1}{2}-\frac{1}{2}}{}^t$	0	1	$(-1)^{J+1}$	$1/\sqrt{t}$	t
$\bar{f}_{10;\frac{1}{2}\frac{1}{2}}{}^t+\bar{f}_{-10;+\frac{1}{2}+\frac{1}{2}}{}^t$	1	0	$(-1)^J$	1	
$\bar{f}_{10;\frac{1}{2}\frac{1}{2}}{}^t-\bar{f}_{-10;+\frac{1}{2}+\frac{1}{2}}{}^t$	1	0	$(-1)^{J+1}$	$1/\sqrt{t}$	t
$\bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t+\bar{f}_{-10;+\frac{1}{2}-\frac{1}{2}}{}^t$	1	1	$(-1)^{J+1}$	1	
$\bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t-\bar{f}_{-10;+\frac{1}{2}-\frac{1}{2}}{}^t$	1	1	$(-1)^J$	$1/\sqrt{t}$	t

For the reaction $\Pi N \to V\Delta$, the $t=0$ conditions obtained by the methods of Appendix A are

$$t^{3/2}[\bar{f}_{10;\frac{3}{2}-\frac{1}{2}}{}^t+\bar{f}_{-10;\frac{3}{2}-\frac{1}{2}}{}^t] = -t^{3/2}[\bar{f}_{10;\frac{3}{2}-\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{3}{2}-\frac{1}{2}}{}^t]=t^{3/2}\bar{f}_{-10\frac{3}{2}-\frac{1}{2}}, \quad (2.6)$$

$$t[\bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t+\bar{f}_{-10;\frac{1}{2}-\frac{1}{2}}{}^t] = -t[\bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{1}{2}-\frac{1}{2}}{}^t]=t\bar{f}_{-10;\frac{1}{2}-\frac{1}{2}}{}^t, \quad (2.7)$$

$$t[\bar{f}_{10;\frac{3}{2}\frac{1}{2}}{}^t+\bar{f}_{-10;\frac{3}{2}\frac{1}{2}}{}^t] = -t[\bar{f}_{10;\frac{3}{2}\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{3}{2}\frac{1}{2}}{}^t]=t\bar{f}_{-10;\frac{3}{2}\frac{1}{2}}{}^t. \quad (2.8)$$

In each of the amplitudes $[\bar{f}_{cd;ab}{}^t \pm \bar{f}_{-c-d;ab}{}^t]$, the Wang kinematic factor K (Table II) allows a maximum singularity $(\sqrt{t})^{-|c-d|-|a-b|}$ at $t=0$—just enough to cancel the explicit factors of t in Eqs. (2.6)–(2.8). Again the $t=0$ conditions can be satisfied either by having the contribution of each parity-conserving amplitude vanish separately like t, or by conspiracy between the coefficients of the singularities.

Examples of solutions and their interpretations are discussed in the following section.

III. CONSPIRACY AND ANGULAR-MOMENTUM CONSERVATION IN THE FORWARD DIRECTION

It has been known for some time that in NN scattering, the double spin-flip amplitude $f_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}}{}^s$ is nonzero in the forward direction only when conspiracy occurs, despite the fact that angular-momentum conservation places no restrictions on this amplitude. The purpose of this section is to demonstrate that conspiracy has the same role in unequal-mass reactions: that of restoring the contributions of double flip amplitudes to full strength in the forward direction.

