

# The $AdS_5 \times S^5$ Superstring

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## Abstract

The duality between the type IIB superstring theory in an  $AdS_5 \times S^5$  background with  $N$  units of five-form flux and  $\mathcal{N} = 4$  super Yang–Mills theory with a  $U(N)$  gauge group has been studied extensively. My version of the construction of the superstring world-sheet action is reviewed here. This paper is dedicated to Michael Duff on the occasion of his 70th birthday.

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# 1 Introduction

I am pleased to contribute to this volume honoring my good friend Michael Duff on the occasion of his 70th birthday. Although we never collaborated, we have had lively scientific discussions, and each of us has influenced the other's research. In the early 1980s, when Michael Green and I were developing superstring theory, extra dimensions were not yet fashionable. However, this began to change as a result of the work of Michael Duff and his collaborators exploring compactifications of 11-dimensional supergravity [1]. This work attracted a community of followers who acquired expertise that would prove useful in the subsequent development of superstring theory and M theory.

Superstring theory was originally developed for a Minkowski spacetime geometry. (For reviews see [2][3][4].) This is the easiest case to handle mathematically, but it is not the only possibility. In particular, there is great interest in Anti de Sitter (AdS) geometries for studies of AdS/CFT duality. This manuscript will review the specific case of the type IIB superstring in an  $AdS_5 \times S^5$  background with  $N$  units of five-form flux, which is dual to  $\mathcal{N} = 4$  super Yang–Mills theory with a  $U(N)$  gauge group [5]. This is an especially interesting example – sometimes referred to as the “hydrogen atom” of AdS/CFT. It has a large group of symmetries. In particular, it is maximally supersymmetric (32 supercharges). In addition, there are two tunable dimensionless parameters: the string coupling constant and the ratio between the curvature radius and the string length scale. These are related by AdS/CFT duality to the Yang–Mills coupling constant and the rank of the gauge group.

In the Minkowski spacetime setting, the free superstring spectrum was identified and shown not to contain ghosts or tachyons in the critical spacetime dimension, which is 10 (nine space and one time). Scattering amplitudes for  $n$  massless external on-shell particles can be constructed perturbatively in the string coupling constant  $g_s$ . The most fundamental examples are the type IIA and type IIB superstring theories, which only involve closed oriented strings and have maximal supersymmetry. For these theories the massless states comprise a supergravity multiplet, and the conserved supercharges consist of two Majorana–Weyl spinors, each of which has 16 real components. In the type IIA case the two spinors have opposite chirality (or handedness), and the theory is parity conserving. In the type IIB case they have the same chirality, and the theory is parity violating. It is a nontrivial fact that the type IIB theory has no gravitational anomalies [6].

String theories have a fundamental length scale,  $l_s$ , called the string scale. In units with  $\hbar = c = 1$ , one also defines the “Regge-slope parameter”  $\alpha' = l_s^2$  and the fundamental string tension  $T = (2\pi\alpha')^{-1}$ . The string coupling constant is determined by the vacuum value

of a massless scalar field  $\phi$ , called the dilaton,  $g_s = \langle e^\phi \rangle$ .  $n$ -particle on-shell scattering amplitudes for both type II theories have a single Feynman diagram at each order of the perturbation expansion. At  $g$  loops the unique string theory Feynman diagram is a genus  $g$  Riemann surface with  $n$  punctures associated to the external particles. This two-dimensional manifold is a Euclideanized string world sheet. The  $g$ -loop amplitude is then given by an integral over the  $3g + n - 3$  complex-dimensional moduli space of such punctured Riemann surfaces. These amplitudes are free of UV divergences.

