BOST–CONNES SYSTEMS AND $\mathbb{F}_1$-STRUCTURES IN GROTHENDIECK RINGS, SPECTRA, AND NORI MOTIVES

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ABSTRACT. We construct geometric lifts of the Bost–Connes algebra to Grothendieck rings and to the associated assembler categories and spectra, as well as to certain categories of Nori motives. These categorifications are related to the integral Bost–Connes algebra via suitable Euler characteristic type maps and zeta functions, and in the motivic case via fiber functors. We also discuss aspects of $\mathbb{F}_1$-geometry, in the framework of torifications, that fit into this general setting.

1. INTRODUCTION AND SUMMARY

This survey/research paper interweaves many different strands that recently became visible in the fabric of algebraic geometry, arithmetics, (higher) category theory, quantum statistics, homotopical "brave new algebra etc., see especially A. Connes and C. Consani [24] [25]; A. Huber, St. MüllerStach [40], etc.

In this sense, our present paper can be considered as a continuation and further extension of [51].

The main difference between the present paper and [51] consists in a change of the categorical environment: the unifying vision we already considered in [51] was provided by I. Zakharevichs notions of assemblers and scissors congruences: cf. [67], [68], [69], and [20]. In this paper, we continue to use the formalism of assemblers and the associated spectra, but we complement it with categories of Nori motives, [40].

As in [51], we focus primarily on various geometrizations of the Bost–Connes algebra(s). Some of these constructions take place in Grothendieck rings, like the previous cases considered in [51], and are aimed at lifting the Bost–Connes endomorphisms to the level of homotopy theoretic spectra through the use of Zakharevich's formalism of assembler categories. In particular we focus on the case of relative Grothendieck rings, endowed with appropriate equivariant Euler characteristic maps, on the case of varieties that admits torifications, for which we consider zeta functions based on the counting of points over $\mathbb{F}_1$ and over extensions $\mathbb{F}_{1^m}$. We also present a more general construction of Bost–Connes type systems associated to exponentiable motivic measures and the associated zeta functions with values in Witt rings, obtained using a lift of the Bost–Connes algebra to Witt rings via Frobenius and Verschiebung maps.

We also consider lifts of the Bost–Connes algebra to Nori motives. We use a (slightly generalized) version of Nori motives, which may be of independent interest in view of possible versions of equivariant periods. In this categorical setting we

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show that the fiber functor from Nori motives maps to a categorification of the Bost–Connes algebra previously constructed by Tabuada and the third author, compatibly with the functors realizing the Bost–Connes structure.

1.1. **Structure of the paper.** Below we will briefly describe contents of subsequent Sections.

In Section 2, Definition 2.4 formally introduces assemblers. We also briefly discuss how motivic vision of algebraic geometry due to Grothendieck can be built into this environment using the recently developed machinery of Nori motives: cf. [3], [40]. In the remaining pages of Section 2, we reproduce the BostConnes formalism, which is lifted from its initial context of type III von Neumann algebras and quantum statistics where it appeared in [15] and moved into our categorical setting, as was already done in [51], [54]. A more detailed view of BostConnes systems in the Nori context is given in Section 8.

Sections 3 and 4 continue this study introducing a relativisation of the underlying geometric category, by consider Bost–Connes structures on relative Grothendieck rings, and on the associated assemblers and spectra.

Sections 5 and 6 essentially are dedicated to the behaviour of varieties with a torus action, torifications etc., in the setting of assemblers and scissors congruences. This behaviour is also directly related to various versions of $\mathbb{F}_1$-geometries: see [42] and [6], [23], [26], [48], [50].

Sections 6 and 7 discuss various versions of zeta functions, and in particular consider the context of spectra in brave new algebra. While Section 6 focuses on zeta functions associated to torified varieties and counting of "points over $\mathbb{F}_1$ and extensions", Section 7 considers the more general case of zeta functions associated to exponentiable motivic measures [43], [59], [60].

Finally, as was already mentioned, in Section 8 we return to the Norifications of BostConnes systems that we briefly discussed in Section 2, which we construct as a lift to a Nori category of motives of the categorification of the BostConnes algebra as a Tannakian category introduced in [54].

2. Grothendieck rings, Assemblers, Nori motives

In this section, we briefly explain that the approach to Grothendieck rings via assemblers can be extended in an interesting way to the domain of motives, namely, Nori motives.

Roughly speaking, the theory of Nori motives starts with lifting the relations

$$[f : X \to S] = [f|_Y : Y \to S] + [f|_{X \smallsetminus Y} : X \smallsetminus Y \to S] \tag{2.1}$$

of (relative) Grothendieck rings $K_0(\mathcal{V}_S)$ to the level of "diagrams", which intuitively can be imagined as "categories without multiplication of morphisms."

More precisely, (cf. Definition 7.1.1 of [40], p. 137), *a diagram* (also called *a quiver*) D is a family consisting of a set of vertices $V(D)$ and a set of oriented edges, $E(D)$. Each edge e either connects two different vertices, going, say, from a vertex $\partial_{\rm out} e = v_1$

to a vertex $\partial_{\rm in} e = v_2$, or else is "an identity", starting and ending with one and the same vertex v. We will consider only diagrams with one identity for each vertex. Diagrams can be considered as objects of a category, with obvious morphisms.

Each small category $\mathcal{C}$ can be considered as a diagram $D(\mathcal{C})$, with $V(D(\mathcal{C})) = \mathrm{Ob}\,\mathcal{C}$, $E(D(\mathcal{C})) = \mathrm{Mor}\,\mathcal{C}$, so that each morphism $X \to Y$ "is" an oriented edge from X to Y. More generally, *a representation* T of a diagram D in a (small) category $\mathcal{C}$ is a morphism of directed graphs $T:\ D \to D(\mathcal{C})$.

We start this section by a brief survey of Zakharevich's formalism of *assemblers* which axiomatizes the "scissors congruence" relations (2.1).

In the next two subsections we pass from an assembler to its geometric diagram and then to its "universal cohomological representation". This is a contemporary embodiment of the primordial Grothendieck's dream that motives constitute a universal cohomology theory of algebraic varieties.

Finally, in the last two subsections we lift this formalism by enriching main relevant objects with an action of a finite cyclic group, with appropriate compatibility conditions. It is this enrichment that provides a framework for the respective lifts of the Bost–Connes algebras, as in the cases discussed in [51] and in the ones we will be discussing in the following sections.

Notice that a considerably more general treatment of graphs with markings, including diagrams etc. in the operadic environment, can be found in [49]. We do not use it here, although it might be highly relevant.

2.1. **Assemblers.** Below we will recall the basics of a general formalism for scissors congruence relations applicable in algebraic geometric contexts defined by I. Zakharevich in [67] and [68]. The abstract form of scissors congruences consists of categorical data called *assemblers*, which in turn determine a homotopy–theoretic *spectrum*, whose homotopy groups embody scissors congruence relations. This formalism is applied in [69] in the framework producing an assembler and a spectrum whose π_0 recovers the Grothendieck ring of varieties. This is used to obtain a characterisation of the kernel of multiplication by the Lefschetz motive, which provides a general explanation for the observations of [14], [55] on the fact that the Lefschetz motive is a zero divisor in the Grothendieck ring of varieties.

Consider a (small) category $\mathcal{C}$ and an object X in $\mathcal{C}$.

Definition 2.1. *A sieve $\mathcal{S}$ over X in $\mathcal{C}$ is a family of morphisms $f_i : X'_i \to X$ (also called "objects over X") satisfying the following conditions:*

a) *Any isomorphism with target X belongs to $\mathcal{S}$ (as a family with one element).*
b) *If a morphism $X' \to X$ belongs to $\mathcal{S}$, then its precomposition with any other morphism in $\mathcal{C}$ with target X'*

$$X'' \to X' \to X$$

also belongs to $\mathcal{S}$.

It follows that composition of any two morphisms in $\mathcal{S}$ composable in $\mathcal{C}$ itself belongs to $\mathcal{S}$ so that any sieve is a category in its own right.

Definition 2.2. *A Grothendieck topology on a category $\mathcal{C}$ consists of the assignment of a collection of sieves $\mathcal{J}(X)$ given for all objects X in $\mathcal{C}$, with the following properties:*

a) *the total overcategory $\mathcal{C}/X$ of morphisms with target X is a member of the collection $\mathcal{J}(X)$.*
b) *The pullback of any sieve in $\mathcal{J}(X)$ under a morphism $f : Y \to X$ exists and is a sieve in $\mathcal{J}(Y)$. Here pullback of a sieve is defined as the family of pullbacks of its objects, $X' \to X$, whereas pullback of such an object w.r.t. $Y \to X$ is defined as $pr_Y : Y \times_X X' \to Y$.*
c) *given $\mathcal{C}' \in \mathcal{J}(X)$ and a sieve $\mathcal{T}$ in $\mathcal{C}/X$, if for every $f : Y \to X$ in $\mathcal{C}'$ the pullback $f^*\mathcal{T}$ is in $\mathcal{J}(Y)$ then $\mathcal{T}$ is in $\mathcal{J}(X)$.*

For more details, see [44], Chapters 16 and 17, or [40], pp. 20–22.

Let $\mathcal{C}$ be a category with a Grothendieck topology. Zakharevich's notion of an assembler category is then defined as follows.

Definition 2.3. *A collection of morphisms $\{f_i : X_i \to X\}_{i\in I}$ in $\mathcal{C}$ is a covering family if the full subcategory of $\mathcal{C}/X$ that contains all the morphisms of $\mathcal{C}$ that factor through the f_i,*

$$\{g : Y \to X \,|\, \exists i \in I \;\; h : Y \to X_i \;\; \textit{such that } f_i \circ h = g\},$$

belongs to the sieve collection $\mathcal{J}(X)$.

In a category $\mathcal{C}$ with an initial object $\emptyset$ two morphisms $f : Y \to X$ and $g : W \to X$ are called *disjoint* if the pullback $Y \times_X W$ exists and is equal to $\emptyset$. A collection $\{f_i : X_i \to X\}_{i\in I}$ in $\mathcal{C}$ is disjoint if f_i and f_j are disjoint for all $i \neq j \in I$.

Definition 2.4. *An assembler category $\mathcal{C}$ is a small category endowed with a Grothendieck topology, which has an initial object $\emptyset$ (with the empty family as covering family), and where all morphisms are monomorphisms, with the property that any two finite disjoint covering families of X in $\mathcal{C}$ have a common refinement that is also a finite disjoint covering family.*

A morphism of assemblers is a functor $F : \mathcal{C} \to \mathcal{C}'$ that is continuous for the Grothendieck topologies and preserves the initial object and the disjointness property, that is, if two morphisms are disjoint in $\mathcal{C}$ their images are disjoint in $\mathcal{C}'$.

For X a finite set, the coproduct of assemblers $\bigvee_{x\in X} \mathcal{C}_x$ is a category whose objects are the initial object $\emptyset$ and all the non–initial objects of the assemblers $\mathcal{C}_x$. Morphisms of non–initial objects are induced by those of $\mathcal{C}_x$.

Consider a pair $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ is an assembler category, and $\mathcal{D}$ is a sieve in $\mathcal{C}$.

One has then an associated assembler $\mathcal{C} \smallsetminus \mathcal{D}$ defined as the full subcategory of $\mathcal{C}$ containing all the objects that are not initial objects of $\mathcal{D}$. The assembler structure on $\mathcal{C} \smallsetminus \mathcal{D}$ is determined by taking as covering families in $\mathcal{C} \smallsetminus \mathcal{D}$ those collections $\{f_i : X_i \to X\}_{i\in I}$ with X_i, X objects in $\mathcal{C} \smallsetminus \mathcal{D}$ that can be completed to a covering family in $\mathcal{C}$, namely such that there exists $\{f_j : X_j \to X\}_{j\in J}$ with X_j in $\mathcal{D}$ such that $\{f_i : X_i \to X\}_{i\in I} \cup \{f_j : X_j \to X\}_{j\in J}$ is a covering family in $\mathcal{C}$.

Moreover, there is a morphism of assemblers $\mathcal{C} \to \mathcal{C} \smallsetminus \mathcal{D}$ that maps objects of $\mathcal{D}$ to $\emptyset$ and objects of $\mathcal{C} \smallsetminus \mathcal{D}$ to themselves and morphisms with source in $\mathcal{C} \smallsetminus \mathcal{D}$ to themselves and morphisms with source in $\mathcal{D}$ to the unique morphism to the same target with source $\emptyset$.

Definition 2.5. *The data $(\mathcal{C}, \mathcal{D}, \mathcal{C} \smallsetminus \mathcal{D})$ are called the abstract scissors congruences.*

The construction of Γ-spaces and homotopy theoretic spectra associated to assembler categories as in [67], which we review later in the paper, provides the formalism we use here and in the previous paper [51] to lift Bost–Connes type algebras to the level of Grothendieck rings and spectra. We describe in the next subsection a general formalism of "enriched assemblers" underlying all the explicit cases of Bost–Connes structures in Grothendieck rings discussed in [51] and in some of the later sections of this paper.

2.2. **Enriched assemblers and Bost–Connes systems.**

Definition 2.6. *Let C be a category. We will call here the enrichment of C the pair consisting of the category $\hat{C}$ and forgetful functor $\hat{C} \to C$ defined as follows:*

- (i) *One object of $\hat{C}$ is any pair (X, v_X) where $X \in \mathrm{Ob}C$ and $v_X : X \to X$ is an automorphism of finite order of X.*
- (ii) *One morphism $\hat{f} : (X, v_X) \to (Y, v_Y)$ in $\hat{C}$ is any morphism $f : X \to Y$ such that $f \circ v_X = v_Y \circ f : X \to Y$ in C.*
- (iii) *The forgetful functor sends $\hat{X}$ to X and $\hat{f}$ to f.*

More generally, we will consider enrichments $\hat{C}$ that only use objects of a particular subcategory of C rather than the full C. This will be stated clearly in the specific cases we discuss later.

We use here a standard categorical notation according to which, say, $f \circ v_X$ is the precomposition of f with v_X.

Now, assume that C is endowed with a structure of assembler. Then a series of constructions presented in §§ 3 and 4 of [51] and in §§ 3–7 of this paper, and restricted there to various categories of schemes, show in fact how this structure of assembler can be lifted from C to $\hat{C}$.

In particular the Bost–Connes type structures we are investigating can be formulated broadly in this setting of enriched assemblers as follows.

2.2.1. *Bost–Connes algebras.* The Bost–Connes algebra was introduced in [15] as a quantum statistical mechanical system that exhibit the Riemann zeta function as partition function, the generators of the cyclotomic extensions of $\mathbb{Q}$ as values of zero-temperature KMS equilibrium states on arithmetic elements in the algebra, and the abelianized Galois group $\hat{\mathbb{Z}}^* \simeq \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{ab}$ as group of quantum symmetries. In particular, the arithmetic subalgebra of the Bost–Connes system is given by the semigroup crossed product

$$\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N} \tag{2.2}$$

of the multiplicative semigroup $\mathbb{N}$ of positive integers acting on the group algebra of the group $\mathbb{Q}/\mathbb{Z}$.

The additive group $\mathbb{Q}/\mathbb{Z}$ can be identified with the multiplicative group ν^* of *roots of unity embedded into* $\mathbb{C}^*$: namely, $r \in \mathbb{Q}/\mathbb{Z}$ corresponds to $e(r) := \exp(2\pi i\, r)$. More generally, the choice of the embedding can be modified by an arbitrary choice of an element in $\hat{\mathbb{Z}}^* = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, as is usually done in representations of the Bost–Connes algebra, see [15]. Thus, we will use here interchangeably the notation ζ or r for elements of $\mathbb{Q}/\mathbb{Z}$ assuming a choice of embedding as above. The group algebra $\mathbb{Q}[\nu^*]$ consists of formal finite linear combinations $\sum_{a_\zeta \in \mathbb{Q}} a_\zeta \zeta$ of roots of unity $\zeta \in \nu^*$. Formality means here that the sum is *not* related to the additive structure of $\mathbb{C}$.