Consideration of unequal-mass kinematics for the reactions $\Pi N \to VN$ and $\Pi N \to V\Delta$ [6] shows that although $\cos\frac{1}{2}\theta_t$ and $\sin\frac{1}{2}\theta_t$ are big and proportional to $s^{1/2}$ for larger t, they are constrained to approach 0 and 1, respectively, at $\theta_s=0°$ ($t=t_{\min}$). This means that all amplitudes

$$f_{co;ab}{}^t=\bar{f}_{co;ab}{}^t(\sin\tfrac{1}{2}\theta_t)^{|c-a+b|}(\cos\tfrac{1}{2}\theta_t)^{|c+a-b|} \quad (3.1)$$

with $(c+a-b)\neq 0$ vanish in the forward direction of the s channel. For $(c+a-b)=0$, the forward amplitude does not vanish but the fast drop in $\sin\frac{1}{2}\theta_t$ toward $t=t_{\min}$ tends to suppress it. Unless $\bar{f}_{co;ab}{}^t$ for $(c+a-b)=0$ compensates by growing like t^{-x} with appropriate x, the suppression will greatly decrease $f_{co;ab}{}^t$ near $t=t_{\min}$.[6]

It is instructive to view these same phenomena from the point of view of the s-channel amplitudes. As shown in Appendix B, at $\theta_s=0°$ the crossing relations reduce to the form

$$f_{co;ab}{}^t=\pm f_{-ca;ob}{}^s. \quad (B5)$$

The vanishing of all $f_{co;ab}{}^t$ with $c+a-b\neq 0$ at $\theta_s=0°$, then, is simply a manifestation of angular momentum conservation along the direction of forward scattering in the s channel. The suppression of $f_{co;ab}{}^t$ with $(c+a-b)=0$ at $\theta_s=0°$, however, is not required by angular momentum conservation. Therefore it is to be expected that $\bar{f}_{co;ab}{}^t$ can have the compensating factor t^{-x} needed to restore the normal size of the angular-momentum-conserving amplitudes at $\theta_s=0°$.

For reactions such as $\Pi N \to V\Delta$, $\sin\frac{1}{2}\theta_t \sim (ts)^{1/2}$ at high s; the forward suppression is caused by the $\sqrt{t}$ factor. Hence if $\bar{f}_{co;ab}{}^t \sim (1/\sqrt{t})^{|c-a+b|}$, we see from Eq. (2.1) that the amplitudes allowed by angular-momentum conservation will contribute with full strength in the forward direction. This is exactly the maximum singularity allowed by the Wang formalism. As we have shown [Eqs. (2.6)–(2.8) and Table II)], the amplitudes will have this maximum singularity only if they conspire at $t=0$.

For $\Pi N \to VN$, $\sin\frac{1}{2}\theta_t \sim t^{1/4}s^{1/2}$ at high s; the forward suppression is caused by the $t^{1/4}$ factor. Hence, for this case, the $t=0$ behavior necessary to ensure full contribution of the angular-momentum-conserving amplitude is $\bar{f}_{co;ab}{}^t \sim (t^{-1/4})^{|c-a+b|}$. Again we see from Table I that the relevant amplitude $f_{-10;\frac{1}{2}-\frac{1}{2}}{}^t$ is allowed just the right singularity $(t^{-1/4})^{|c-a+b|} \sim t^{-1/2}$ in the Wang formalism, but [in view of Eq. (2.3)] can achieve it only through conspiracy.

Hence we conclude that the conspiracy relations play a very similar role in equal- and unequal-mass reactions. In both cases nonconspiring flip-flip amplitudes vanish

TABLE II. Amplitudes for $\Pi N \to V\Delta$.