There are two basic approaches to incorporating the fermionic degrees of freedom.<sup>2</sup> The first one, called the RNS formalism, involves fermionic (*i.e.*, Grassmann valued) world-sheet fields that transform as world-sheet spinors and spacetime vectors. One of the shortcomings of the RNS formalism is that nonzero backgrounds for fields belonging to the RR sector are difficult to incorporate. The second basic approach, called the GS formalism, utilizes fermionic world-sheet fields that transform as world-sheet scalars and spacetime spinors. It can handle background RR fields and it makes spacetime supersymmetry manifest. However, the rules for constructing multiloop amplitudes have not been worked out in the GS formalism. This paper will describe type IIB superstring theory in an  $AdS_5 \times S^5$  background. Since this background includes a nonzero RR field, a five-form field strength, the GS formalism is best suited to this problem.

Maldacena's original paper proposing AdS/CFT duality drew attention to three (previously known) maximally supersymmetric geometries containing  $AdS$  factors [5].  $AdS_4 \times S^7$  and  $AdS_7 \times S^4$  are M-theory backgrounds, whereas  $AdS_5 \times S^5$  is a type IIB superstring theory background. The latter case can be studied in greatest detail, because superstring theory is better understood than M theory, which does not have a dimensionless coupling constant. The dual conformal field theory (CFT) in this case is four-dimensional  $\mathcal{N} = 4$  super Yang–Mills theory with a  $U(N)$  gauge group. The integer parameter  $N$  corresponds to the amount of self-dual five-form flux, the nonzero RR field, in the 10d geometry.

The goal of this paper is to describe the world-sheet action of a type IIB superstring in the  $AdS_5 \times S^5$  background geometry, with the appropriate self-dual five-form flux. The isometry group of type IIB superstring theory in this background is given by the supergroup  $PSU(2, 2|4)$ . This problem was originally studied in the GS formalism [7][8][9] using a formulation based on the quotient space  $PSU(2, 2|4)/SO(4, 1) \times SO(5)$ . This paper will summarize closely related much later work using a slightly different approach [10]. It is based on three world-sheet one-form supermatrices whose complete dependence on the ten bosonic coordinates and the 32 Grassmann coordinates is explicit. We will also review the

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<sup>2</sup>There are also other approaches, such as Berkovits' "pure spinor" formalism.

proof that this action describes an integrable theory [11][12][13][14].

## 1.1 The bosonic truncation

The unit-radius sphere  $S^5$  can be described as a submanifold of  $\mathbb{R}^6$

$$\hat{z} \cdot \hat{z} = (z^1)^2 + (z^2)^2 + \dots + (z^6)^2 = 1. \quad (1)$$

Similarly, the unit-radius anti de Sitter space  $AdS_5$  can be described by<sup>3</sup>

$$\hat{y} \cdot \hat{y} = -(y^0)^2 + (y^1)^2 + \dots + (y^4)^2 - (y^5)^2 = -1, \quad (2)$$

which is a submanifold of  $\mathbb{R}^{4,2}$ . These formulas make the symmetries  $SO(6)$  and  $SO(4,2)$ , respectively, manifest. When we add fermions these groups will be replaced by their covering groups, which are  $SU(4)$  and  $SU(2,2)$ . In this notation the  $AdS_5 \times S^5$  metric of radius  $R$  takes the form

$$ds^2 = R^2(d\hat{y} \cdot d\hat{y} + d\hat{z} \cdot d\hat{z}). \quad (3)$$

The induced world-volume metric of a probe  $p$ -brane with local coordinates  $\sigma^\alpha$ ,  $\alpha = 0, 1, \dots, p$  is

$$G_{\alpha\beta}(\sigma) = \partial_\alpha \hat{z}(\sigma) \cdot \partial_\beta \hat{z}(\sigma) + \partial_\alpha \hat{y}(\sigma) \cdot \partial_\beta \hat{y}(\sigma). \quad (4)$$