The action of the semigroup $\mathbb{N}$ on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ that defines the crossed product (2.2) is given by the endomorphisms

$$\rho_n(\sum a_\zeta \zeta) := \sum a_\zeta \, \frac{1}{n} \sum_{\zeta'^n = \zeta} \zeta'. \tag{2.3}$$

Equivalently, the algebra (2.2) is generated by elements $e(r)$ with the relations $e(0) = 1$, $e(r + r') = e(r)e(r')$, and elements μ_n and μ_n^* satisfying the relations

$$\begin{array}{lll} \mu_n^* \mu_n = 1, \, \forall n; & \mu_n \mu_n^* = \pi_n, \, \forall n & \text{with } \pi_n = \frac{1}{n} \sum_{nr=0} e(r); \\ \mu_{nm} = \mu_n \mu_m, \, \forall n, m; & \mu_{nm}^* = \mu_n^* \mu_m^*, \, \forall n, m; & \mu_n^* \mu_m = \mu_m \mu_n^* \text{ if } (n, m) = 1. \end{array} \tag{2.4}$$

The semigroup action (2.3) is then equivalently written as $\rho_n(a) = \mu_n \, a \, \mu_n^*$, for all $a = \sum a_\zeta \zeta$ in $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. The element $\pi_n \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ is an idempotent, hence the generators μ_n are isometries but not unitaries. See [15] and §3 of [27] for a detailed discussion of the Bost–Connes system and the role of the arithmetic subalgebra (2.2).

In [26] an integral model of the Bost–Connes algebra was constructed in order to develop a model of $\mathbb{F}_1$-geometry in which the Bost–Connes system encodes the extensions $\mathbb{F}_{1^m}$, in the sense of [42], of the "field with one element" $\mathbb{F}_1$.

The integral Bost–Connes algebra is obtained by considering the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, which we can again implicitly identify with $\mathbb{Z}[\nu^*]$ for a choice of embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}$ as roots of unity.

Define its *ring endomorphisms* σ_n:

$$\sigma_n(\sum a_\zeta \zeta) := \sum a_\zeta \zeta^n. \tag{2.5}$$

Define *additive maps* $\tilde{\rho}_n$: $\mathbb{Z}[\nu^*] \to \mathbb{Z}[\nu^*]$:

$$\tilde{\rho}_n\big(\sum a_\zeta \zeta\big) := \sum a_\zeta \sum_{\zeta'^{\,n}=\zeta} \zeta'. \tag{2.6}$$

When there is no danger of misunderstanding, we may write the r.h.s. of (2.6) in the shorthand notation

$$\tilde{\rho}_n\big(\sum a_\zeta \zeta\big) := \sum a_\zeta \zeta^{1/n}. \tag{2.7}$$

The maps σ_n and $\tilde{\rho}_n$ satisfy the relations

$$\sigma_n \circ \tilde{\rho}_n = n\,\mathrm{id}, \quad \tilde{\rho}_n \circ \sigma_n = n\,\pi_n. \tag{2.8}$$

The integral Bost–Connes algebra is then defined as the algebra generated by the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and generators $\tilde{\mu}_n$ and μ_n^* with the relations

$$\begin{aligned} &\tilde{\mu}_n\, a\, \mu_n^* = \tilde{\rho}_n(a),\ \forall n; \quad \mu_n^*\, a = \sigma_n(a)\, \mu_n^*,\ \forall n; \qquad a\, \tilde{\mu}_n = \tilde{\mu}_n\, \sigma_n(a),\ \forall n; \\ &\tilde{\mu}_{nm} = \tilde{\mu}_n \tilde{\mu}_m,\ \forall n,m; \quad \mu_{nm}^* = \mu_n^* \mu_m^*,\ \forall n,m; \quad \tilde{\mu}_n \mu_m^* = \mu_m^* \tilde{\mu}_n \text{ if } (n,m)=1. \end{aligned} \tag{2.9}$$

where the relations in the first line hold for all $a = \sum a_\zeta \zeta \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, with σ_n and $\tilde{\rho}_n$ as in (2.5) and (2.6).

The maps $\tilde{\rho}_n$ of the integral Bost–Connes algebra and the semigroup action ρ_n in the rational Bost–Connes algebra (2.2) are related by

$$\rho_n = \frac{1}{n} \tilde{\rho}_n$$

with $\tilde{\rho}_n$ defined as in (2.6) and (2.7).

Briefly returning to enriched assemblers, one can say that relevant lifts of Bost–Connes algebras study the the interaction between various endomorphisms v_X. In the following we discuss a related setting, which will use Nori–type correspondences and motives instead of assembler categories and Grothendieck rings and will lead to similar lifts of the Bost–Connes algebra to the motivic level.

2.2.2. *Bost–Connes systems on categories.* Let $\hat{C}$ be an enrichment of a category C, in the sense of Definition 2.6. We assume here that C is an additive (symmetric) monoidal category and that the enrichment $\hat{C}$ is compatible with this structure.

Definition 2.7. *A Bost–Connes system in an enrichment $\hat{C}$ of an additive (symmetric) monoidal category C consists of two families of endofunctors $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\rho}_n\}_{n\in\mathbb{N}}$ of $\hat{C}$ with the following properties:*

1. *The functors σ_n are compatible with both the additive and the (symmetric) monoidal structure, while the functors $\tilde{\rho}_n$ are functors of additive categories.*
2. *For all $n, m \in \mathbb{N}$ these endofunctors satisfy*

$$\sigma_{nm} = \sigma_n \circ \sigma_m, \qquad \tilde{\rho}_{nm} = \tilde{\rho}_n \circ \tilde{\rho}_m.$$

(3) *The compositions satisfy*

$$\sigma_n \circ \tilde{\rho}_n(X, v_X) = (X, v_X)^{\oplus n} \quad \text{and} \quad \tilde{\rho}_n \circ \sigma_n(X, v_X) = (X, v_X) \otimes (Z_n, v_n), \tag{2.10}$$

for some object (Z_n, v_n) in $\hat{C}$, and similarly on morphisms.

This definition covers the main examples considered in §§ 3 and 4 of [51] obtained using the assembler categories associated to the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ of varieties with a good $\hat{\mathbb{Z}}$-action factoring through some finite cyclic quoetient and to the equivariant version $\mathrm{Burn}^{\hat{\mathbb{Z}}}$ of the Kontsevich–Tschinkel Burnside ring. This same definition also accounts for the construction we will discuss in §5 of this paper, based on assembler categories associated to torified varieties. In all of these cases, the endofunctors σ_n and $\tilde{\rho}_n$ of Definition 2.7 have the form

$$\sigma_n(X, v_X) = (X, v_X \circ \sigma_n) \quad \text{and} \quad \tilde{\rho}_n(X, v_X) = (X \times Z_n, \Phi_n(v_X)),$$

where the endomorphism v_X is the action of a generator of some finite cyclic group $\mathbb{Z}/N\mathbb{Z}$ quotient of $\hat{\mathbb{Z}}$ and the action satisfies $v_X \circ \sigma_n(\zeta, x) = v_X(\sigma_n(\zeta), x)$, where $\sigma_n(\zeta) = \zeta^n$ is the Bost–Connes map of (2.5), while the action $\Phi_n(v_X)$ on $X \times Z_n$ is a geometric form of the Verschiebung, as will be discussed more explicitly in §3.3. The object (Z_n, v_n) in Definition 2.7 plays the role of the element $n\pi_n$ in the integral Bost–Connes algebra and the relations (2.10) play the role of the relations (2.8).

A more general version of Definition 2.7 is needed in cases like the construction presented in §3 below, on assemblers associated to relative equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. These require modifying Definition 2.7 as follows.

Definition 2.8. *Let $\hat{\mathcal{I}}$ be an enrichment of an additive (symmetric) monoidal category $\mathcal{I}$ as above, endowed with a Bost–Connes system given by endofunctors $\{\sigma_n^{\mathcal{I}}\}$ and $\{\tilde{\rho}_n^{\mathcal{I}}\}$ of $\hat{\mathcal{I}}$ as in Definition 2.7, with α_n the object in $\hat{\mathcal{I}}$ with $\tilde{\rho}_n \circ \sigma_n(\alpha) = \alpha \otimes \alpha_n$. Let $\{\hat{C}_\alpha\}_{\alpha \in \hat{\mathcal{I}}}$ be a collection of enrichments of additive (symmetric) monoidal categories C_α, indexed by the objects of the auxiliary category $\hat{\mathcal{I}}$, endowed with functors $f_n : \hat{C}_{\alpha^{\oplus n}} \to \hat{C}_\alpha$ and $h_n : \hat{C}_{\alpha \times \beta} \to \hat{C}_\alpha$. Let $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{\tilde{\rho}_n\}_{n \in \mathbb{N}}$ be two collections of functors*

$$\sigma_n : \hat{C}_\alpha \to \hat{C}_{\sigma_n^{\mathcal{I}}(\alpha)} \quad \text{and} \quad \tilde{\rho}_n : \hat{C}_\alpha \to \hat{C}_{\tilde{\rho}_n^{\mathcal{I}}(\alpha)}$$

satisfying the properties:

(1) *The functors σ_n are compatible with both the additive and the (symmetric) monoidal structure, while the functors $\tilde{\rho}_n$ are functors of additive categories.*
(2) *For all $n, m \in \mathbb{N}$ these functors satisfy*

$$\sigma_{nm} = \sigma_n \circ \sigma_m, \qquad \tilde{\rho}_{nm} = \tilde{\rho}_n \circ \tilde{\rho}_m.$$

(3) *The compositions*

$$\sigma_n \circ \tilde{\rho}_n : \hat{C}_\alpha \to \hat{C}_{\alpha^{\oplus n}} \quad \text{and} \quad \tilde{\rho}_n \circ \sigma_n : \hat{C}_\alpha \to \hat{C}_{\alpha \otimes \alpha_n}$$

satisfy

$$
\begin{array}{rcl}
f_n \circ \sigma_n \circ \tilde{\rho}_n(X, v_X)_\alpha & = & (X, v_X)_\alpha^{\oplus n} \quad \textit{and} \\
h_n \circ \tilde{\rho}_n \circ \sigma_n(X, v_X)_\alpha & = & (X, v_X)_\alpha \otimes (Z_n, v_n)_\alpha,
\end{array} \tag{2.11}
$$

for some object $(Z_n, v_n)_\alpha$ *in* $\hat{C}_\alpha$.

While we have focused so far on assembler categories, as those were at the basis of our constructions of Bost–Connes systems in [51], we consider here also a different categorical setting that will allow us to identify analogous structures at a motivic level. We follow the formalism of geometric diagrams and Nori motives, which we review in the next subsections.

2.3. **From geometric diagrams to Nori motives.** We recall the main idea in the construction of Nori motives from geometric diagrams. For more details, see [40], pp. 140–144.

Start with the following data:

a) a diagram D;
b) a noetherian commutative ring with unit R and the category of finitely generated R–modules R–Mod;
c) a representation T of D in R–Mod.

Produce from them the category $C(D,T)$ defined in the following way:

d1) If D is finite, then $C(D,T)$ is the category of finitely generated R–modules equipped with an R–linear action of $End(T)$.

d2) If D is infinite, first consider its all finite subdiagrams F.

For each F construct $C(F, T|_F)$ as in d1). Then apply the following limiting procedure.

- Objects of $C(D,T)$ will be all objects of the categories $C(F, T|_F)$. If $F \subset F'$, then each object X_F of $C(F, T|_F)$ can be canonically extended to an object of $C(F', T|_{F'})$.
- Morphisms from X to Y in $C(D,T)$ will be defined as colimits over F of morphisms from X_F to Y_F with respect to these extensions.

The result is called *the diagram category* $C(D,T)$.

It is an R–linear abelian category which is endowed with R–linear faithful exact forgetful functor

$$f_T : C(D,T) \to R - Mod.$$

2.3.1. *Universal diagram category.* The following results explain why abstract diagram categories play a central role in the formalism of Nori motives: they formalise the Grothendieck intuition of motives as objects of the universal cohomology theory.

Theorem 2.9. (i) *Any representation $T : D \to R - Mod$ can be presented as precomposition of the forgetful functor f_T with an appropriate representation $\tilde{T} : D \to C(D,T)$:*

$$T = f_T \circ \tilde{T}.$$

with the following universal property:

Given any R–linear abelian category A with a representation $F : D \to A$ and R–linear faithful exact functor $f : A \to R - Mod$ with $T = f \circ F$, it factorizes through a faithful exact functor $L(F) : C(D,T) \to A$ compatibly with the decomposition

$$T = f_T \circ \tilde{T}.$$

(ii) *The functor $L(F)$ is unique up to unique isomorphism of exact additive functors.*

For proofs, cf. [40], pp. 140–141 and p. 167.

2.3.2. *Nori geometric diagrams.* If we start not with an abstract category but with a "geometric" category $\mathcal{C}$ of spaces/varieties/schemes, possibly endowed with additional structures, in which one can define morphisms of closed embeddings $Y \hookrightarrow X$ (or $Y \subset X$) and morphisms of complements to closed embeddings $X \setminus Y \to X$, as in the environment of (2.1) above, we can define the Nori diagram of *effective pairs* $D(\mathcal{C})$ in the following way (see [40], pp. 207–208).

a) One vertex of $D(\mathcal{C})$ is a triple (X, Y, i) where $Y \hookrightarrow X$ is a closed embedding, and i is an integer.
b) Besides obvious identities, there are edges of two types.
b1) Let (X, Y) and (X', Y') be two pairs of closed embeddings. Every morphism $f : X \to X'$ such that $f(Y) \subset Y'$ produces functoriality edges f^* (or rather (f^*, i)) going from (X', Y', i) to (X, Y, i).
b2) Let $(Z \subset Y \subset X)$ be a stair of closed embeddings. Then it defines coboundary edges ∂ from (Y, Z, i) to $(X, Y, i + 1)$.

2.3.3. *(Co)homological representatons of Nori geometric diagrams.* If we start not just from the initial category of spaces $\mathcal{C}$, but rather from a pair $(\mathcal{C}, H)$ where H is a cohomology theory, then assuming reasonable properties of this pair, we can define the respective representation T_H of $D(\mathcal{C})$ that we will call a *(co)homological representation of $D(\mathcal{C})$*.

For a survey of such pairs $(\mathcal{C}, H)$ that were studied in the context of Grothendieck's motives, see [40], pp. 31–133. The relevant cohomology theories include, in particular, singular cohomology, and algebraic and holomorphic de Rham cohomologies.

Below we will consider the basic example of cohomological representations of Nori diagrams that leads to Nori motives.

2.3.4. *Effective Nori motives.* We follow [40], pp. 207–208. Take as a category $\mathcal{C}$, starting object in the definition of Nori geometric diagrams above, the category of varieties X defined over a subfield $k \subset \mathbb{C}$.

We can then define the Nori diagram $D(\mathcal{C})$ as above. This diagram will be denoted Pairs^{eff} from now on.

The category of effective mixed Nori motives is the diagram category $C(\mathrm{Pairs}, H^*)$ where $H^i(X, \mathbb{Z})$ is the respective singular cohomology of the analytic space X^{an} (cf. [40], pp. 31–34 and further on).

Define the diagram of effective pairs Pairs^{eff} exactly as in the general case.

It turns out (see [40], Proposition 9.1.2. p. 208) that the map

$$H^* : \mathrm{Pairs}^{eff} \to \mathbb{Z} - Mod$$

sending (X, Y, i) to the relative singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$, naturally extends to a representation of the respective Nori diagram in the category of finitely generated abelian groups $\mathbb{Z}$–Mod.

2.4. **Bost–Connes systems in categories of Nori motives.** In the constructions described in §§ 3 and 4 of [51] and in §§ 3–7 of the present paper we obtain lifts of the integral Bost–Connes algebra to various assembler categories and associated spectra, in the form described in Definitions 2.7 and 2.8, starting from a ring homomorphism (motivic measure) from the relevant Grothendieck ring to the group ring $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the integral Bost–Connes algebra, that is equivariant with respect to the maps σ_n and $\tilde{\rho}_n$ of (2.5) and (2.6) of the Bost–Connes algebra and the maps (also denoted by σ_n and $\tilde{\rho}_n$) on the Grothendieck ring induced by a Bost–Connes system on the corresponding assembler category. The motivic measure provides in this way a map that lifts the Bost–Connes structure. We provide several examples of such measures.