Amplitudes	μ	λ	Dominant parity	Kinematic factor	Extra factor in no-conspiracy case
$f_{00;\frac{1}{2}\frac{1}{2}}{}^t$	0	0	$(-1)^{J+1}$	1	
$\bar{f}_{00;\frac{3}{2}\frac{1}{2}}{}^t$	0	1	$(-1)^{J+1}$	$1/\sqrt{t}$	
$\bar{f}_{00;\frac{3}{2}-\frac{1}{2}}{}^t$	0	2	$(-1)^{J+1}$	$1/t$	
$\bar{f}_{00;\frac{1}{2}-\frac{1}{2}}{}^t$	0	1	$(-1)^{J+1}$	$1/\sqrt{t}$	
$\bar{f}_{10;\frac{1}{2}\frac{1}{2}}{}^t+\bar{f}_{-10;\frac{1}{2}\frac{1}{2}}{}^t$	1	0	$(-1)^J$	$1/\sqrt{t}$	
$\bar{f}_{10;\frac{1}{2}\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{1}{2}\frac{1}{2}}{}^t$	1	0	$(-1)^{J+1}$	$1/\sqrt{t}$	
$\bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t+\bar{f}_{-10;\frac{1}{2}-\frac{1}{2}}{}^t$	1	1	$(-1)^{J+1}$	$1/t$	t
$\bar{f}_{10;\frac{1}{2}-\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{1}{2}-\frac{1}{2}}{}^t$	1	1	$(-1)^J$	$1/t$	t
$\bar{f}_{10;\frac{3}{2}\frac{1}{2}}{}^t+\bar{f}_{-10;\frac{3}{2}\frac{1}{2}}{}^t$	1	1	$(-1)^{J+1}$	$1/t$	t
$\bar{f}_{10;\frac{3}{2}\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{3}{2}\frac{1}{2}}{}^t$	1	1	$(-1)^J$	$1/t$	t
$\bar{f}_{10;\frac{3}{2}-\frac{1}{2}}{}^t+\bar{f}_{-10;\frac{3}{2}-\frac{1}{2}}{}^t$	1	2	$(-1)^{J+1}$	$1/t^{3/2}$	t
$\bar{f}_{10;\frac{3}{2}-\frac{1}{2}}{}^t-\bar{f}_{-10;\frac{3}{2}-\frac{1}{2}}{}^t$	1	2	$(-1)^J$	$1/t^{3/2}$	t

at $t=0$; the main difference is that $t=0$ corresponds exactly to forward scattering in the equal-mass case, but is only approached asymptotically by forward production in the unequal-mass case.

Conspiracy relation (2.4) is in a separate category, since the $\lambda=\mu=0$ amplitude is not multiplied by half-angle factors and thus not subject to quite the same suppression in the forward direction. In the no-conspiracy case, however, it does contain a $\sqrt{t}$, which along the curve $\theta_s=0^0$ contributes an effective suppression $\sim s^{-1}$. In the case of conspiracy, Eq. (2.4) equates $f_{00;\frac{1}{2}\frac{1}{2}}{}^t$ to $\frac{1}{2}i\bar{f}_{00;\frac{1}{2}-\frac{1}{2}}$ which grows like $s^{\alpha-1}t^{-1/2}$. Since $t^{-1/2}\sim s$ along the boundary curve, $f_{00;\frac{1}{2}\frac{1}{2}}{}^t$ will grow like s^α at $\theta_s=0°$, i.e., a conspiring singularity contributes full strength to the $\lambda=\mu=0$ amplitude along the forward direction.

APPENDIX A: DERIVATION OF CONSPIRACY RELATIONS

The method which we use has been formulated by Cohen-Tannoudji, Morel, and Navelet.[3] One defines the $\bar{f}^t$ as "parity-conserving" helicity amplitudes with the kinematic singularities found by Wang[8] removed [Eq. (2.2)]. For these amplitudes, the crossing matrix can be written in the form

$$\bar{f}_i{}^s=\sum_j \tilde{X}_{ij}{}^{-1}\tilde{f}_j{}^t. \tag{A1}$$

In general, near $t=0$ the matrix elements of $\tilde{X}_{ij}{}^{-1}$ have the form[10]

$$\tilde{X}_{ij}{}^{-1}=(C_{ij}/t)+\text{terms regular at } t=0). \tag{A2}$$

Thus the condition

$$\sum_j C_{ij}\tilde{f}_j{}^t(t=0)=0 \tag{A3}$$

must hold, since neither $\bar{f}_i{}^s$ nor the individual elements $\tilde{f}_j{}^t$ have kinematic singularities or zeros at $t=0$. The equations (A3) are the desired conspiracy relations.