As usual, the functions  $\hat{z}(\sigma)$  and  $\hat{y}(\sigma)$  describe the spacetime embedding of the brane. In this work we are concerned with the superstring for which  $p = 1$ . The bosonic part of the radius  $R$  superstring action can be written in the general coordinate invariant form

$$S = -\frac{R^2}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} G_{\alpha\beta}, \quad (5)$$

where  $h_{\alpha\beta}(\sigma)$  is an auxiliary world-sheet metric. This metric is related to the induced metric  $G_{\alpha\beta}(\sigma)$ , up to a Weyl rescaling, by the  $h$  equation of motion. The AdS/CFT dictionary implies that

$$R^2 = \alpha' \sqrt{\lambda} \quad \text{and} \quad g_{YM}^2 = 4\pi g_s, \quad (6)$$

where  $\lambda = g_{YM}^2 N$  is the 't Hooft parameter of the dual CFT, which is  $\mathcal{N} = 4$  super Yang–Mills theory with gauge group  $U(N)$ .

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<sup>3</sup>Strictly speaking, this describes the Poincaré patch of  $AdS_5$ .

## 2 Supermatrices and supergeometry

In order to add fermionic degrees of freedom, it is convenient to introduce Grassmann coordinates. Towards this end, let us first discuss supermatrices, which we write in the form<sup>4</sup>

$$S = \begin{pmatrix} a & \zeta b \\ \zeta c & d \end{pmatrix}, \quad \zeta = e^{-i\pi/4} \quad (7)$$

$a$  and  $d$  are  $4 \times 4$  matrices of Grassmann-even numbers referring to  $SU(4)$  and  $SU(2, 2)$ . On the other hand,  $b$  and  $c$  are  $4 \times 4$  matrices of Grassmann-odd numbers that transform as bifundamentals,  $(\mathbf{4}, \bar{\mathbf{4}})$  and  $(\bar{\mathbf{4}}, \mathbf{4})$ , of the two groups.

The “superadjoint” is defined by

$$S^\dagger = \begin{pmatrix} a^\dagger & -\zeta c^\dagger \\ -\zeta b^\dagger & d^\dagger \end{pmatrix}. \quad (8)$$

This definition ensures that  $(S_1 S_2)^\dagger = S_2^\dagger S_1^\dagger$  and  $(S^\dagger)^\dagger = S$ , as one can easily verify. By definition, a unitary supermatrix satisfies  $S S^\dagger = I$ , where  $I$  is the unit matrix, and an antihermitian supermatrix satisfies  $S + S^\dagger = 0$ .<sup>5</sup> In this way one defines the super Lie group  $SU(2, 2|4)$  and the super Lie algebra  $\mathfrak{su}(2, 2|4)$ . The “supertrace” is defined (as usual) by

$$\text{str} S = \text{tr} a - \text{tr} d. \quad (9)$$

The main virtue of this definition is that

$$\text{str}(S_1 S_2) = \text{str}(S_2 S_1). \quad (10)$$

Note also that  $\text{str} I = 0$ , since the  $a$  and  $d$  blocks are both  $4 \times 4$ . The  $\mathfrak{psu}(2, 2|4)$  algebra, which is the one that is required, does not have a supermatrix realization. Rather, it is described by  $\mathfrak{su}(2, 2|4)$  supermatrices modded out by the equivalence relation

$$S \sim S + \lambda I. \quad (11)$$

In addition to the bosonic  $y$  and  $z$  coordinates, described in the introduction, we will utilize  $\theta$  coordinates, which are 16 complex Grassmann numbers that transform under  $SU(4) \times SU(2, 2)$  as  $(\mathbf{4}, \bar{\mathbf{4}})$ . It is natural to describe them by  $4 \times 4$  matrices, rather than by 32-component spinors as is traditionally done for the flat-space theory. This has the advantage that no Fierz transformations are required in the analysis. Infinitesimal symmetry transformations are given by the rule

$$\delta\theta = \omega\theta - \theta\tilde{\omega} + \varepsilon + i\theta\varepsilon^\dagger\theta, \quad (12)$$

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<sup>4</sup>Other authors use slightly different, but equivalent, conventions.