When we replace the formalism of assembler categories and homotopy theoretic spectra underlying the Grothendieck rings with geometric diagrams and associated Tannakian categories of Nori motives, with the same notion of categorical Bost–Connes systems introduced in Definitions 2.7 and 2.8, we can lift the Euler characteristic type motivic measures to the level of categorifications. Indeed, it is known from earlier results of Tabuada and the third author, [54], that the Bost–Connes algebra admits a categorification in terms of a Tannakian category of $\mathbb{Q}/\mathbb{Z}$-graded vector spaces endowed with Frobenius and Verschiebung type endofunctors.

In §8 in this paper we construct Bost–Connes systems in categories of Nori motives, based on the same notion of enrichements and the structures described in Definitions 2.7 and 2.8, and we show that the fiber functor maps to the categorification of the Bost–Connes algebra constructed in [54] compatibly with the respective Bost–Connes endofunctors.

We discuss in the next subsection some further considerations and questions about assemblers, diagrams, and Nori motives.

2.5. **Nori geometric diagrams for assemblers, and a challenge.** According to the Nori formalism as it is presented in [HuM-S17], we must start with a "geometric" category C of spaces/varieties/schemes, possibly endowed with additional structures, in which one can define morphisms of closed embeddings $Y \hookrightarrow X$ (or $Y \subset X$) and morphisms of complements to closed embeddings $X \smallsetminus Y \to X$, Then the Nori diagram of *effective pairs* $D(C)$ is defined as in [40], pp. 207–208, see §2.3.2 above.

In the current context, *objects* of our category C will be *assemblers* $\mathcal{C}$ (of course, described in terms of a category of lower level). In particular, each such $\mathcal{C}$ is endowed with a Grothendieck topology.

A *vertex* of the Nori diagram $D(C)$ will be a triple $(\mathcal{C}, \mathcal{C} \setminus \mathcal{D}, i)$ where its first two terms are taken from an abstract scissors congruence in C (cf. above), and i is an integer. Intuitively, this means that we we are considering the canonical embedding $\mathcal{C} \setminus \mathcal{D} \hookrightarrow \mathcal{C}$ as an analog of closed embedding. This intuition makes translation of the remaining components of Nori's diagrams obvious, except for one: *what is the geometric meaning of the integer i in $(\mathcal{C}, \mathcal{C} \setminus \mathcal{D}, i)$?*

The answer in the general context of assemblers, seemingly, was not yet suggested, and already in the algebraic–geometric contexts is non–obvious and non–trivial. Briefly, i translates to the level of Nori geometric diagrams the *weight filtration* of various cohomology theories (cf. [40], 10.2.2, pp. 238–241), and the existence of such translation and its structure are encoded in several versions of *Nori's Basic Lemma* independently and earlier discovered by A. Beilinson and K. Vilonen (cf. [40], 2.5, pp. 45–59).

The most transparent and least technical version of the Basic Lemma ([40], Theorem 2.5.2 , p. 46) shows that in algebraic geometry the existence of weight filtration is based upon special properties of *affine schemes.* As we will see in the last section, lifts of Bost–Connes algebras to the level of cohomology based upon the techniques of *enrichment* also require a definition of *affine* assemblers. Since we do not know its combinatorial version, the enrichments that we can study now, force us to return to algebraic geometry.

This challenge suggests to think about other possible geometric contexts in which dimensions/weights of the relevant objects may take, say, p–adic values (as in the theory of p–adic weights of automorphic forms inaugurated by J. P. Serre), or rational values (as it happens in some corners of "geometries below $\mathrm{Spec}\,\mathbb{Z}$"), or even real values (as in various fractal geometries).

Can one transfer the scissors congruences imagery there?

See, for example, the formalism of Farey semi–intervals as base of ∞–adic topology.

3. Bost–Connes algebra in the relative Grothendieck ring

In this section we generalize the lift of the integral Bost–Connes algebra to the Grothendieck ring described in [51] to the case of relative Grothendieck rings.

3.1. **Relative Grothendieck ring.** We describe here a variant of construction of [51], where we work with relative Grothendieck rings and with an Euler characteristic with values in a Grothendieck ring of locally constant sheaves. We show that this relative setting provides ways of lifting to the level of Grothendieck classes certain subalgebras of the integral Bost–Connes algebras associated to the choice of a finite set of non-archimedean places.

The relative Grothendieck ring $K_0(\mathcal{V}_S)$ of varieties over a base variety S over a field $\mathbb{K}$ is generated by the isomorphism classes of data $f : X \to S$ of a variety X over S with the relations

$$[f : X \to S] = [f|_Y : Y \to S] + [f|_{X \smallsetminus Y} : X \smallsetminus Y \to S]$$

as in (2.1) for a closed embedding $Y \hookrightarrow X$ of varieties over S. The product is given by the fibered product $X \times_S Y$. We will write $[X]_S$ as shorthand notation for the class $[f : X \to S]$ in $K_0(\mathcal{V}_S)$.

A morphism $\phi : S \to S'$ induces a base change ring homomorphism $\phi^* : K_0(\mathcal{V}_{S'}) \to K_0(\mathcal{V}_S)$ and a direct image map $\phi_* : K_0(\mathcal{V}_S) \to K_0(\mathcal{V}_{S'})$ which is a group homomorphisms and a morphism of $K_0(\mathcal{V}_{S'})$-modules, but not a ring homomorphism. The class $[\phi : S \to S']$ as an element in $K_0(\mathcal{V}_{S'})$ is the image of $1 \in K_0(\mathcal{V}_S)$ under ϕ_*.

When $S = \mathrm{Spec}(\mathbb{K})$ one recovers the ordinary Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$.

3.2. **Equivariant relative Grothendieck ring.** Let X be a variety with a good action $\alpha : G \times X \to X$ by a finite group G and X' a variety with a good action α' by G'. As morphisms we then consider pairs (ϕ, φ) of a morphism $\phi : X \to X'$ and a group homomorphism $\varphi : G \to G'$ such that $\phi(\alpha(g, x)) = \alpha'(\varphi(g), \phi(x))$, for all $g \in G$ and $x \in X$. Thus, isomorphisms of varieties with good G-actions are pairs of an isomorphism $\phi : X \to X'$ of varieties and a group automorphism $\varphi \in \mathrm{Aut}(G)$ with the compatibility condition as above.

Given a base variety (or scheme) S with a given good action α_S of a finite group G, and varieties X, X' over S, with good G-actions $\alpha_X, \alpha_{X'}$ and G-equivariant maps $f : X \to S$ and $f' : X' \to S$, we consider morphisms given by a triple (ϕ, φ, ϕ_S) of a morphism $\phi : X \to X'$, a group homomorphism $\varphi : G \to G$ with the compatibility as above, and an endomorphism $\phi_S : S \to S$ such that $f' \circ \phi = \phi_S \circ f$. Then these maps also satisfy $\phi_S(\alpha_S(g, f(x)) = \alpha_S(\varphi(g), \phi_S(f(x)))$.

We then consider the abelian group generated by isomorphism classes $[f : X \to S]$ of varieties over S with compatible good G-actions, with respect to isomorphisms (ϕ, φ, ϕ_S) as above, with the inclusion-exclusion relations generated by equivariant embeddings with compatible G-equivariant maps

$$
\begin{tikzcd}
Y \arrow[r, hook] \arrow[dr, "f|_Y"'] & X \arrow[d, "f"] & X \smallsetminus Y \arrow[l, hook'] \arrow[dl, "f|_{X \smallsetminus Y}"] \\
& S &
\end{tikzcd}
\tag{3.1}
$$

and isomorphisms. This means that we have $[f : X \to S] = [f_Y : Y \to S] + [f_{X\smallsetminus Y} : X \smallsetminus Y \to S]$ if there are isomorphisms $(\phi_Y, \varphi_Y, \phi_{S,Y})$ and $(\phi_{X\smallsetminus Y}, \varphi_{X\smallsetminus Y}, \phi_{S,X\smallsetminus Y})$, such that the diagram commutes

$$
\begin{tikzcd}
Y \arrow[r, "\phi_Y"] \arrow[d, "f_Y"] & Y \arrow[r, hook] \arrow[dr, "f|_Y"'] & X \arrow[d, "f"] & X \smallsetminus Y \arrow[l, hook'] \arrow[dl, "f|_{X\smallsetminus Y}"] & X \smallsetminus Y \arrow[l, "\phi'_{X\smallsetminus Y}"'] \arrow[d, "f_{X\smallsetminus Y}"'] \\
S \arrow[rr, "\phi_{S,Y}"'] & & S & & S \arrow[ll, "\phi_{S,X\smallsetminus Y}"]
\end{tikzcd}
\tag{3.2}
$$

The product $[f : X \to S] \cdot [f' : X' \to S]$ given by $[\tilde{f} : X \times_S X' \to S]$ with $\tilde{f} = f \circ \pi_X = f' \circ \pi_{X'}$ is well defined on isomorphism classes, with the diagonal action $\tilde{\alpha}(g, (x, x')) = (\alpha_X(g, x), \alpha_{X'}(g, x'))$ satisfying $f(\alpha_X(g, x)) = \alpha_S(g, f(x)) = \alpha_S(g, f'(x')) = f'(\alpha_{X'}(g, x'))$. We denote the resulting Grothendieck ring by $K_0^G(\mathcal{V}_S)$.

In the case of the equivariant Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ we considered in [51], we can then also consider a relative version $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, with S a variety with a good $\hat{\mathbb{Z}}$-action as above. We consider the Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ given by the isomorphism classes of S-varieties $f : X \to S$ with good $\hat{\mathbb{Z}}$-actions with respect to which f is equivariant, with the notion of isomorphism described above. The product is given by the fibered product over S with the diagonal $\hat{\mathbb{Z}}$-action. The inclusion-exclusion relations are as in (2.1) where $Y \hookrightarrow X$ and $X \smallsetminus Y \hookrightarrow X$ are equivariant embeddings with compatible $\hat{\mathbb{Z}}$-equivariant maps as in (3.2).

3.3. **The geometric Verschiebung action.** We recall here how to construct the geometric Verschiebung action used in [51] to lift the Bost–Connes maps to the level of Grothendieck rings. This has the effect of transforming an action of $\hat{\mathbb{Z}}$ on X that factors through some $\mathbb{Z}/N\mathbb{Z}$ into an action of $\hat{\mathbb{Z}}$ on $X \times Z_n$, with $Z_n = \{1, \ldots, n\}$, that factors through $\mathbb{Z}/Nn\mathbb{Z}$. For $x \in X$, let $\underline{x} = (x, a_i)_{a_i \in Z_n} = (x_i)_{i=1}^n$ be the subset $\{x\} \times Z_n$. For ζ_N a primmitive N-th root of unity, we write in matrix form

$$
V_n(\zeta_{Nn}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha(\zeta_N) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}
$$

so that we can write

$$
V_n(\zeta_{Nn}) \cdot \underline{x} = \begin{cases} (x, a_{i+1}) & i = 1, \ldots, n-1 \\ (\alpha(\zeta_N) \cdot x, a_1) & i = n \end{cases}
\tag{3.3}
$$

which satisfies $V_n(\zeta_{Nn})^n = \alpha(\zeta_N) \times \mathrm{Id}_{Z_n}$. The resulting action $\Phi_n(\alpha)$ of $\hat{\mathbb{Z}}$ on $X \times Z_n$ that factors through $\mathbb{Z}/Nn\mathbb{Z}$ is specified by setting

$$\Phi_n(\alpha)(\zeta_{Nn}) \cdot (x, a) = (V_n(\alpha(\zeta_N)) \cdot \underline{x})_a. \tag{3.4}$$

3.4. **Lifting the Bost–Connes endomorphisms.** Consider a base scheme S with a good action of $\hat{\mathbb{Z}}$ that factors through some fixed $\mathbb{Z}/N\mathbb{Z}$. Let $f : X \to S$ be a variety over S with a good $\hat{\mathbb{Z}}$ action such that the map is $\hat{\mathbb{Z}}$-equivariant. We denote by $\alpha_S : \hat{\mathbb{Z}} \times S \to S$ the action on S and by $\alpha_X : \hat{\mathbb{Z}} \times X \to X$. We write the equivariant relative Grothendieck ring as $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$ to explicitly remember the fixed (up to isomorphisms as in §3.2) action on S.

We know from [51] that the integral Bost–Connes algebra lifts to the equivariant Grothendieck ring $K^{\hat{\mathbb{Z}}}(\mathcal{V}_{\mathbb{Q}})$ via maps σ_n and $\tilde{\rho}_n$ that, respectively, precompose the action with the Bost–Connes endomorphism σ_n and apply a geometric form of the Verschiebung map. We extend the same construction here to the relative case, by similarly transforming in a compatible way the actions on X and on S. The lift of the endomorphisms $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and the group homomorphisms $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the integral Bost–Connes algebra to the setting of equivariant relative Grothendieck rings is determined by the maps (3.5) and (3.6) below.

Proposition 3.1. *For all $n \in \mathbb{N}$ there are ring homomorphisms*

$$\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S \circ \sigma_n)}) \tag{3.5}$$

and group homomorphisms

$$\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\Phi_n(\alpha_S))}), \tag{3.6}$$

where $Z_n = \mathrm{Spec}(\mathbb{Q}^n)$ and $\Phi_n(\alpha_S)$ is the action of $\hat{\mathbb{Z}}$ obtained as in (3.3) *and* (3.4), *with compositions satisfying*

$$\tilde{\rho}_n \circ \sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\alpha_S\times\alpha_n)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$$

$$\sigma_n \circ \tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)^{\oplus n}}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}),$$

with $\sigma_n \circ \tilde{\rho}_n = n\, id$ and $\tilde{\rho}_n \circ \sigma_n$ is the product by (Z_n, α_n).

Proof. Given a class $[f : (X, \alpha_X) \to (S, \alpha_S)]$ in $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$, with α_X the compatible $\hat{\mathbb{Z}}$-action on X, we set

$$\sigma_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f : (X, \alpha_X \circ \sigma_n) \to (S, \alpha_S \circ \sigma_n)].$$

Since the group $\hat{\mathbb{Z}}$ is commutative and so is its endomorphism ring, the transformation above respects isomorphism classes since for an isomorphism (ϕ, φ, ϕ_S) the actions satisfy

$$\phi_X(\alpha_X(\sigma_n(g), x)) = \alpha'_X(\varphi(\sigma_n(g)), \phi(x)) = \alpha'_X(\sigma_n(\varphi(g)), \phi(x)),$$

and similarly for the actions α_S, α'_S, so that (ϕ, φ, ϕ_S) is also an isomorphism of the images under σ_n. We then set

$$\tilde{\rho}_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f \times \mathrm{id} : (X \times Z_n, \Phi_n(\alpha_X)) \to (S \times Z_n, \Phi_n(\alpha_S))].$$

Again this is well defined on the isomorphism classes.

As in [51] we see that $\sigma_n \circ \tilde{\rho}_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f : (X, \alpha_X) \to (S, \alpha_S)]^{\oplus n}$ and $\tilde{\rho}_n \circ \sigma_n[f : (X, \alpha_X) \to (S, \alpha_S)] = [f \times \mathrm{id} : (X \times Z_n, \alpha_X \times \alpha_n) \to (S \times Z_n, \alpha_S \times \alpha_n)]$. The Grothendieck groups $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n, \alpha_S\times\alpha_n)})$ and $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)^{\oplus n}})$ map to $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$ via the morphism induced by composition with the natural maps of the respective base varieties to (S, α_S). $\square$

The fact that the ring homomorphisms (3.5) and (3.6) determine a lift of the ring endomorphism $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and group homomorphisms $\tilde{\rho}_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ of the integral Bost–Connes algebra is discussed in Proposition 3.3 and §3.8 below.