Cohen-Tannoudji *et al.* found that in reactions (such as $\Pi N\rightarrow V\Delta$) with unequal masses in both t-channel states, this technique does not lead to any conspiracy relations between different $\bar{f}_{cd;ab}{}^t$'s. It does, however, lead to relations between different parity-conserving amplitudes.[11] If the amplitudes are dominated by Regge poles, different poles will dominate different parity-conserving amplitudes; thus the relations obtained are of a nontrivial nature. For this reason we present the simple derivation in detail.

For each individual amplitude $\bar{f}_{cd;ab}{}^t$ in reactions where the masses are related as in $\Pi N\rightarrow V\Delta$, the Wang kinematic factor at $t=0$ allows a maximum singularity $(1/\sqrt{t})^{|\lambda-\mu|}$, where $\lambda=a-b$, $\mu=c-d$. Thus $\bar{f}_{cd;ab}{}^t$ and $\bar{f}_{-c-d;ab}{}^t$ have different maximum singularities unless at least one of μ and λ vanishes. The parity-conserving combinations $(\bar{f}_{cd;ab}{}^t\pm\bar{f}_{-c-d;ab}{}^t)$ are allowed the larger of the two singularities, i.e., $(1/\sqrt{t})^{|\lambda|+|\mu|}$. Hence if $P=\big||\mu|-|\lambda|\big|$, $Q=|\mu|+|\lambda|$ and $\bar{f}_{cd;ab}{}^t$ is the individual amplitude which is only allowed the smaller singularity, we have

$$\bar{f}_{cd;ab}{}^t\sim(1/\sqrt{t})^P \tag{A4}$$

and

$$\bar{f}_{cd;ab}{}^t=\tfrac{1}{2}[(\bar{f}_{cd;ab}{}^t+\bar{f}_{-c-d;ab}{}^t)+(\bar{f}_{cd;ab}{}^t-\bar{f}_{-c-d;ab}{}^t)]$$
$$\sim\frac{1}{2}\left[\frac{C_1}{(\sqrt{t})^Q}+\frac{C_2}{(\sqrt{t})^Q}\right]. \tag{A5}$$

To make (A5) consistent with (A4), C_1+C_2 must have a zero of order $(\sqrt{t})^{Q-P}$ at $t=0$. As with all $t=0$ constraint equations this can be satisfied either by conspiracy (C_1 and C_2 nonvanishing and correlated) or nonconspiracy [C_1 and C_2 separately vanishing like $(\sqrt{t})^{Q-P}$].[12]

In the derivation just presented, we did not make explicit use of the crossing matrix, but it is easy to verify that conditions of type (A3) do arise if one works with parity-conserving amplitudes, and lead to the same $t=0$ relations as described above.

As a particularly simple example of this type of conspiracy, consider the s-channel reaction $\Pi\Pi\rightarrow\gamma\gamma$. The conspiracy relations can be derived as in Eqs. (A4)–(A5), or from the invariant amplitudes for this case, but it is instructive to work through the crossing method of Cohen-Tannoudji *et al.* The crossing relations for the helicity amplitudes are

$$f_{+0;+0}{}^t=f_{+-;00}{}^s,\quad f_{+0;-0}{}^t=f_{++;00}{}^s. \tag{A6}$$

Rewriting Eqs. (A6) in terms of $\bar{f}$'s one finds

$$\bar{f}_{+0;+0}{}^t=\frac{(\sin\theta_s)^2}{4(\cos\frac{1}{2}\theta_t)^2}\bar{f}_{+-;00}{}^s,$$
$$\bar{f}_{+0;-0}{}^t=\frac{1}{(\sin\frac{1}{2}\theta_t)^2}\bar{f}_{++;00}{}^s, \tag{A7}$$

which can be reexpressed in terms of s and t:

$$\bar{f}_{+0;+0}{}^t=\frac{-(t-\mu^2)^2}{4s(s-4\mu^2)}\bar{f}_{+-;00}{}^s,$$
$$\bar{f}_{+0;-0}{}^t=\frac{-(t-\mu^2)^2}{st}\bar{f}_{++;00}{}^s. \tag{A8}$$

[10] More generally there may also be terms C_{ij}'/t^N, where $N=2, 3, \cdots$. For such terms, the condition analogous to (A3) is that $\sum_j C_{ij}'\tilde{f}_j{}^t$ must vanish like t^N at $t=0$.