<sup>5</sup>Additional minus signs, needed to take account of the indefinite signature of  $SU(2, 2)$  are suppressed in this discussion.

where  $\omega$  belongs to the  $\mathfrak{su}(2, 2)$ ,  $\tilde{\omega}$  belongs to  $\mathfrak{su}(4)$ , and  $\varepsilon$  is a bifundamental matrix of Grassmann numbers. It is straightforward to verify that iteration of this formula closes on the  $\mathfrak{psu}(2, 2|4)$  algebra. It is reminiscent of Goldstino transformations in theories with spontaneously broken supersymmetry.

Supermatrices  $\Gamma(\theta) \in SU(2, 2|4)$  can be written in the form

$$\Gamma(\theta) = \begin{pmatrix} I & \zeta\theta \\ \zeta\theta^\dagger & I \end{pmatrix} \begin{pmatrix} f^{-1} & 0 \\ 0 & \tilde{f}^{-1} \end{pmatrix} \quad (13)$$

by choosing  $f$  and  $\tilde{f}$  such that  $\Gamma\Gamma^\dagger = I$ . This is achieved for

$$f = \sqrt{I + u} = I + \frac{1}{2}u + \dots \quad (14)$$

$$\tilde{f} = \sqrt{I + \tilde{u}} = I + \frac{1}{2}\tilde{u} + \dots, \quad (15)$$

where

$$u = i\theta\theta^\dagger \quad \text{and} \quad \tilde{u} = i\theta^\dagger\theta. \quad (16)$$

It then follows that

$$\delta_\varepsilon\Gamma = \begin{pmatrix} M(\varepsilon) & 0 \\ 0 & \tilde{M}(\varepsilon) \end{pmatrix} \Gamma + \Gamma \begin{pmatrix} 0 & \zeta\varepsilon \\ \zeta\varepsilon^\dagger & 0 \end{pmatrix}, \quad (17)$$

where

$$M(\varepsilon) = (\delta_\varepsilon f - i f \varepsilon \theta^\dagger) f^{-1}, \quad (18)$$

$$\tilde{M}(\varepsilon) = (\delta_\varepsilon \tilde{f} - i \tilde{f} \varepsilon^\dagger \theta) \tilde{f}^{-1}. \quad (19)$$

Now consider the superconnection

$$\mathcal{A} = \Gamma^{-1} d\Gamma = \begin{pmatrix} K & \zeta\Psi \\ \zeta\Psi^\dagger & \tilde{K} \end{pmatrix}. \quad (20)$$

This one-form supermatrix, constructed entirely out of  $\theta$ , is super-antihermitian and flat ( $d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$ ). This formula defines the even one-forms  $K$  and  $\tilde{K}$ , and the odd one-forms  $\Psi$  and  $\Psi^\dagger$ . Under a supersymmetry transformation

$$\delta_\varepsilon\mathcal{A} = -d \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} - \left[ \mathcal{A}, \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} \right], \quad (21)$$

where  $M$  and  $\tilde{M}$  are the functions defined above.

Let us now recast the bosonic coordinates  $\hat{y}$  and  $\hat{z}$ , defined in the introduction, in matrix form.  $S^5$  can be represented by the antisymmetric  $SU(4)$  matrix:

$$Z = \begin{pmatrix} 0 & u & v & w \\ -u & 0 & -\bar{w} & \bar{v} \\ -v & \bar{w} & 0 & -\bar{u} \\ -w & -\bar{v} & \bar{u} & 0 \end{pmatrix} = \Sigma_a z^a, \quad (22)$$

where  $u = z^1 + iz^2$ ,  $v = z^3 + iz^4$ , and  $w = z^5 + iz^6$ . Requiring  $|u|^2 + |v|^2 + |w|^2 = 1$ , it then follows that

$$Z = -Z^T, \quad ZZ^\dagger = I, \quad \det Z = 1. \quad (23)$$

The main purpose in displaying all the elements of the matrix  $Z$  is to establish the existence of a matrix with these properties. The matrix  $Z$  defines a codimension 10 map of  $S^5$  into  $SU(4)$ . There is a very similar construction for  $Y : AdS_5 \rightarrow SU(2, 2)$ .