Because the maps σ_n and $\tilde{\rho}_n$ constructed in this way simultaneously modify the action on the varieties and on the base scheme S, they do not give endomorphisms of the same $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$. However, given (S, α_S), it is possible to identify a subalgebra of the integral Bost–Connes algebra that lift to endomorphisms of a corresponding subring of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)})$. We discuss this in the following subsections.

3.5. **Prime decomposition of the Bost–Connes algebra.** As in [26], for each prime p, we can decompose the group $\mathbb{Q}/\mathbb{Z}$ into a product $\mathbb{Q}_p/\mathbb{Z}_p \times (\mathbb{Q}/\mathbb{Z})^{(p)}$, where $\mathbb{Q}_p/\mathbb{Z}_p$ is the Prüfer group, namely the subgroup of elements of $\mathbb{Q}/\mathbb{Z}$ where the denominator is a power of p, isomorphic to $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, while $(\mathbb{Q}/\mathbb{Z})^{(p)}$ consists of the elements with denominator prime to p.

Similarly, given a finite set F of primes, we can decompose $\mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})_F \times (\mathbb{Q}/\mathbb{Z})^F$, where the first term $(\mathbb{Q}/\mathbb{Z})_F$ is identified with fractions in $\mathbb{Q}/\mathbb{Z}$ whose denominator has prime factor decomposition consisting only of primes in F, while elements in $(\mathbb{Q}/\mathbb{Z})^F$ have denominators prime to all $p \in F$. The group ring decomposes accordingly as $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F] \otimes \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$.

The subsemigroup $\mathbb{N}_F \subset \mathbb{N}$ generated multiplicatively by the primes $p \in F$ acts on $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})_F]$ by endomorphisms

$$\rho_n(e(r)) = \frac{1}{n} \sum_{nr'=r} e(r'), \quad n \in \mathbb{N}_F,\ r \in (\mathbb{Q}/\mathbb{Z})_F.$$

The corresponding morphisms $\sigma_n(e(r)) = e(nr)$ and maps $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ act on $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F]$ and we can consider the associated algebra $\mathcal{A}_{\mathbb{Z},F}$ generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_F]$ and $\tilde{\mu}_n$, μ_n^* with $n \in \mathbb{N}_F$, with the relations

$$\tilde{\mu}_{nm} = \tilde{\mu}_n\tilde{\mu}_m, \quad \mu_{nm}^* = \mu_n^*\mu_m^*, \quad \mu_n^*\tilde{\mu}_n = n, \quad \tilde{\mu}_n\mu_m^* = \mu_m^*\tilde{\mu}_n, \tag{3.7}$$

where the first two relations hold for arbitrary $n, m \in \mathbb{N}$, the third for arbitrary $n \in \mathbb{N}$ and the forth for $n, m \in \mathbb{N}$ satisfying $(n, m) = 1$, and the relations

$$x\tilde{\mu}_n = \tilde{\mu}_n\sigma_n(x) \quad \mu_n^* x = \sigma_n(x)\mu_n^*, \quad \tilde{\mu}_n x \mu_n^* = \tilde{\rho}_n(x), \tag{3.8}$$

for any $x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, where $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$, and with

$$\mathcal{A}_{\mathbb{Z},F} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})_F] \rtimes \mathbb{N}_F.$$

We refer to $\mathcal{A}_{\mathbb{Z},F}$ as the F-part of the integral Bost–Connes algebra.

The decomposition $\mathbb{N} = \mathbb{N}_F \times \mathbb{N}^{(F)}$, where $\mathbb{N}^{(F)}$ is generated by all primes $p \notin F$, gives also an algebra $\mathcal{A}_{\mathbb{Z}}^{(F)}$ generated by $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$ and the $\tilde{\mu}_n$ and μ_n^* as in (3.8) with $p \notin F$ with

$$\mathcal{A}_{\mathbb{Z}}^{(F)} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[(\mathbb{Q}/\mathbb{Z})^F] \rtimes \mathbb{N}^{(F)}.$$

We refer to $\mathcal{A}_{\mathbb{Z}}^{(F)}$ as the F-coprime part of the integral Bost–Connes algebra.

3.6. **Equivariant Euler characteristic.** There is an Euler characteristic map given by a ring homomorphism

$$\chi_S^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S) \tag{3.9}$$

to the Grothendieck ring of constructible sheaves over S with $\hat{\mathbb{Z}}$-action, [38], [47], [56], [66].

Lemma 3.2. *Let S be a variety with a good $\hat{\mathbb{Z}}$-action that factors through a finite level $\mathbb{Z}/N\mathbb{Z}$. Given a constructible sheaf $[\mathcal{F}]$ in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$, let $\mathcal{F}|_{S^g}$ denote the restrictions to the fixed point sets S^g, for $g \in \mathbb{Z}/N\mathbb{Z}$. These determine classes in $K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. One obtains in this way a map*

$$\chi : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to \bigoplus_{g \in \mathbb{Z}/N\mathbb{Z}} K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]. \tag{3.10}$$

Proof. The $\hat{\mathbb{Z}}$ action on S factors through some $\mathbb{Z}/N\mathbb{Z}$, hence the fixed point sets are given by S^g for $g \in \mathbb{Z}/N\mathbb{Z}$. Given a constructible sheaves $\mathcal{F}$ over S with $\hat{\mathbb{Z}}$-action, consider the restrictions $\mathcal{F}|_{S^g}$. The subgroup $\langle g \rangle$ generated by g acts trivially on S^g, hence for each $s \in S^g$ it acts on the stalk $\mathcal{F}_s$. Thus, these restrictions define classes $[\mathcal{F}|_{S^g}] \in K_0(\mathbb{Q}_{S^g}) \otimes R(\langle g \rangle) \subset K_0(\mathbb{Q}_{S^g}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. By precomposing with the Euler characteristic (3.9) one then obtains the map (3.10). $\square$

We will also consider the map $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0(\mathbb{Q}_{S^G}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ given by the Euler characteristic followed by restriction of sheaves to the fixed point set S^G of the group action.

3.6.1. *Fixed points and delocalized homology.* Equivariant characteristic classes from constructible sheaves to delocalized homology are constructed in [56].

For a variety S with a good action by a finite group G, and a (generalized) homology theory H, the associated delocalized equivariant theory is given by

$$H^G(S) = (\oplus_{g \in G} H(S^g))^G$$

where the disjoint union $\sqcup_g S^g$ of the fixed point sets S^g has an induced G-action $h : S^g \to S^{hgh^{-1}}$. In the case of an abelian group we have $H^G(S) = \oplus_{g \in G} H(S^g)^G$.

Let S be a variety with a good $\hat{\mathbb{Z}}$-action that factors through a finite level $\mathbb{Z}/N\mathbb{Z}$. If S has the trivial $\mathbb{Z}/N\mathbb{Z}$-action we have $H^{\mathbb{Z}/N\mathbb{Z}}(S) = H(S) \otimes \mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$. In particular, if S is just a point, then this is $\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}]$. More generally, there is a morphism

$$\mathbb{Z}[\mathbb{Z}/N\mathbb{Z}] \times H^{\mathbb{Z}/N\mathbb{Z}}(S) \to H^{\mathbb{Z}/N\mathbb{Z}}(S)$$

induced by $H^{\mathbb{Z}/N\mathbb{Z}}(pt) \times H^{\mathbb{Z}/N\mathbb{Z}}(S) \to H^{\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}}(pt \times S) \to H^{\mathbb{Z}/N\mathbb{Z}}(S)$ with the restriction to the diagonal subgroup as the last map.

3.7. **Lifting the F_N-coprime Bost–Connes algebra.** Let $F = F_N$ be the set of prime factors of N and let $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^F]$ denote, as before, the part of the group ring of $\mathbb{Q}/\mathbb{Z}$ involving only denominators relatively prime to N. The semigroup $\mathbb{N}^{(F)}$ is generated by primes $p \not| N$ and we consider the morphisms $\sigma_n(e(r)) = e(nr)$ and maps $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ with $n \in \mathbb{N}^{(F)}$ and $r \in (\mathbb{Q}/\mathbb{Z})^F$ as discussed above.

Proposition 3.3. *Let S be a base scheme with a good action of $\hat{\mathbb{Z}}$ that factors through $\mathbb{Z}/N\mathbb{Z}$. The endomorphisms $\sigma_n : \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}] \to \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ with $n \in \mathbb{N}^{(F_N)}$ of the F_N-coprime part of the integral Bost–Connes algebra lift to endomorphisms $\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, which define a semigroup action of the multiplicative group $\mathbb{N}^{(F_N)}$ on the Grothendieck ring $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The maps $\tilde{\rho}_n$, for $n \in \mathbb{N}^{(F_N)}$, lift to group homomorphisms $\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ so that $\sigma_n \circ \tilde{\rho}_n[f : X \to S] = [f : X \to S]^{\oplus n}$ and $\tilde{\rho}_n \circ \sigma_n[f : X \to S] = [f : X \to S] \cdot \mathcal{Z}_{n,S}$, where $\mathcal{Z}_{n,S}$ is defined as $\mathcal{Z}_{n,S} = S \times Z_n$, with $Z_n = \mathrm{Spec}(\mathbb{Q}^n)$ with the action $\Phi_n(\alpha_S)$ obtained as in* (3.3) *and* (3.4)*.*

Proof. Given the base variety S with a good $\hat{\mathbb{Z}}$-action factoring through a finite level $\mathbb{Z}/N\mathbb{Z}$, let $F = F_N$ denote the set of prime factors of N. Let X be a variety over S, with a $\hat{\mathbb{Z}}$-equivariant map $f : (X, \alpha_X) \to (S, \alpha_S)$, where we explicitly write the actions, satisfying $f(\alpha_X(\zeta, x)) = \alpha_S(\zeta, f(x))$. For $(N, n) = 1$, we let $\sigma_n : [f : (X, \alpha_X) \to (S, \alpha_S)] = [f : (X, \alpha_X \circ \sigma_n) \to (S, \alpha_S \circ \sigma_n)]$, where $(S, \alpha_S \circ \sigma_n) \simeq (S, \alpha_S)$ with the notion of isomorphism discussed in §3.2, since $\zeta \mapsto \sigma_n(\zeta)$ is an automorphism of $\mathbb{Z}/N\mathbb{Z}$. Thus, the maps σ_n, for $n \in \mathbb{N}^{(F_N)}$ determine a semigroup action of $\mathbb{N}^{(F_N)}$ by endomorphisms of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$.

Consider then $(\mathcal{Z}_{n,N}, \Phi_n(\alpha_S))$ as above, which we write equivalently as $\tilde{\rho}_n(S, \alpha_S)$ where $\tilde{\rho}_n$ is the lift of the Bost–Connes map to $K^{\hat{\mathbb{Z}}}(\mathcal{V})$ as in [51]. We know that $\tilde{\rho}_n \circ \sigma_n[S, \alpha_S] = [S, \alpha_S] \cdot [Z_n, \alpha_n]$ in $K^{\hat{\mathbb{Z}}}(\mathcal{V})$. Since for $(n, N) = 1$ we have $(S, \alpha_S \circ \sigma_n) \simeq (S, \alpha_S)$, this gives $(\mathcal{Z}_{n,N}, \Phi_n(\alpha_S)) \simeq (S \times Z_n, \alpha_S \times \gamma_n)$. Then setting $\tilde{\rho}_n(f : X \to S) = (\tilde{f} : X \times_S \mathcal{Z}_{n,S} \to S)$ with $\tilde{f} = f \circ \pi_X$ gives $X \times_S \mathcal{Z}_{n,S} \simeq X \times Z_n$, and the composition properties for $\tilde{\rho}_n \circ \sigma_n$ and $\sigma_n \circ \tilde{\rho}_n$ are satisfied.

Given a class $[f : X \to S]$, let $[\mathcal{F}_{X,S}]$ be the class in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$ of the constructible sheaf given by the Euler characteristic (3.9) of $[f : X \to S]$. Let $[\mathcal{F}_{X,S}|_{S^{\mathbb{Z}/N\mathbb{Z}}}]$ be the resulting class in $K_0(S^{\mathbb{Z}/N\mathbb{Z}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ obtained by restriction to the fixed point set $S^{\mathbb{Z}/N\mathbb{Z}}$ with the element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ specifying the representation of $\hat{\mathbb{Z}}$ on the stalks of the sheaf $\mathcal{F}_{X,S}|_{S^{\mathbb{Z}/N\mathbb{Z}}}$. For $(N, n) = 1$, the action of σ_n by automorphisms of $\mathbb{Z}/N\mathbb{Z}$ wih the resulting action by isomorphisms of S induces an action by isomorphisms on the $K_0(S^{\mathbb{Z}/N\mathbb{Z}})$ part and the usual Bost–Connes action on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. The restriction of the semigroup action of $\mathbb{N}^{(F_N)}$ to the subring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ is then the image of the action of the maps σ_n and $\tilde{\rho}_n$ on the preimage of this subring under the morphism $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0(\mathbb{Q}_{S^G}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. □

While this construction captures a lift of the $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ part of the Bost–Connes algebra with the semigroup action of $\mathbb{N}^{(F_N)}$, the fact that the endomorphisms σ_n acting on the roots of unity in $\mathbb{Z}/N\mathbb{Z}$ are automorphisms when $(N, n) = 1$ loses some of the interesting structure of the Bost–Connes algebra, which stems from the partial invertibility of these morphisms. Thus, one also wants to recover the structure of the complementary part of the Bost–Connes algebra with the group ring $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})_{F_N}]$ and the semigroup $\mathbb{N}_{F_N}$.

3.8. **Lifting the full Bost–Connes algebra.** Unlike the $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{F_N}]$ part of the Bost–Connes algebra described above, when one considers the full Bost–Connes algebra, including the F_N-part, the lift to the Grothendieck ring no longer consists of endomorphisms of a fixed $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$, but is given as in Proposition 3.1 by homomorphisms as in (3.5) and (3.6),

$$\sigma_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S\circ\sigma_n)}),$$

$$\tilde{\rho}_n : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha_S)}) \to K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\Phi_n(\alpha_S))}).$$

For G a finite abelian group with a good action $\alpha : G \times S \to S$ on a variety S, let $(S, \alpha)^G_k = \{s \in S : \alpha(g^k, s) = s, \forall g \in G\}$ denote the set of periodic points of period k, with $(S, \alpha)^G_1 = (S, \alpha)^G$ the set of fixed points. We always have $(S, \alpha)^G_k \subseteq (S, \alpha)^G_{km}$ for all $m \in \mathbb{N}$, hence in particular a copy of the fixed point set $(S, \alpha)^G$ is contained in all $(S, \alpha)^G_k$. For $G = \mathbb{Z}/N\mathbb{Z}$, with ζ_N a primitive N-th root of unity generator, the set of k-periodic points is given by $(S, \alpha)^{\mathbb{Z}/N\mathbb{Z}}_k = \{s \in S : \alpha(\zeta_N^k, s) = s\}$.