[11] These appear to be similar to the relations derived by Toller from group-theoretical considerations: M. Toller, Rome University Report (unpublished).

[12] Note that the property $\bar{f}_{cd;ab}{}^t\sim(\pm)\bar{f}_{-c-d;ab}{}^t$, which holds for the contribution of a Regge pole of definite parity to the amplitude at asymptotically large s, is consistent with the singularities at $t=0$ only if the no-conspiracy choice

$$[\bar{f}_{cd;ab}{}^t\sim(1/\sqrt{t})^P,\ \bar{f}_{-c-d;ab}{}^t\sim(1/\sqrt{t})^P]$$

is made. This is not surprising since, as explained above, the conspiracies in these amplitudes are necessarily between trajectories of opposite parity.

By standard reasoning,[8,13] the $\bar{f}^s$ have no kinematic singularities or zeros in t. Thus it follows from the crossing relations (A8) that $\bar{f}_{+0;+0}{}^t \sim$ constant at $t=0$ and $\bar{f}_{+0;-0}{}^t \sim t^{-1}$. The amplitudes

$$a=-\frac{1}{(t-\mu^2)^2}\bar{f}_{+0;+0}{}^t\,,\qquad b=-\frac{t}{(t-\mu^2)^2}\bar{f}_{+0;-0}{}^t\,, \tag{A9}$$

are therefore free of kinematic singularities or zeros at $t=0$.[14] The counterpart of Eq. (A1) for this case is

$$\begin{pmatrix} \bar{f}_{+-;00}{}^s \\ \bar{f}_{++;00}{}^s \end{pmatrix} = \begin{pmatrix} 4s(s-4\mu^2) & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{A10}$$

which, analyzed by the procedure of Eqs. (A2)–(A3), gives no $t=0$ relations. This is in agreement with the conclusions of Cohen-Tannoudji *et al.* about the lack of conspiracy between different $\bar{f}_{cd;ab}{}^t$'s. But if we turn to parity-conserving combinations $(\bar{f}_{+0;+0}{}^t+\bar{f}_{-0;+0}{}^t)$ and $(\bar{f}_{+0;+0}{}^t-\bar{f}_{-0;+0}{}^t)$, the reasoning following Eq. (A8) indicates that both parity-conserving amplitudes $\sim t^{-1}$ at $t=0$, and that in this case the amplitudes which are free of kinematic singularities or zeros at $t=0$ can be written[15]

$$\bar{f}_{+0;+0}{}^{t(+)}=\frac{t}{(t-\mu^2)^2}(\bar{f}_{+0;+0}{}^t+\bar{f}_{-0;+0}{}^t)\,,\qquad \bar{f}_{+0;+0}{}^{t(-)}=\frac{t}{(t-\mu^2)^2}(\bar{f}_{+0;+0}{}^t-\bar{f}_{-0;+0}{}^t) \tag{A11}$$

(note that $\bar{f}_{-0;+0}{}^t=\bar{f}_{+0;-0}{}^t$). Working out the crossing matrix connecting $\bar{f}^t$ to $\bar{f}^s$, we find

$$\begin{pmatrix} \bar{f}_{+-;00}{}^s \\ \bar{f}_{++;00}{}^s \end{pmatrix} = \begin{pmatrix} -2s(s-4\mu^2)/t & -2s(s-4\mu^2)/t \\ -\frac{1}{2}s & \frac{1}{2}s \end{pmatrix} \times \begin{pmatrix} \bar{f}_{+0;+0}{}^{t(+)} \\ \bar{f}_{+0;+0}{}^{t(-)} \end{pmatrix}. \tag{A12}$$

For this matrix, the procedure of Eqs. (A1)–(A3) gives the condition

$$t[\bar{f}_{+0;+0}{}^t+\bar{f}_{-0;+0}{}^t]=-t[\bar{f}_{+0;+0}{}^t-\bar{f}_{-0;+0}{}^t] \tag{A13}$$

at $t=0$. As usual, this relation can be satisfied either by separate zeros in both parity-conserving amplitudes, or by conspiracy between them.