The supersymmetry transformations of the bosonic coordinates are

$$\delta_\varepsilon Z = MZ + ZM^T \quad \text{and} \quad \delta_\varepsilon Y = \tilde{M}Y + Y\tilde{M}^T. \quad (24)$$

It then follows that the antihermitian connections

$$\Omega = ZdZ^{-1} - K - ZK^T Z^{-1}, \quad (25)$$

$$\tilde{\Omega} = YdY^{-1} - \tilde{K} - Y\tilde{K}^T Y^{-1} \quad (26)$$

transform nicely under supersymmetry transformations:

$$\delta_\varepsilon \Omega = [M, \Omega] \quad \text{and} \quad \delta_\varepsilon \tilde{\Omega} = [\tilde{M}, \tilde{\Omega}]. \quad (27)$$

The  $PSU(2, 2|4)$  invariant metric with the correct bosonic truncation is

$$G_{\alpha\beta} = -\frac{1}{4} \left( \text{tr}(\Omega_\alpha \Omega_\beta) - \text{tr}(\tilde{\Omega}_\alpha \tilde{\Omega}_\beta) \right). \quad (28)$$

The next step is to split the Grassmann-odd matrix  $\Psi$  in the superconnection  $\mathcal{A}$ , transforming as  $(\mathbf{4}, \bar{\mathbf{4}})$  under  $SU(4) \times SU(2, 2)$ , into two pieces that correspond to Majorana–Weyl (MW) spinors while respecting the group theory. To do this, we define an involution

$$\Psi \rightarrow \Psi' = Z\Psi^*Y^{-1}. \quad (29)$$

$\Psi'$  also transforms as  $(\mathbf{4}, \bar{\mathbf{4}})$ . Then we can write

$$\Psi = \Psi_1 + i\Psi_2 \quad \text{and} \quad \Psi' = \Psi_1 - i\Psi_2, \quad (30)$$

where  $\Psi_1$  and  $\Psi_2$  are “MW matrices” for which  $\Psi'_I = \Psi_I$ ,  $I = 1, 2$ .

Let us now define three antihermitian supermatrix one-forms out of quantities introduced above:

$$A_1 = \begin{pmatrix} \Omega & 0 \\ 0 & \tilde{\Omega} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \zeta\Psi \\ \zeta\Psi^\dagger & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & \zeta\Psi' \\ \zeta\Psi'^\dagger & 0 \end{pmatrix}. \quad (31)$$

In all three cases, infinitesimal supersymmetry transformations take the form

$$\delta_\varepsilon A_i = \left[ \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix}, A_i \right] \quad i = 1, 2, 3. \quad (32)$$

Next, we define

$$J_i = \Gamma^{-1} A_i \Gamma \quad i = 1, 2, 3, \quad (33)$$

to obtain supermatrices that transform under  $\mathfrak{psu}(2, 2|4)$  in the natural way, namely

$$\delta_\Lambda J_i = [\Lambda, J_i], \quad (34)$$

where the infinitesimal parameters are incorporated in the supermatrix

$$\Lambda = \begin{pmatrix} \omega & \zeta \varepsilon \\ \zeta \varepsilon^\dagger & \tilde{\omega} \end{pmatrix}. \quad (35)$$