Lemma 3.4. *The sets of periodic points satisfy $(S, \alpha \circ \sigma_n)^G_k = (S, \alpha)^G_{nk}$. The sets $(S \times Z_n, \Phi_n(\alpha))^G_k$ can be non-empty when $n|k$ with $(S \times Z_n, \Phi_n(\alpha))^G_k = ((S, \alpha)^G_{k/n})^n$.*

Proof. Under the action $\alpha \circ \sigma_n$ the periodicity condition means $\alpha \circ \sigma_n(\zeta^k, s) = \alpha(\zeta^{nk}, s) = s$ for all $\zeta \in G$ hence the identification $(S, \alpha\circ\sigma_n)^G_k = (S, \alpha)^G_{nk}$. In the case of the geometric Verschiebung action $\Phi_n(\alpha)$ on $S \times Z_n$, the k-periodicity condition $\Phi_n(\alpha)(\zeta^k, (s, z)) = (s, z)$ implies that $n|k$ for the k-periodicity in the $z \in Z_n$ variable and that $\alpha(\zeta^{k/n}, s) = s$. □

The identification $(S, \alpha\circ\sigma_n)^G_k = (S, \alpha)^G_{nk}$ implies the inclusion $(S, \alpha)^G_k \subseteq (S, \alpha\circ\sigma_n)^G_k$ and in particular the inclusion of the fixed point sets $(S, \alpha)^G \subseteq (S, \alpha\circ\sigma_n)^G$. Similarly, $(S \times Z_n, \Phi_n(\alpha))^G \subseteq ((S, \alpha)^G)^n$. Since these inclusions will in general be strict, due to the fact that the endomorphisms σ_n are not automorphisms, one cannot simply use the map given by the equivariant Euler characteristic followed by the restriction to the fixed point set

$$K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0(\mathbb{Q}_{S^{\hat{\mathbb{Z}}}}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

to lift the Bost–Connes endomorphisms to the maps (3.5) and (3.6) of Proposition 3.1. However, a simple variant of the same idea, where we consider sets of periodic points, gives the lift of the full Bost–Connes algebra to the equivariant relative Grothendieck rings $K_0^G(\mathcal{V}_{(S,\alpha)})$.

Consider the equivariant Euler chatacteristic map followed by the restrictions to the sets of periodic points

$$K_0^G(\mathcal{V}_{(S,\alpha)}) \stackrel{\chi_S^G}{\to} K_0^G(\mathbb{Q}_{(S,\alpha)}) \to \bigoplus_{k\geq 1} K_0^G(\mathbb{Q}_{(S,\alpha)^G_k}). \tag{3.11}$$

Also, for a given $n \in \mathbb{N}$, consider the same map composed with the projection to the summands with $n|k$

$$\chi^G_{S,n} : K_0^G(\mathcal{V}_{(S,\alpha)}) \stackrel{\chi_S^G}{\to} K_0^G(\mathbb{Q}_{(S,\alpha)}) \to \bigoplus_{k\geq 1\,:\,n|k} K_0^G(\mathbb{Q}_{(S,\alpha)^G_k}). \tag{3.12}$$

For simplicity we consider below the case where the fixed point set and periodic points sets of the action (S,α) are all finite sets.

Proposition 3.5. *Let (S,α) be a variety with a good $\hat{\mathbb{Z}}$-action factoring through some finite level $\mathbb{Z}/N\mathbb{Z}$, such that the set $(S,\alpha)^{\hat{\mathbb{Z}}}_k$ of k-periodic points for this action is finite, for all $k \geq 1$. Then the maps* (3.11) *and* (3.12) *intertwine the maps σ_n of* (3.5) *with the endomorphisms σ_n of the integral Bost–Connes algebra, and the maps $\tilde{\rho}_n$ of* (3.6) *and the maps $\tilde{\rho}_n$ of the integral Bost–Connes algebra.*

Proof. Under the assumptions that all the $(S,\alpha)^G_k$ for $k \geq 0$ are finite sets, we can identify the target of the map with $\oplus_k K_0(\mathbb{Q}_{(S,\alpha)^G_k}) \otimes R(G)$. In the case of varieties with good $\hat{\mathbb{Z}}$ actions factoring through some finite $\mathbb{Z}/N\mathbb{Z}$, we obtain in this way ring homomorphisms

$$\chi^{\hat{\mathbb{Z}}}_{(S,\alpha)} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \to \bigoplus_{k\geq 1} K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_k}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

$$\chi^{\hat{\mathbb{Z}}}_{(S,\alpha),n} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \to \bigoplus_{k\geq 1\,:\,n|k} K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_k}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}].$$

These maps fit in the following commutative diagrams of ring homomorphisms

$$\begin{tikzcd}
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \arrow[r, "\chi^{\hat{\mathbb{Z}}}_{(S,\alpha),n}"] \arrow[d, "\sigma_n"] & \oplus_{n|k} K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_k}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \arrow[d, "J_n\otimes\sigma_n"] \\
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha\circ\sigma_n)}) \arrow[r, "\bar{\chi}^{\hat{\mathbb{Z}}}_{(S,\alpha\circ\sigma_n)}"'] & \bigoplus_\ell K_0(\mathbb{Q}_{(S,\alpha\circ\sigma_n)^{\hat{\mathbb{Z}}}_\ell}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]
\end{tikzcd}$$

where the map $(J_n)_{k,\ell}$ is non-trivial for $k = \ell n$ and identifies $K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_\ell})$ with $K_0(\mathbb{Q}_{(S,\alpha\circ\sigma_n)^{\hat{\mathbb{Z}}}_k})$, while the maps $\sigma_n : \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ are the Bost–Connes endomorphisms. Similarly, we obtain commutative diagrams of group homomorphisms

$$\begin{tikzcd}
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)}) \arrow[r, "\chi^{\hat{\mathbb{Z}}}_{(S,\alpha)}"] \arrow[d, "\tilde{\rho}_n"] & \oplus_\ell K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_\ell}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \arrow[d, "\tilde{J}_n \otimes \tilde{\rho}_n"] \\
K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S\times Z_n,\Phi_n(\alpha))}) \arrow[r, "\chi^{\hat{\mathbb{Z}}}_{(S\times Z_n,\Phi_n(\alpha)),n}"'] & \oplus_{n|k} K_0(\mathbb{Q}_{(S\times Z_n,\Phi_n(\alpha))^{\hat{\mathbb{Z}}}_k}) \otimes \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]
\end{tikzcd}$$

where $(\tilde{J}_n)_{\ell,k}$ is non-trivial for $k = \ell n$ and maps $K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_k})$ to $K_0(\mathbb{Q}_{(S,\alpha)^{\hat{\mathbb{Z}}}_k})^{\oplus n}$ and identifies the latter with $K_0(\mathbb{Q}_{(S\times Z_n,\Phi_n(\alpha))^{\hat{\mathbb{Z}}}_\ell})$. $\square$

A similar argument can be given using a map obtained by composing the equivariant Euler characteristic considered here with values in $K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$ with equivariant characteristic classes from constructible sheaves to delocalized equivariant homology as in [56], see §3.6.1 above.

4. From Rings to Spectra

In this section we show that the Bost–Connes structure can be lifted further from the level of the relative Grothendieck ring to the level of spectra, using the assembler category construction of [67].

4.1. **Assemblers for the relative Grothendieck ring.** The relative Grothendieck ring $K_0(\mathcal{V}_S)$ of varieties over a base variety S over a field $\mathbb{K}$ is generated by the isomorphism classes of data $f : X \to S$ of a variety X over S with the relations

$$[f : X \to S] = [f|_Y : Y \to S] + [f|_{X\smallsetminus Y} : X \smallsetminus Y \to S]$$

for a closed embedding $Y \hookrightarrow X$ of varieties over S. The product is given by the fibered product $X \times_S Y$. We will write $[X]_S$ as shorthand notation for the class $[f : X \to S]$ in $K_0(\mathcal{V}_S)$.

A morphism $\phi : S \to S'$ induces a base change ring homomorphism $\phi^* : K_0(\mathcal{V}_{S'}) \to K_0(\mathcal{V}_S)$ and a direct image map $\phi_* : K_0(\mathcal{V}_S) \to K_0(\mathcal{V}_{S'})$ which is a group homomorphisms and a morphism of $K_0(\mathcal{V}_{S'})$-modules, but not a ring homomorphism. The class $[\phi : S \to S']$ as an element in $K_0(\mathcal{V}_{S'})$ is the image of $1 \in K_0(\mathcal{V}_S)$ under ϕ_*.

When $S = \mathrm{Spec}(\mathbb{K})$ one recovers the ordinary Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$ of varieties over $\mathbb{K}$.

An assembler $\mathcal{C}_S$ such that the associated spectrum $K(\mathcal{C}_S)$ has $K_0(\mathcal{C}_S) = \pi_0 K(\mathcal{C}_S)$ given by the relative Grothendieck ring $K_0(\mathcal{V}_S)$ can be obtained as a slight modification of the construction given in [69] for the ordinary Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$.

Lemma 4.1. *The assembler $\mathcal{C}_S$ for the relative Grothendieck ring $K_0(\mathcal{V}_S)$ has objects $f : X \to S$ that are varieties over S and morphisms that are locally closed embeddings of varieties over S. The Grothendieck topology on $\mathcal{C}_S$ is generated by the covering families $\{Y \hookrightarrow X, X \smallsetminus Y \hookrightarrow X\}$ with compatible maps* (3.1)

$$
\begin{tikzcd}
Y \arrow[r, hook] \arrow[dr, "f|_Y"'] & X \arrow[d, "f"] & X \smallsetminus Y \arrow[l, hook'] \arrow[dl, "f|_{X \smallsetminus Y}"] \\
& S &
\end{tikzcd}
\tag{4.1}
$$

Proof. The argument is the same as in [67], [69] and in [51]. In this setting finite disjoint covering families are maps

$$
\begin{tikzcd}
X_i \arrow[r, hook] \arrow[dr, "f_i"'] & X \arrow[d, "f"] \\
& S
\end{tikzcd}
$$

where $X_i = Y_i \smallsetminus Y_{i-1}$ with commutative diagrams

$$
\begin{tikzcd}
Y_0 \arrow[r, hook] \arrow[drrr, "f_0"'] & Y_1 \arrow[r, hook] \arrow[drr, "f_1"] & \cdots \arrow[r, hook] & Y_n = X \arrow[d, "f"] \\
& & & S
\end{tikzcd}
$$

The category has pullbacks, hence as shown in [67] this suffices to obtain that any two finite disjoint covering families have a common refinement. Morphisms are embeddings compatible with the structure maps as in (4.1) hence in particular monomorphisms. Theorem 2.3 of [67] then shows that the spectrum $K(\mathcal{C}_S)$ associated to this assembler category has $\pi_0 K(\mathcal{C}_S) = K_0(\mathcal{V}_S)$. □

In a similar way we obtain an assembler category and spectrum for the equivariant version $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$. The argument is as in the previous case and in Lemma 4.5.1 of [51], using the inclusion-exclusion relations (3.2).

Corollary 4.2. *An assembler category $\mathcal{C}^{\hat{\mathbb{Z}}}_{(S,\alpha)}$ for $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_{(S,\alpha)})$ is constructed as in Lemma 4.1 with objects the $\hat{\mathbb{Z}}$-equivariant $f : X \to S$, morphisms given by $\hat{\mathbb{Z}}$-equivariant locally closed embeddings of varieties over S and with Grothendieck topology generated by the covering families given by $\hat{\mathbb{Z}}$-equivariant maps as in* (3.1) *and* (3.2).

As in [51], we show that the lifting of the integral Bost–Connes algebra obtained in Propositions 3.1 and 3.5 further lifts to functors of the associated assembler categories, with the σ_n compatible with the monoidal structure, but not the $\tilde{\rho}_n$.

Proposition 4.3. *The maps $\sigma_n : (f : (X, \alpha_X) \to (S, \alpha)) \mapsto (f : (X, \alpha_X \circ \sigma_n) \to (S, \alpha \circ \sigma_n))$ and $\tilde{\rho}_n : (f : (X, \alpha_X) \to (S, \alpha)) \mapsto (f \times \mathrm{id} : (X \times Z_n, \Phi_n(\alpha_X)) \to (S \times Z_n, \Phi_n(\alpha)))$ define functors of the assembler categories $\sigma_n : \mathcal{C}^{\hat{\mathbb{Z}}}_{(S,\alpha)} \to \mathcal{C}^{\hat{\mathbb{Z}}}_{(S,\alpha\circ\sigma_n)}$*

and $\tilde{\rho}_n : \mathcal{C}^{\hat{\mathbb{Z}}}_{(S,\alpha)} \to \mathcal{C}^{\hat{\mathbb{Z}}}_{(S\times Z_n,\Phi_n(\alpha))}$. *The functors* σ_n *are compatible with the monoidal structure.*

Proof. The functors σ_n defined as above on objects are compatibly defined on morphisms by assigning to a locally closed embedding

$$\sigma_n : \quad \begin{tikzcd} (Y,\alpha_Y) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X,\alpha_X) \arrow[d, "f_X"] \\ & (S,\alpha) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} (Y,\alpha_Y\circ\sigma_n) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X,\alpha_X\circ\sigma_n) \arrow[d, "f_X"] \\ & (S,\alpha\circ\sigma_n) \end{tikzcd}$$

Similarly, we define the $\tilde{\rho}_n$ on morphisms by

$$\tilde{\rho}_n : \quad \begin{tikzcd} (Y,\alpha_Y) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X,\alpha_X) \arrow[d, "f_X"] \\ & (S,\alpha) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} (Y\times Z_n,\Phi_n(\alpha_Y)) \arrow[r, "j"] \arrow[dr, "f_Y"'] & (X\times Z_n,\Phi_n(\alpha_X)) \arrow[d, "f_X"] \\ & (S\times Z_n,\Phi_n(\alpha)) \end{tikzcd}$$

The functors σ_n are compatible with the monoidal structure since $\sigma_n(X,\alpha_X)\times \sigma_n(X',\alpha_{X'}) = (X\times X',(\alpha\times\alpha')\circ\Delta\circ\sigma_n) = \sigma_n((X,\alpha_X)\times(X',\alpha_{X'}))$. ☐

The functor of assembler categories determines an induced map of spectra and in turn an induced map of homotopy groups. By construction the induced maps on the π_0 homotopy agree with the maps (3.5) and (3.6) of Proposition 3.1.

5. Torifications, $\mathbb{F}_1$-points, zeta functions, and spectra

In this section we related the point of view developed in [51] to the approach to $\mathbb{F}_1$-geometry based on torifications. This was first introduced in [48]. Weaker forms of torification were also considered in [50], which allow for the development of a form of $\mathbb{F}_1$-geometry suitable for the treatment of certain classical moduli spaces.

The approach we follow here, in order the relate the case of torified geometry with the Bost–Connes systems on Grothendieck rings, assemblers, and spectra discussed in [51], is based on the following simple setting. instead of working with the equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ and $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$, where one assumes the varieties have a good $\hat{\mathbb{Z}}$-action that factors through some finite level, we consider here a variant that connects to the torifications point of view on $\mathbb{F}_1$-geometry of [48]. We replace varieties with $\hat{\mathbb{Z}}$-actions that factor through some finite $\mathbb{Z}/N\mathbb{Z}$ with varieties with a $\mathbb{Q}/\mathbb{Z}$-action induced by a torification, where the group schemes $\mathfrak{m}_n$ of n-th roots of unity, given by the kernels

$$1 \to \mathfrak{m}_n \to \mathbb{G}_m \xrightarrow{\lambda\mapsto\lambda^n} \mathbb{G}_m \to 1$$

determine a diagonal embedding in each torus and an action by multiplication. This is a very restrictive class of varieties, because the existence of a torification on a variety implies that the Grothendieck class is a sum of classes of tori with non-negative coefficients. The resulting construction will be more restrictive than the one considered in [51]. We will see, however, that one can still see in this context several interesting phenomena, especially in connection with the "dynamical" approach to $\mathbb{F}_1$-geometry proposed in [51].

5.1. **Torifications.** A torification of an algebraic variety defined over $\mathbb{Z}$ is a decomposition $X = \sqcup_{i\in\mathcal{I}} T_i$ into algebraic tori $T_i = \mathbb{G}_m^{d_i}$. Weaker to stronger forms of torification [50] include

(1) *torification of the Grothendieck class*: $[X] = \sum_{i\in\mathcal{I}} (\mathbb{L}-1)^{d_i}$ with $\mathbb{L}$ the Lefschetz motive;
(2) *geometric torification*: $X = \sqcup_{i\in\mathcal{I}} T_i$ with $T_i = \mathbb{G}_m^{d_i}$;
(3) *affine torification*: the existence of an affine covering compatible with the geometric torification, [48];
(4) *regular torification*: the closure of each torus in the geometric torification is also a union of tori of the torification, [48].

Similarly, there are different possibilities when one considers morphisms of torified varieties, see [50]. In view of describing associated Grothendieck rings, we review the different notions of morphisms. The Grothendieck classes are then defined with respect to the corresponding type of isomorphism.