[13] Y. Hara, Phys. Rev. **136**, B507 (1964).
[14] D. Horn, Caltech report No. CALT-68-131/Internal Report 34, 1967 (unpublished).
[15] As described in Appendix A of Ref. 7, reactions with photons have special features, and other arguments give $\bar{f}^t$ a different factor at $t=\mu^2$. However, this will not affect our present discussion of behavior at $t=0$.

APPENDIX B: CROSSING RELATIONS IN THE FORWARD DIRECTION FOR UNEQUAL-MASS REACTIONS

The Trueman-Wick[16] crossing relations between t-channel and s-channel helicity amplitudes may be put in the form[8]

$$f_{cd;ab}{}^t=\sum d_{A'a}{}^{J_a}(\chi_a)d_{b'b}{}^{J_b}(\chi_b)d_{c'c}{}^{J_c}(\chi_c) \times d_{D'd}{}^{J_d}(\chi_d)f_{c'A';D'b'}{}^s\,, \tag{B1}$$

where

$$\sin\chi_a=\frac{2m_a[\phi(s,t)]^{1/2}}{t_{ab}s_{ac}}\,,\quad \sin\chi_c=\frac{2m_c[\phi(s,t)]^{1/2}}{t_{cd}s_{ac}}\,,\quad \sin\chi_b=\frac{2m_b[\phi(s,t)]^{1/2}}{t_{ab}s_{bd}}\,,\quad \sin\chi_d=\frac{2m_d[\phi(s,t)]^{1/2}}{t_{cd}s_{bd}}\,, \tag{B2}$$

$$t_{ij}=[t-(m_i+m_j)^2]^{1/2}[t-(m_i-m_j)^2]^{1/2}\,,\qquad s_{ij}=[s-(m_i+m_j)^2]^{1/2}[s-(m_i-m_j)^2]^{1/2}\,, \tag{B3}$$

$$\phi(s,t)=st(\sum m_i^2-s-t)-t(m_b^2-m_d^2)(m_a^2-m_c^2) -s(m_a^2-m_b^2)(m_c^2-m_d^2)-(m_a^2m_d^2-m_c^2m_b^2) \times(m_a^2+m_b^2+m_c^2+m_d^2)\,, \tag{B4}$$

and ϕ vanishes on the boundary of the physical region.

For those cases where the masses are unequal in either the final or the initial state of the t channel, $\phi=0$ does not coincide with $t=0$ at finite s. At $\phi=0$, therefore, the sines of all the crossing angles vanish; hence these angles must assume the values 0 or π.[8] When these values are substituted into the crossing matrix, it is easily seen that each s-channel helicity amplitude crosses to only one t-channel amplitude along the curve $\phi=0$, and that this crossing is such that each helicity index in the amplitude either remains the same or changes sign.

It has been shown by Shepard[17] that (a) for the particles at a t-channel vertex connecting *unequal* nonzero masses, both helicities flip (do not flip) if the mass of the particles whose line is reversed under crossing is less (greater) than the mass of the uncrossed particle; (b) at a vertex connecting *equal* nonzero masses, both helicities flip (do not flip) if (for the unequal-mass pair at the other vertex) the mass of the particle which gets crossed is greater (less) than the mass of the uncrossed particle. Hence for the s-channel reactions $\Pi N\rightarrow VN$ and $\Pi N\rightarrow V\Delta$ the crossing relarelations along the curve $\phi=0$ take the form

$$f_{co;ab}{}^t=\pm f_{-ca;ob}{}^s\,. \tag{B5}$$

[16] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).
[17] H. K. Shepard, Phys. Rev. **159**, 3331 (1967).