We will formulate the superstring action and its equations of motion in terms of the three one-forms  $J_i$ . Using the explicit formulas given above, one obtains the Maurer–Cartan equations<sup>6</sup>

$$dJ_1 = -J_1 \wedge J_1 + J_2 \wedge J_2 + J_3 \wedge J_3 - J_1 \wedge J_2 - J_2 \wedge J_1, \quad (36)$$

$$dJ_2 = -2J_2 \wedge J_2, \quad (37)$$

$$dJ_3 = -(J_1 + J_2) \wedge J_3 - J_3 \wedge (J_1 + J_2). \quad (38)$$

### 3 The superstring world-sheet theory

The superstring action contains two pieces: the metric term discussed previously and a Wess–Zumino (WZ) term. The WZ term for the fundamental string can be expressed in terms of a  $\mathfrak{psu}(2, 2|4)$  invariant two-form. Candidate invariant two-forms are

$$\text{str}(J_i \wedge J_j) = -\text{str}(J_j \wedge J_i). \quad (39)$$

However, the only one of these that is nonzero is  $\text{str}(J_2 \wedge J_3)$ . Therefore the WZ term of the fundamental string is proportional to  $\int \text{str}(J_2 \wedge J_3)$ . This term does not contribute to the bosonic truncation.

The supersymmetric extension of the induced world-volume metric in the introduction can now be recast in the form

$$G_{\alpha\beta} = -\frac{1}{4} \text{str}(J_{1\alpha} J_{1\beta}). \quad (40)$$

The complete superstring world-sheet action in the  $AdS_5 \times S^5$  background is then

$$S = \frac{\sqrt{\lambda}}{16\pi} \int \text{str}(J_1 \wedge \star J_1) - \frac{\sqrt{\lambda}}{8\pi} \int \text{str}(J_2 \wedge J_3). \quad (41)$$

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<sup>6</sup>The relation to the notation of [11] is  $J_1 = 2p$ ,  $J_2 = q$ ,  $J_3 = q'$ .

This action is manifestly invariant under  $\delta_\Lambda J_i = [\Lambda, J_i]$ , and thus it has  $PSU(2, 2|4)$  symmetry. It allows one to deduce that the  $PSU(2, 2|4)$  Noether current is

$$J_N = J_1 + \star J_3. \quad (42)$$

The conservation of this current encodes all of the superstring equations of motion, which take the remarkably concise form

$$d \star J_N = d \star J_1 + dJ_3 = 0. \quad (43)$$

Even so, they are fairly complicated when written out in term of the matrices  $Z$ ,  $Y$ , and  $\theta$ .

This theory resembles a sigma model for a symmetric space. However, it has two features that are specific to string theory. The first is the fact that the Hodge duals ( $\star J$ ) in the preceding paragraph are defined using the auxiliary metric  $h_{\alpha\beta}$ . As a consequence, the theory has reparametrization invariance and Weyl symmetry, which are local symmetries. The second feature is a local fermionic symmetry, called kappa symmetry. It determines the ratio of the coefficients of the two terms in the action (up to a sign). There are well-known ways of analyzing the consequences of these features for the flat space ( $R \rightarrow \infty$ ) limit of this theory, which generalize to the present setting. Reparametrization invariance of the world-sheet theory allows one to choose a gauge in which  $h_{\alpha\beta}$  is conformally flat, *i.e.*,  $h_{\alpha\beta}(\sigma) = \Lambda(\sigma)\eta_{\alpha\beta}$ , where  $\eta$  is the 2d Minkowski metric. In critical string theories the decoupling of the conformal factor  $\Lambda$ , *i.e.*, the Weyl symmetry, remains valid at the quantum level. This implies that the trace of the stress tensor  $T_{\alpha\beta}$  vanishes. In world-sheet light-cone coordinates ( $\sigma^\pm = \sigma^0 \pm \sigma^1$ ), the remaining  $h_{\alpha\beta}$  equations of motion are the stress tensor constraints  $T_{++} = T_{--} = 0$ , which amount to  $\text{str}(J_{1+}J_{1+}) = \text{str}(J_{1-}J_{1-}) = 0$ . In the quantum treatment, these operators are required to have vanishing matrix elements between physical states. Also, local kappa symmetry allows one to choose a gauge in which half of the  $\theta$  coordinates are set to zero. The flat-space limit should give the type IIB superstring world-sheet theory in 10d Minkowski spacetime, which is a free theory in an appropriate gauge [15].