A torified morphism in the sense of [48] between torified varieties $f : (X,T) \to (Y,T')$ is a morphism $f : X \to Y$ of varieties together with a map $h : I \to J$ of the indexing sets of the torifications $X = \sqcup_{i\in I} T_i$ and $Y = \sqcup_{j\in J} T'_j$ such that the restriction of f to tori T_i is a morphism of algebraic groups $f_i : T_i \to T'_{h(i)}$.

We then have the following classes of morphisms of torified varieties from [50]:

(1) *strong morphisms*: these are torified morphisms in the sense of [48], namely morphisms that restrict to morphisms of tori of the respective torifications.
(2) *ordinary morphisms*: an ordinary morphism of torified varieties (X,T) and (Y,T') is a morphism $f : X \to Y$ such that becomes a torified morphism after composing with isomorphisms, that is, $\phi_Y \circ f \circ \phi_X : (X,T) \to (Y,T')$ is a strong morphism of torified varieties, for some isomorphisms $\phi_X : X \to X$ and $\phi_Y : Y \to Y$.
(3) *weak morphisms*: the torified varieties (X,T) and (Y,T') admit decompositions $X = \sqcup_i X_i$ and $Y = \sqcup_j Y_i$, compatible with the torifications, such that there exist ordinary morphisms $f_i : (X_i, T_i) \to (Y_{f(i)}, T'_{f(i)})$ of these subvarieties.

Correspondingly, we can construct Grothendieck rings $K_0(\mathcal{T})^s$, $K_0(\mathcal{T})^o$, and $K_0(\mathcal{T})^w$ in the following way.

As an abelian group $K_0(\mathcal{T})^s$ is generated by isomorphism classes $[X,T]_s$ of pairs of a torifiable variety X and a torification T modulo strong isomorphisms, with the inclusion-exclusion relations $[X,T]_s = [Y,T_Y]_s + [X \smallsetminus Y, T_{X\smallsetminus Y}]_s$ whenever $(Y,T_Y) \hookrightarrow (X,T)$ is a strong morphism (that is, the inclusion of Y in X is compatible with the torification: Y is a union of tori of the torification of X) and (Y,T_Y) is a *complemented* *subvariety* in (X,T), which means that the complement $X \smallsetminus Y$ is also a union of tori of the torification so that $(X \smallsetminus Y, T_{X\smallsetminus Y})$ is also a strong morphism. This complemented condition is very strong. Indeed, one can see that, for example, there are in general very few complemented points in a torified variety. The product operation is $[X,T]_s \cdot [Y,T']_s = [X \times Y, T \times T']_s$ with the torification $T \times T'$ given by the product tori $T_{ij} = T_i \times T'_j = \mathbb{G}_m^{d_i+d_j}$.

The abelian group $K_0(\mathcal{T})^o$ is generated by isomorphism classes $[X]_o$ varieties that admit a torification with respect to ordinary isomorphisms, with the inclusion-exclusion relations $[X]_o = [Y]_o + [X \smallsetminus Y]_o$ whenever the inclusions $Y \hookrightarrow X$ and $X \smallsetminus Y \hookrightarrow X$ are ordinary morphisms. The product is the class of the Cartesian product $[X]_o \cdot [Y]_o = [X \times Y]_o$.

The abelian group $K_0(\mathcal{T})^w$ is generated by the isomorphism classes $[X]_w$ of torifiable varieties X with respect to weak morphisms, with the inclusion-exclusion relations $[X]_w = [Y]_w + [X \smallsetminus Y]_w$ whenever the inclusions $Y \hookrightarrow X$ and $X \smallsetminus Y \hookrightarrow X$ are weak morphisms. The product structure is again given by $[X]_w \cdot [Y]_w = [X \times Y]_w$.

The reader can consult the explicit examples given in [50] to see how these notions (and the resulting Grothendieck rings) can be different. Note however that, in all these cases, the Grothendieck classes $[X]_a$ with $a = s, o, w$ have the form $[X]_a = \sum_{n\geq 0} a_n \mathbb{T}^n$ with $a_n \in \mathbb{Z}_+$ and $\mathbb{T}^n = [\mathbb{G}_m^n]$.

5.1.1. *Relative case.* In a similar way, we can construct relative Grothendieck rings $K_S(\mathcal{T})^a$ with $a = s, o, w$ where in the strong case $S = (S, T_S)$ is a choice of a variety with an assigned torification, with $K_S(\mathcal{T})^s$ generated as an abelian group by isomorphisms classes $[f : (X,T) \to (S,T_S)]$ where f is a strong morphism of torified varieties and the isomorphism class is taken with respect to strong isomorphisms ϕ, ϕ_S such that the diagram commutes

$$\begin{tikzcd} (X,T) \arrow[r, "\phi"] \arrow[d, "f"'] & (X',T') \arrow[d, "f'"] \\ (S,T_S) \arrow[r, "\phi_S"] & (S,T_S) \end{tikzcd}$$

with inclusion-exclusion relations

$$[f : (X,T) \to (S,T_S)] =$$

$$[f|_{(Y,T_Y)} : (Y,T_Y) \to (S,T_S)] + [f|_{(X\smallsetminus Y, T_{X\smallsetminus Y})} : (X \smallsetminus Y, T_{X\smallsetminus Y}) \to (S,T_S)]$$

where $\iota_Y : (Y,T_Y) \hookrightarrow (X,T)$ is a strong morphism and (Y,T_Y) is complemented with $\iota_{X\smallsetminus Y} : (X \smallsetminus Y, T_{X\smallsetminus Y}) \hookrightarrow (X,T)$ also a strong morphism and both these inclusions are compatible with the map $f : (X,T) \to (S,T_S)$, so that $f_Y = f \circ \iota_Y$ and $f|_{(X\smallsetminus Y, T_{X\smallsetminus Y})} = f \circ \iota_{X\smallsetminus Y}$ are strong morphisms. The construction for ordinary and weak morphism is similar, with the appropriate changes in the definition.

5.2. **Group actions.** In order to operate on Grothendieck classes with Bost–Connes type endomorphisms, we introduce appropriate group actions.

Torified varieties carry natural $\mathbb{Q}/\mathbb{Z}$ actions, since the roots of unity embed diagonally in each torus of the torification and act on it by multiplication. However, we will also be interested in considering good $\hat{\mathbb{Z}}$-actions, in the sense already discussed in [51], that is, actions of $\hat{\mathbb{Z}}$ that factor through some finite $\mathbb{Z}/N\mathbb{Z}$.

Remark 5.1. The main reason for working with $\hat{\mathbb{Z}}$-actions rather than with $\mathbb{Q}/\mathbb{Z}$ actions lies in the fact that, in the construction of the geometric Vershiebung action discussed in §3.3 we need to be able to describe the cyclic permutation action of $\mathbb{Z}/n\mathbb{Z}$ on the finite set Z_n as an action factoring through $\mathbb{Z}/n\mathbb{Z}$. This cannot be done in the case of $\mathbb{Q}/\mathbb{Z}$-actions because there are no nontrivial group homomorphisms $\mathbb{Q}/\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ since $\mathbb{Q}/\mathbb{Z}$ is infinitely divisible.

In the case of the natural $\mathbb{Q}/\mathbb{Z}$-actions on torifications, we consider objects of the form (X, T, α) where X is a torifiable variety, T a choice of a torification, and $\alpha : \mathbb{Q}/\mathbb{Z} \times X \to X$ an action of $\mathbb{Q}/\mathbb{Z}$ determined by an embedding of $\mathbb{Q}/\mathbb{Z}$ as roots of unity in $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$, which act on each torus $T_i = \mathbb{G}_m^{k_i}$ diagonally by multiplication. An embedding of $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{G}_m$ is determined by an invertible element in $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{G}_m) = \hat{\mathbb{Z}}$, hence the action α is uniquely determined by the torification T and by the choice of an element in $\hat{\mathbb{Z}}^*$.

The corresponding morphisms are, respectively, strong, ordinary, or weak morphisms of torified varieties compatible with the $\mathbb{Q}/\mathbb{Z}$-actions, in the sense that the resulting torified morphism (after composing with isomorphisms or with local isomorphisms in the ordinary and weak case) are $\mathbb{Q}/\mathbb{Z}$-equivariant. We can then proceed as above and obtain equivariant Grothendieck rings $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^s$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^o$, and $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^w$ of torified varieties.

In the case of good $\hat{\mathbb{Z}}$-actions (which simply means $\mathbb{Z}/N\mathbb{Z}$-actions where the value of N is not a priori specified), the setting is essentially the same. We consider objects of the form (X, T, α) where X is a torifiable variety, T a choice of a torification, and $\alpha : \mathbb{Z}/N\mathbb{Z} \times X \to X$ is given by the action of the N-th roots of unity on the tori $T_i = \mathbb{G}_m^{k_i}$ by multiplication. Thus, a good $\hat{\mathbb{Z}}$-action is determined by T, by the choice of an embedding of roots of unity in $\mathbb{G}_m$ (an element of $\hat{\mathbb{Z}}^*$) as above, and by the choice of $N \in \mathbb{N}$ that determines which subgroup of roots of unity is acting.

This choice of good $\hat{\mathbb{Z}}$-actions, with strong, ordinary, or weak morphisms whose associated torified morphisms are $\mathbb{Z}/N\mathbb{Z}$-equivariant, determine equivariant Grothendieck rings $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ of torified varieties with good $\hat{\mathbb{Z}}$-actions.

5.3. **Assembler and spectrum of torified varieties.** As in the previous cases of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V})$ of [51] and in the case of $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S)$ discussed above, we consider the Grothendieck rings $K_0(\mathcal{T})^s$, $K_0(\mathcal{T})^o$, and $K_0(\mathcal{T})^w$ and their corresponding equivariant versions $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^s$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^o$, $K_0^{\mathbb{Q}/\mathbb{Z}}(\mathcal{T})^w$, and $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^s$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^o$, $K_0^{\hat{\mathbb{Z}}}(\mathcal{T})^w$ from the point of view of assemblers and spectra developed in [67], [68], [69].

Proposition 5.2. *For $a = s, o, w$, the category $\mathcal{C}^a_{\mathcal{T}}$ has objects that are pairs (X, T) of a torifiable variety and a torification, with morphisms the locally closed embeddings that are, respectively, strong, ordinary, or weak morphisms of torified varieties. The Grothendieck topology is generated by the covering families*

$$\{(Y, T_Y) \hookrightarrow (X, T_X), (X \smallsetminus Y, T_{X\smallsetminus Y}) \hookrightarrow (X, T_X)\} \tag{5.1}$$

where both embeddings are strong, ordinary, or weak morphisms, respectively. The category $\mathcal{C}^a_{\mathcal{T}}$ is an assembler with spectrum $K(\mathcal{C}^a_{\mathcal{T}})$ satisfying $\pi_0 K(\mathcal{C}^a_{\mathcal{T}}) = K_0(\mathcal{T})^a$. Similarly, for $G = \mathbb{Q}/\mathbb{Z}$ or $G = \hat{\mathbb{Z}}$ let $\mathcal{C}^{G,a}_{\mathcal{T}}$ be the category with objects (X, T, α) given by a torifiable variety X with a torification T and a G-action α of the kind discussed in §5.2 and morphisms the locally closed embeddings that are G-equivariant strong, ordinary, or weak morphisms. The Grothendieck topology is generated by covering families (5.1) *with G-equivariant embeddings. The category $\mathcal{C}^{G,a}_{\mathcal{T}}$ is also an assembler, whose associated spectrum $K(\mathcal{C}^{G,a}_{\mathcal{T}})$ satisfies $\pi_0 K(\mathcal{C}^{G,a}_{\mathcal{T}}) = K^G_0(\mathcal{T})^a$.*

Proof. The argument is again as in [67], see Lemma 4.1 above. We check that the category admits pullbacks. In the strong case, if (Y, T_Y) and $(Y', T_{Y'})$ are objects with morphisms $f : (Y, T_Y) \hookrightarrow (X, T_X)$ and $f' : (Y', T_{Y'}) \hookrightarrow (X, T_X)$ given by embeddings that are strong morphisms of torified varieties. This means that the tori of the torification T_Y are restrictions to Y of tori of the torification T_X of X. Thus, both Y and Y' are unions of subcollections of tori of T_X. Their intersection $Y \cap Y'$ will then also inherit a torification consisting of a subcollection of tori of T_X and the resulting embedding $(Y \cap Y', T_{Y\cap Y'}) \hookrightarrow (X, T_X)$ is a strong morphism of torified varieties. In the ordinary case, we consider embeddings $f : Y \hookrightarrow X$ and $f' : Y' \hookrightarrow X$ that are ordinary morphisms of torified varieties, which means that, for isomorphisms ϕ_X, ϕ'_X, ϕ_Y, $\phi_{Y'}$, the compositions

$$\phi_X \circ f \circ \phi_Y : (Y, T_Y) \hookrightarrow (X, T_X), \qquad \phi'_X \circ f' \circ \phi_{Y'} : (Y', T_{Y'}) \hookrightarrow (X, T_X)$$

are (strong) torified morphisms. Thus, the tori of the torifications T_Y and $T_{Y'}$ are subcollections of tori of X, under the embeddings $\phi_X \circ f \circ \phi_Y$ and $\phi'_X \circ f' \circ \phi_{Y'}$. The intersection $\phi_X \circ f \circ \phi_Y(Y) \cap \phi'_X \circ f' \circ \phi_{Y'}(Y') \subset X$ is isomorphic to a copy of $Y \cap Y'$ and has an induced torification $T_{Y\cap Y'}$ by a subcollection of tori of T_X. The embedding of $Y \cap Y'$ in X with this image is an ordinary morphism with respect to this torification. The weak case is constructed similarly to the ordinary case on the pieces of the decomposition. The equivariant cases are constructed analogously, as discussed in the case of equivariant Grothendieck rings of varieties in [51]. □

5.4. **Lifting of the Bost–Connes system for torifications.**

We consider here lifts of the integral Bost–Connes algebra to the Grothendieck rings $K^{\hat{\mathbb{Z}}}_0(\mathcal{T})^s$, $K^{\hat{\mathbb{Z}}}_0(\mathcal{T})^o$, and $K^{\hat{\mathbb{Z}}}_0(\mathcal{T})^w$ and to the assemblers and spectra $K^{\hat{\mathbb{Z}}}(\mathcal{C}^s_{\mathcal{T}})$, $K^{\hat{\mathbb{Z}}}(\mathcal{C}^o_{\mathcal{T}})$, and $K^{\hat{\mathbb{Z}}}(\mathcal{C}^w_{\mathcal{T}})$.

We regard the zero-dimensional variety Z_n as a torified variety with the torification consisting of n zero dimensional tori and with a good $\hat{\mathbb{Z}}$ action factoring through $\mathbb{Z}/n\mathbb{Z}$ that cyclically permutes the points of Z_n. We write (Z_n, T_0, γ) for this object.

Proposition 5.3. *Setting* $\sigma_n(X,T,\alpha) = (X,T,\alpha\circ\sigma_n)$ *for all* $n \in \mathbb{N}$ *and* $\tilde{\rho}_n(X,T,\alpha) = (X \times Z_n, \sqcup_{a\in Z_n} T, \Phi_n(\alpha))$ *determines endofunctors of the assembler categories* $\mathcal{C}_{\mathcal{T}}^{\hat{\mathbb{Z}},a}$ *that induce, respectively, ring homomorphisms* $\sigma_n : K^{\hat{\mathbb{Z}}}(\mathcal{C}^a_{\mathcal{T}}) \to K^{\hat{\mathbb{Z}}}(\mathcal{C}^a_{\mathcal{T}})$ *and group homomorphisms* $\tilde{\rho}_n : K^{\hat{\mathbb{Z}}}(\mathcal{C}^a_{\mathcal{T}}) \to K^{\hat{\mathbb{Z}}}(\mathcal{C}^a_{\mathcal{T}})$ *with the Bost–Connes relations*

$$\tilde{\rho}_n \circ \sigma_n(X,T,\alpha) = (X,T,\alpha) \times (Z_n, T_0, \gamma) \qquad \sigma_n \circ \tilde{\rho}_n(X,T,\alpha) = (X,T,\alpha)^{\oplus n}.$$

Proof. The proof is completely analogous to the case discussed in Proposition 4.3 and to the similar cases discussed in [51]. □

Remark 5.4. Bost–Connes type quantum statistical mechanical systems associated to individual toric varieties (and more generally to varieties admitting torifications) were constructed in [41]. Here instead of Bost–Connes endomorphisms of individual varieties we are interested in a Bost–Connes system over the entire Grothendieck ring and its associated spectrum.

Remark 5.5. Variants of the construction above can be obtained by considering the multivariable versions of the Bost–Connes system discussed in [52], with actions of subsemigroups of $M_N(\mathbb{Z})^+$ on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes N}$, that is, subalgebras of the crossed product algebra

$$\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]^{\otimes N} \rtimes_\rho M_N(\mathbb{Z})^+$$

generated by $e(\underline{r})$ and μ_α, μ_α^* with

$$\rho_\alpha(e(\underline{r})) = \mu_\alpha e(\underline{r})\mu_\alpha^* = \frac{1}{\det\alpha} \sum_{\alpha(\underline{s})=\underline{r}} e(\underline{s})$$

$$\sigma_\alpha(e(\underline{r})) = \mu_\alpha^* e(\underline{r})\mu_\alpha = e(\alpha(\underline{r})).$$

The relevance of this more general setting to $\mathbb{F}_1$-geometries lies in a result of Borger and de Smit [13] showing that every torsion free finite rank Λ-ring embeds in some $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^{\otimes N}$ with the action of $\mathbb{N}$ determined by the Λ-ring structure compatible with the diagonal subsemigroup of $M_N(\mathbb{Z})^+$.

5.5. **Counting $\mathbb{F}_1$-points.** Assuming that a variety X over $\mathbb{Z}$ admits an $\mathbb{F}_1$-structure, regarded here as one of several possible forms of torified structure recalled above, [48], [50], the number of points of X over $\mathbb{F}_1$ is computed as the $q \to 1$ limit of the counting function $N_X(q)$ of points over $\mathbb{F}_q$ of the mod p reduction of X, for q a power of p. Any form of torified structure in particular implies that the variety is polynomially countable, hence that the counting function $N_X(q)$ is a polynomial in q with $\mathbb{Z}$-coefficients. The limit $\lim_{q\to 1} N_X(q)$, possibly normalized by a power of $q-1$, is interpreted as the number of $\mathbb{F}_1$-points of X, see [64]. Similarly, one can define "extensions" $\mathbb{F}_{1^m}$ of $\mathbb{F}_1$, in the sense of [42] (see also [26]). These corresponds to actions of the groups $\mathfrak{m}_m$ of m-th roots of unity. In terms of a torified structure, the points over $\mathbb{F}_{1^m}$ count m-th roots of unity in each torus of the decomposition. In terms of the counting function $N_X(q)$ the counting of points of X over the extension $\mathbb{F}_{1^m}$ is obtained as the value $N_X(m+1)$, see Theorem 4.10 of [23] and Theorem 1 of [28]). Summarizing, we have the following.

Lemma 5.6. *Let X be a variety over $\mathbb{Z}$ with torified Grothendieck class*

$$[X] = \sum_{i=0}^{N} a_i \mathbb{T}^i \tag{5.2}$$

with coefficients $a_i \in \mathbb{Z}_+$ and $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$. Then the number of points over $\mathbb{F}_{1^m}$ of X is given by

$$\#X(\mathbb{F}_{1^m}) = \sum_{i=0}^{N} a_i\, m^i. \tag{5.3}$$

In particular, $\#X(\mathbb{F}_1) = a_0 = \chi(X)$ the Euler characteristic.

5.6. **Bialynicki-Birula decompositions and torified geometries.** As shown in [5], [18], the motive of a smooth projective variety with action of the multiplicative group admits a decomposition, obtained via the method of Bialynicki-Birula, [9], [10], [11]. We recall the result here, in a particular case which gives rise to examples of torified varieties.

Lemma 5.7. *Let X be a smooth projective k-variety X endowed with a $\mathbb{G}_m$ action such that the fixed point locus $X^{\mathbb{G}_m}$ admits a torification of the Grothendieck class. Then X also admits a torification of the Grothendieck class. Consider the filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset \emptyset$ with affine fibrations $\phi_i : X_i \smallsetminus X_{i-1} \to Z_i$ over the components $X^{\mathbb{G}_m} = \sqcup_i Z_i$, associated to the Bialynicki-Birula decomposition. If the fixed point locus $X^{\mathbb{G}_m}$ admits a geometric torification such that the restrictions of the fibrations ϕ_i to the individual tori of the torification of Z_i are trivializable, then X also admits a geometric torification.*

Proof. The Bialynicki-Birula decomposition, [9], [10], [11], see also [39], shows that a smooth projective k-variety X endowed with a $\mathbb{G}_m$ action has smooth closed fixed point locus $X^{\mathbb{G}_m}$ which decomposes into a finite union of components $X^{\mathbb{G}_m} = \sqcup_i Z_i$, of dimensions $\dim Z_i$ the dimension of $TX_z^{\mathbb{G}_m}$ at $z \in Z_i$. The variety X has a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset \emptyset$ with affine fibrations $\phi_i : X_i \smallsetminus X_{i-1} \to Z_i$ of relative dimension d_i equal to the dimension of the positive eigenspace of the $\mathbb{G}_m$-action on the tangent space of X at points of Z_i. The scheme $X_i \smallsetminus X_{i-1}$ is identified with $\{x \in X : \lim_{t\to 0} tx \in Z_i\}$ under the $\mathbb{G}_m$-action $t : x \mapsto tx$, with $\phi_i(x) = \lim_{t\to 0} tx$. As shown in [18], the object $M(X)$ in the category of correspondences Corr_k with integral coefficients (and in the category of Chow motives) decomposes as

$$M(X) = \bigoplus_i M(Z_i)(d_i), \tag{5.4}$$

where $M(Z_i)$ are the motives of the components of the fixed point set and $M(Z_i)(d_i)$ are Tate twists. The class in the Grothendieck ring $K_0(\mathcal{V}_k)$ decomposes then as

$$[X] = \sum_i [Z_i] \mathbb{L}^{d_i}. \tag{5.5}$$

It is then immediate that, if the components Z_i admit a geometric torification (respectively, a torification of the Grothendieck class) then the variety X also does. If $Z_i = \cup_{j=1}^{n_i} T_{ij}$ with $T_{ij} = \mathbb{G}_m^{a_{ij}}$ or, respectively $[Z_i] = \sum_{j=1}^{n_i} (\mathbb{L}-1)^{a_{ij}}$, then $X = \cup_{i=0}^{n} (X_i \smallsetminus X_{i-1}) = \cup_{i=0}^{n} \mathcal{F}^{d_i}(Z_i)$, where $\mathcal{F}^{d_i}(Z_i)$ denotes the total space of the affine fibration $\phi_i : X_i \smallsetminus X_{i-1} \to Z_i$ with fibers $\mathbb{A}^{d_i}$. The Grothendieck class is then torified by

$$[X] = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \mathbb{T}^{a_{ij}} (1 + \sum_{k=1}^{d_i} \binom{d_i}{k} \mathbb{T}^k),$$

with $\mathbb{T} = \mathbb{L} - 1$ the class of the multiplicative group $\mathbb{T} = [\mathbb{G}_m]$, and where the affine spaces are torified by

$$\mathbb{L}^n - 1 = \sum_{k=1}^{n} \binom{n}{k} \mathbb{T}^k.$$

If the restriction of the fibration $\mathcal{F}^{d_i}(Z_i)$ to the individual tori T_{ij} of the torification of Z_i is trivial, then it can be torified by a products $T_{ij} \times T_k$ of the torus T_{ij} and the tori T_k of a torification of the fiber affine space $\mathbb{A}^{d_i}$. This determines a a geometric torification of the affine fibrations $\mathcal{F}^{d_i}(Z_i)$, hence of X. $\square$

5.7. **An example of torified varieties.** A physically significant example of torified varieties of the type described in Lemma 5.7 arises in the context of BPS state counting of [21]. Refined BPS state counting computes the multiplicities of BPS particles with charges in a lattice (K-theory changes of even D-branes) for assigned spin quantum numbers of a $\mathrm{Spin}(4) = SU(2) \times SU(2)$ representation, see [21], [22], [32].

We mention here the following explicit example from [22], namely the case of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1)$ of Gieseker semi-stable shaved on $\mathbb{P}^2$ with Hilbert polynomial equal to $4m+1$. In this case, it is proved in [22] that $\mathcal{M}_{\mathbb{P}^2}(4,1)$ has a torus action of $\mathbb{G}_M^2$ for which the fixed point locus consists of 180 isolated points and 6 components isomorphic to $\mathbb{P}^1$. The Grothendieck class, obtained through the Bialynicki-Birula decomposition [22] is given by

$$\begin{aligned} [\mathcal{M}_{\mathbb{P}^2}(4,1)] = {} & 1 + 2\mathbb{L} + 6\mathbb{L}^2 + 10\mathbb{L}^3 + 14\mathbb{L}^4 + 15\mathbb{L}^5 \\ & +16\mathbb{L}^6 + 16\mathbb{L}^7 + 16\mathbb{L}^8 + 16\mathbb{L}^9 + 16\mathbb{L}^{10} + 16\mathbb{L}^{11} \\ & +15\mathbb{L}^{12} + 14\mathbb{L}^{13} + 10\mathbb{L}^{14} + 6\mathbb{L}^{15} + 2\mathbb{L}^{16} + \mathbb{L}^{17}. \end{aligned}$$

Note that, for a smooth projective variety with Grothendieck class that is a polynomial in the Lefschetz motive $\mathbb{L}$, the Poincaré polynomial and the Grothendieck class are related by replacing x^2 with $\mathbb{L}$, since the variety is Hodge–Tate. In torified form the above gives

$$\begin{aligned} [\mathcal{M}_{\mathbb{P}^2}(4,1)] = {} & \mathbb{T}^{17} + 19\,\mathbb{T}^{16} + 174\,\mathbb{T}^{15} + 1020\,\mathbb{T}^{14} + 4284\,\mathbb{T}^{13} + 13665\,\mathbb{T}^{12} + 34230\,\mathbb{T}^{11} \\ & +68678\,\mathbb{T}^{10} + 111606\,\mathbb{T}^9 + 147653\,\mathbb{T}^8 + 159082\,\mathbb{T}^7 + 139008\,\mathbb{T}^6 \\ & +97643\,\mathbb{T}^5 + 54320\,\mathbb{T}^4 + 23370\,\mathbb{T}^3 + 7468\,\mathbb{T}^2 + 1632\,\mathbb{T} + 192, \end{aligned}$$

where $192 = \chi(\mathcal{M}_{\mathbb{P}^2}(4,1))$ is the Euler characteristics, which is also the number of points over $\mathbb{F}_1$. The number of points over $\mathbb{F}_{1^m}$ gives 864045 for $m = 1$ (the number of tori in the torification), 383699680 for $m = 2$ (roots of unity of order two), 36177267945 for $m = 3$ (roots of unity of order three), etc.

In this example, the Euler characteristic $\chi(\mathcal{M}_{\mathbb{P}^2}(4,1))$, which can also be seen as the number of $\mathbb{F}_1$-points, is interpreted physically as determining the BPS counting. It is natural to ask whether the counting of $\mathbb{F}_{1^m}$-points, which corresponds to the counting of roots of unity in the tori of the torification, can also carry physically significant information.

Other examples of torified varieties relevant to physics can be found in the context of quantum field theory, see [6] and [57].

5.7.1. *BPS counting and the virtual motive.* The formulation of the refined BPS counting given in [21] can be summarized as follows. The virtual motive $[X]_{\rm vir} = \mathbb{L}^{-n/2}[X]$, with $n = \dim(X)$, is a class in the ring of motivic weights $K_0(\mathcal{V})[\mathbb{L}^{-1/2}]$, see [12]. When X admits a $\mathbb{G}_m$ action and a Bialynicki-Birula decomposition as discussed in the previous section, where all the components Z_i of the fixed point locus of the $\mathbb{G}_m$-action have Tate classes $[Z_i] = \sum_j c_{ij}\mathbb{L}^{b_{ij}} \in K_0(\mathcal{V})$, with $c_{ij} \in \mathbb{Z}$ and $b_{ij} \in \mathbb{Z}_+$, the virtual motive $[X]_{\rm vir}$ is a Laurent polynomial in the square root $\mathbb{L}^{1/2}$ of the Lefschetz motive,

$$[X]_{\rm vir} = \sum_{i,j} c_{ij}\, \mathbb{L}^{b_{ij}+d_i-1/2},$$

where, as before, d_i is the dimension of the positive eigenspace of the $\mathbb{G}_m$-action on the tangent space of X at points of Z_i. In applications to BPS counting, one considers the virtual motive of a moduli space M that admits a perfect obstruction theory, so that it has virtual dimension zero and an associated invariant $\#_{\rm vir}M$ which is computed by a virtual index

$$\#_{\rm vir}M = \chi_{\rm vir}(M, K^{1/2}_{M,{\rm vir}}) = \chi(M, K^{1/2}_{M,{\rm vir}} \otimes \mathcal{O}_{M,{\rm vir}}),$$

where $\mathcal{O}_{M,{\rm vir}}$ is the virtual structure sheaf and $K^{1/2}_{M,{\rm vir}}$ is a square root of the virtual canonical bundle, see [34].

The formal square root $\mathbb{L}^{1/2}$ of the Leftschetz motive can be introduced, at the level of the category of motives, as shown in §3.4 of [45], using the Tannakian formalism, [29]. Given $\mathcal{C} = {\rm Num}^{\dagger}_{\mathbb{Q}}$, the Tannakian category of pure motives with the numerical equivalence relation and the Koszul sign rule twist $\dagger$ in the tensor structure, with motivic Galois group $G = {\rm Gal}(\mathcal{C})$. The inclusion of the Tate motives (with motivic Galois group $\mathbb{G}_m$) determines a group homomorphism $t : G \to \mathbb{G}_m$, which satisfies $t \circ w = 2$ with the weight homomorphism $w : \mathbb{G}_m \to G$ (see §5 of [30]). The category $\mathcal{C}(\mathbb{Q}(\frac{1}{2}))$ obtained by adjoining a square root of the Tate motive to $\mathcal{C}$ is then obtained as the Tannakian category whose Galois group is the fibered product

$$G^{(2)} = \{(g,\lambda) \in G \times \mathbb{G}_m \,:\, t(g) = \lambda^2\}.$$

The construction of square roots of Tate motives described in [45] was generalized in [46] to arbitrary n-th roots of Tate motives, obtained via the same Tannakian construction, with the category $\mathcal{C}(\mathbb{Q}(\frac{1}{n}))$ obtained by adjoining an n-th root of the Tate motive determined by its Tannakian Galois group

$$G^{(n)} = \{(g,\lambda) \in G \times \mathbb{G}_m \,:\, t(g) = \sigma_n(\lambda)\},$$

with $\sigma_n : \mathbb{G}_m \to \mathbb{G}_m$, $\sigma_n(\lambda) = \lambda^n$. The category $\hat{\mathcal{C}}$ obtained by adjoining to $\mathcal{C} = \mathrm{Num}^{\dagger}_{\mathbb{Q}}$ arbitrary roots of the Tate motives is the Tannakian category with Galois group $\hat{G} = \varprojlim_n G^{(n)}$. The category $\hat{\mathcal{C}}$ has an action of $\mathbb{Q}^*_+$ by automorphisms induced by the endomorphisms σ_n of $\mathbb{G}_m$. These roots of Tate motives give rise to a good formalism of $\mathbb{F}_\zeta$-geometry, with ζ a root of unity, lying "below" $\mathbb{F}_1$-geometry and expressed at the motivic level in terms of a Habiro ring type object associated to the Grothendieck ring of orbit categories of $\hat{\mathcal{C}}$, see [46].