The  $AdS_5 \times S^5$  superstring theory is not a free theory, but it is integrable. The key to the proof of this fact is to establish that there is a one-parameter family of flat connections  $J(t)$ ,

$$dJ(t) + J(t) \wedge J(t) = 0. \quad (44)$$

Given a flat connection  $J(t)$ , there is a standard procedure for constructing an infinite family of conserved charges and arguing that this implies integrability [16][17].

Before constructing  $J(t)$  for the superstring action, let us first consider its bosonic truncation. Classically, integrability of the supersymmetric theory requires integrability of the bosonic truncation. At the quantum level, the supersymmetric theory is expected to be better behaved than its bosonic truncation, because it is in its critical dimension.<sup>7</sup> In the bosonic truncation, we have  $J_2 = J_3 = 0$  and  $J_1$  is flat:  $dJ_1 = -J_1 \wedge J_1$ . Moreover, the Noether equations (and the equations of motion) are simply  $d \star J_1 = 0$ . Let us now consider linear combinations of the form

$$J(t) = c_1(t)J_1 + d_1(t) \star J_1. \quad (45)$$

This is a flat connection provided that  $c_1 = (c_1)^2 - (d_1)^2$ , which is solved for the one-parameter family of choices

$$c_1(t) = -\sinh^2 t, \quad d_1(t) = \sinh t \cosh t. \quad (46)$$

Let us now generalize the preceding to include the fermionic degrees of freedom. The Maurer–Cartan equations, together with  $d \star J_1 + dJ_3 = 0$ , imply that

$$J(t) = c_1(t)J_1 + d_1(t) \star J_1 + c_2(t)J_2 + c_3(t)J_3 \quad (47)$$

is flat for a one-parameter family given by<sup>8</sup>

$$c_2(t) = 1 - \cosh t, \quad c_3(t) = \sinh t \quad (48)$$

together with the previous choices of  $c_1(t)$  and  $d_1(t)$  [11]. To verify this result it is useful to know that the equations of motion imply that [10]

$$\star J_1 \wedge J_2 + J_2 \wedge \star J_1 = -(J_1 \wedge J_3 + J_3 \wedge J_1), \quad (49)$$

$$\star J_1 \wedge J_3 + J_3 \wedge \star J_1 = -(J_1 \wedge J_2 + J_2 \wedge J_1). \quad (50)$$

Each of the two terms in the superstring action has manifest  $PSU(2, 2|4)$  global symmetry and reparametrization invariance. As mentioned earlier, the requirement of kappa symmetry determines the ratio of the coefficients of the two terms up to a sign. Fortunately, this ratio is the same as the one required for integrability.

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<sup>7</sup>The quantum theory of this 2d string action describes the classical behavior of the spacetime string in 10d. In other words, it gives the full  $\sqrt{\lambda} = R^2/\alpha'$  dependence at leading order in  $g_s$ .

<sup>8</sup>The alternative choices  $c'_1 = c_1$ ,  $d'_1 = -d_1$ ,  $c'_2 = 2 - c_2$ ,  $c'_3 = c_3$  in [10] and [11] are obtained from these by  $t \rightarrow i\pi - t$ .

## 4 Conclusion

The  $AdS_5 \times S^5$  superstring action described here exhibits the complete  $\theta$  dependence of all quantities, and it has manifest  $PSU(2, 2|4)$  symmetry. In contrast to its flat-space limit, this is an interacting world-sheet theory, so it is much more challenging to give a complete quantum description of its spectrum and other properties. Nevertheless, it has been studied in great detail (using earlier formulations) and compared, with remarkable success, to the dual CFT by taking advantage of the fact that it is integrable [14].

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