6. Torified varieties and zeta functions

We discuss in this section the connection between the dynamical point of view on $\mathbb{F}_1$-geometry proposed in [51] and the point of view based on torifications.

6.1. **Counting $\mathbb{F}_1$-points and zeta function.** For a variety X over $\mathbb{Z}$ that is polynomially countable (that is, the counting functions $N_X(q) = \#X_p(\mathbb{F}_q)$ with X_p the mod p reduction is a polynomial in q with $\mathbb{Z}$ coefficients) the counting of points over the "extensions" $\mathbb{F}_{1^m}$ (in the sense of [42]) can be obtained as the values $N_X(m+1)$ (see Theorem 4.10 of [23] and Theorem 1 of [28]). As we discussed earlier, in the case of a torified variety, with Grothendieck class $[X] = \sum_{i\geq 0} a_i \mathbb{T}^i$ with $a_i \in \mathbb{Z}_+$, this corresponds to the counting given in (5.3). This is the counting of the number of m-th roots of unity in each torus $\mathbb{T}^i = [\mathbb{G}_m^i]$ of the torification.

For a variety X over a finite field $\mathbb{F}_q$ the Hasse–Weil zeta function is given, in logarithmic form by

$$\log Z_{\mathbb{F}_q}(X,t) = \sum_{m\geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m. \tag{6.1}$$

In the case of torified varieties, there is an analogous zeta function over $\mathbb{F}_1$. We think of this $\mathbb{F}_1$-zeta function as defined on torified Grothendieck classes, $Z_{\mathbb{F}_1}([X],t)$. In the case of geometric torifications, we can regard it as a function of the variety and the torification, $Z_{\mathbb{F}_1}((X,T),t)$. For simplicity of notation, we will simply write $Z_{\mathbb{F}_1}(X,t)$ by analogy to the Hasse–Weil zeta function.

Lemma 6.1. *Let X be a variety over $\mathbb{Z}$ with a torified Grothendieck class $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$ with $a_k \in \mathbb{Z}_+$. Then the $\mathbb{F}_1$-zeta function is given by*

$$\log Z_{\mathbb{F}_1}(X,t) = \sum_{k=0}^{N} a_k \,\mathrm{Li}_{1-k}(t), \tag{6.2}$$

where $\mathrm{Li}_s(t)$ *is the polylogarithm function with* $\mathrm{Li}_1(t) = -\log(1-t)$ *and for* $k \geq 1$

$$\mathrm{Li}_{1-k}(t) = (t\frac{d}{dt})^{k-1}\frac{t}{1-t}.$$

Proof. For $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$ with $a_k \in \mathbb{Z}_+$ as above, we can consider a similar zeta function based on the counting of $\mathbb{F}_{1^m}$-points described above. Using (5.3), we obtain an expression of the form

$$\log Z_{\mathbb{F}_1}(X,t) = \sum_{m\geq 1} \frac{\#X(\mathbb{F}_{1^m})}{m} t^m = \sum_{k=0}^{N} a_k \sum_{m\geq 1} m^{k-1} t^m = \sum_{k=0}^{N} a_k \,\mathrm{Li}_{1-k}(t),$$

given by a linear combination of polylogarithm functions $\mathrm{Li}_s(t)$ at integer values $s \leq 1$. Such polylogarithm functions can be expressed explicitly in the form $\mathrm{Li}_1(t) = -\log(1-t)$ and for $k \geq 1$

$$\mathrm{Li}_{1-k}(t) = (t\frac{d}{dt})^{k-1}\frac{t}{1-t} = \sum_{\ell=0}^{k-1} \ell!\, S(k,\ell+1)\left(\frac{t}{1-t}\right)^{\ell+1},$$

with $S(k,r)$ the Stirling numbers of the second kind

$$S(k,r) = \frac{1}{r!}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} j^k.$$

$\square$

As in the case of the Hasse–Weil zeta function over $\mathbb{F}_q$ (see [59]), the $\mathbb{F}_1$-zeta function gives an exponentiable motivic measure.

Proposition 6.2. *The* $\mathbb{F}_1$*-zeta function is an exponentiable motivic measure, that is, a ring homomorphism* $Z_{\mathbb{F}_1} : K_0(\mathcal{T})^a \to W(\mathbb{Z})$ *from the Grothendieck ring of torified varieties (with either* $a = w, o, s$*) to the Witt ring.*

Proof. Clearly with respect to addition in the Grothendieck ring of torified varieties we have $[X] + [X'] = \sum_{i\geq 0} a_i \mathbb{T}^i + \sum_{j\geq 0} a'_j \mathbb{T}^j = \sum_{k\geq 0} b_k \mathbb{T}^k$ with $b_k = a_k + a'_k$, hence

$$\log Z_{\mathbb{F}_1}([X]+[X'],t) = \sum_{k=0}^{N} b_k \,\mathrm{Li}_{1-k}(t) = \log Z_{\mathbb{F}_1}([X],t) + \log Z_{\mathbb{F}_1}([X'],t).$$

The behavior with respect to products $[X]\cdot[Y]$ in the Grothendieck ring of torified varieties can be analyzed as in [59] for the Hasse–Weil zeta function. We view the $\mathbb{F}_1$-zeta function

$$Z_{\mathbb{F}_1}(X,t) = \exp(\sum_{k=0}^{N} a_k \,\mathrm{Li}_{1-k}(t))$$

as the element in the Witt ring $W(\mathbb{Z})$ with ghost components $\#X(\mathbb{F}_{1^m}) = \sum_{k=0}^{N} m^k$, by writing the ghost map $\mathrm{gh} : W(\mathbb{Z}) \to \mathbb{Z}^{\mathbb{N}}$ as

$$\mathrm{gh} : Z(t) = \exp(\sum_{m\geq 1}\frac{N_m}{m} t^m) \mapsto t\frac{d}{dt}\log Z(t) = \sum_{m\geq 1} N_m t^m \mapsto (N_m)_{m\geq 1}.$$

The ghost map is an injective ring homomorphism. Thus, it suffices to see that on the ghost components $N_m(X \times Y) = N_m(X) \cdot N_m(Y)$. If $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$ and $[Y] = \sum_{\ell\geq 0} b_\ell \mathbb{T}^\ell$ then $[X \times Y] = \sum_{n\geq 0}\sum_{k+\ell=n} a_k b_\ell \mathbb{T}^n$ and $N_m(X \times Y) = \sum_{n\geq 0}\sum_{k+\ell=n} a_k b_\ell m^n = N_m(X) \cdot N_m(Y)$. □

6.2. **Dynamical zeta functions.** The dynamical approach to $\mathbb{F}_1$-structures proposed in [51] is based on the existence if an endomorphism $f : X \to X$ that induces a quasi-unipotent morphism f_* on the homology $H_*(X,\mathbb{Z})$. In particular, this means that the map f_* has eigenvalues that are roots of unity.

In the case of a variety X endowed with a torification $X = \sqcup_i T^{d_i}$, one can consider in particular endomorphisms $f : X \to X$ that preserve the torification and that restrict to endomorphisms of each torus T^{d_i}.

In general to a self-map $f : X \to X$, one can associate the dynamical Artin–Mazur zeta function and the homological Lefschetz zeta function. A particular class of maps with the property that they induce quasi-unipotent morphisms in homology is given by the Morse–Smale diffeomorphisms of smooth manifolds, see [63]. These are diffeomorphisms characterized by the properties that the set of nonwandering points is finite and hyperbolic, consisting of a finite number of periodic points, and for any pair of these points x, y the stable and unstable manifolds $W^s(x)$ and $W^u(y)$ intersect transversely. Morse–Smale diffeomorphisms are structurally stable among all diffeomorphisms, [35], [63].

The Lefschetz zeta function

$$\zeta_{\mathcal{L},f}(t) = \exp\left(\sum_{m\geq 1}\frac{L(f^m)}{m}t^m\right), \tag{6.3}$$

with $L(f^m)$ the Lefschetz number of the m-th iterate f^m,

$$L(f^m) = \sum_{k\geq 0}(-1)^k \mathrm{Tr}((f^m)_* \,|\, H_k(X,\mathbb{Q})),$$

which for a function with finitely many fixed points is also equal to

$$L(f^m) = \sum_{x\in \mathrm{Fix}(f^m)} \mathcal{I}(f^m, x),$$

with $\mathcal{I}(f^m, x)$ the index of the fixed point. This is a rational function of the form

$$\zeta_{\mathcal{L},f}(t) = \prod_k \det(1 - t\, f_*|H_k(X,\mathbb{Q}))^{(-1)^{k+1}}.$$

In the case of a map f with finitely many periodic points, all hyperbolic, the Lefschetz zeta function can be equivalently written (see [35]) as the rational function

$$\zeta_{\mathcal{L},f}(t) = \prod_\gamma (1 - \Delta_\gamma t^{p(\gamma)})^{(-1)^{u(\gamma)+1}},$$

with the product over periodic orbits γ with least period $p(\gamma)$ and with $u(\gamma) = \dim E^u_x$ for $x \in \gamma$, the dimension of the span of eigenvectors of $Df_x^{p(\gamma)} : T_xM \to T_xM$ with eigenvalues λ with $|\lambda| > 1$, and $\Delta_\gamma = \pm 1$ according to whether $Df_x^{p(\gamma)}$ is orientation preserving or reversing. The relation comes from the identity $\mathcal{I}(f^m, x) = (-1)^{u(\gamma)}\Delta_\gamma$. The Artin–Mazur zeta function is given by

$$\zeta_{AM,f}(t) = \exp\left(\sum_{m\geq 1} \frac{\#\mathrm{Fix}(f^m)}{m} t^m\right). \tag{6.4}$$

The case of Morse–Smale diffeomorphisms can be treated as in [36] to obtain rationality and a description in terms of the homological zeta functions.

In the setting of real tori $\mathbb{R}^d/\mathbb{Z}^d$, one can considers the case of a toral endomorphism specified by a matrix $M \in M_d(\mathbb{Z})$. In the hyperbolic case, the counting of isolated fixed points of M^m is given by $|\det(1 - M^m)|$ and the dynamical Artin–Mazur zeta function is expressible in terms of the Lefschetz zeta function, associated to the signed counting of fixed points, through the fact that the Lefschetz zeta function agrees with the zeta function

$$\zeta_M(t) = \exp(\sum_{n\geq 1} \frac{t^n}{n} a_n), \quad \text{with} \quad a_n = \det(1 - M^n), \tag{6.5}$$

where $a_n = \det(1-M^n)$ is a signed fixed point counting. The general relation between the zeta functions for the signed $\det(1 - M^n)$ and for $|\det(1 - M^m)|$ is shown in [4] for arbitrary toral endomorphisms, with $M \in M_d(\mathbb{Z})$.

In the case of complex algebraic tori $T^d = \mathbb{G}_m^d(\mathbb{C})$, one can similarly consider the endomorphisms action of the semigroup of matrices $M \in M_d(\mathbb{Z})^+$ by the linear action on $\mathbb{C}^d$ preserving $\mathbb{Z}^d$ and the exponential map $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$ so that, for $M = (m_{ab})$ and $\lambda_a = \exp(2\pi i u_a)$, with the action given by

$$\lambda = (\lambda_a) \mapsto M(\lambda) = \exp(2\pi i \sum_b m_{ab} u_b).$$

The subgroup $\mathrm{SL}_n(\mathbb{Z}) \subset M_n(\mathbb{Z})^+$ acts by automorphisms. These generalize the Bost–Connes endomorphisms $\sigma_n : \mathbb{G}_m \to \mathbb{G}_m$, which correspond to the ring homomorphisms of $\mathbb{Z}[t, t^{-1}]$ given by $\sigma_n : P(t) \mapsto P(t^n)$ and determine multivariable versions of the Bost–Connes algebra, see [52]. We can consider in this way maps of complex algebraic tori $T^d_{\mathbb{C}} = \mathbb{G}_m^d(\mathbb{C})$ that induce maps of the real tori obtained as the subrgroup $T^d_{\mathbb{R}} = U(1)^d \subset \mathbb{G}_m^d(\mathbb{C})$, and associate to these maps the Lefschetz and Artin–Mazur zeta functions of the induced map of real tori.

In the case of a variety with a torification, we consider endomorphisms $f : X \to X$ that preserves the tori of the torification and restricts to each torus T^{d_i} to a diffeomorphism $f_i : T^{d_i}_{\mathbb{R}} \to T^{d_i}_{\mathbb{R}}$. In particular, we consider toral endomorphism with matrix $M_i \in M_{d_i}(\mathbb{Z})$, we can associate to the pair (X, f) a zeta function of the form

$$\zeta_{\mathcal{L},f}(X, t) = \prod_i \zeta_{\mathcal{L},f_i}(t), \qquad \zeta_{AM,f}(X, t) = \prod_i \zeta_{AM,f_i}(t). \tag{6.6}$$

Proposition 6.3. *The zeta functions* (6.3) *and* (6.4) *define exponentiable motivic measures on the Grothendieck ring* $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ *of §6 of* [51] *with values in the Witt ring* $W(\mathbb{Z})$. *The zeta functions* (6.6) *define exponentiable motivic measures on the Grothendieck ring* $K_0(\mathcal{T})^a$ *of torified varieties with values in* $W(\mathbb{Z})$.

Proof. The Grothendieck ring $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ considered in §6 of [51] consists of pairs (X,f) of a complex quasi-projective variety and an automorphism $f : X \to X$ that induces a quasi-uniponent map f_* in homology. The addition is simply given by the disjoin union, and both the counting of periodic points $\#\mathrm{Fix}(f^m)$ and the Lefschetz numbers $L(f^m)$ behave additively under disjoint unions. Thus, the zeta functions $\zeta_{\mathcal{L},f}(t)$ and $\zeta_{AM,f}(t)$, seen as elements in the Witt ring $W(\mathbb{Z})$ add

$$\zeta_{\mathcal{L},f_1\sqcup f_2}(t) = \exp\left(\sum_{m\geq 1} \frac{L((f_1\sqcup f_2)^m)}{m} t^m\right) =$$

$$\exp\left(\sum_{m\geq 1} \frac{L(f_1^m)}{m} t^m\right) \cdot \exp\left(\sum_{m\geq 1} \frac{L(f_2^m)}{m} t^m\right) = \zeta_{\mathcal{L},f_1}(t) +_W \zeta_{\mathcal{L},f_2}(t)$$

and similarly for $\zeta_{AM,f_1\sqcup f_2}(t) = \zeta_{AM,f_1}(t) +_W \zeta_{AM,f_2}(t)$. The product is given by the Cartesian product $(X_1,f_1)\times(X_2,f_2)$. Since $\mathrm{Fix}((f_1\times f_2)^m) = \mathrm{Fix}(f_1^m)\times\mathrm{Fix}(f_2^m)$ and the same holds for Lefschetz numbers since

$$L((f_1\times f_2)^m) = \sum_{k\geq 0}\sum_{\ell+r=k} (-1)^{\ell+r}\mathrm{Tr}((f_1^m)_*\otimes(f_2^m)_* \,|\, H_\ell(X_1,\mathbb{Q})\otimes H_r(X_2,\mathbb{Q}))$$

which gives $L(f_1^m)\cdot L(f_2^m)$. Thus, we can use as in Proposition 6.2 the fact that the ghost map $\mathrm{gh} : W(\mathbb{Z}) \to \mathbb{Z}^{\mathbb{N}}$

$$\mathrm{gh} : \exp(\sum_{m\geq 1} \frac{N_m}{m} t^m) \mapsto \sum_{m\geq 1} N_m t^m \mapsto (N_m)_{m\geq 1}$$

is an injective ring homomorphism to obtain the multiplicative property. The case of the torified varieties and the zeta functions (6.6) is analogous, combining the additive and multiplicative behavior of the fixed point counting and the Lefschetz numbers on each torus and of the decomposition into tori as in Proposition 6.2. ☐

In the case of quasi-unipotent maps of tori the Lefschetz zeta function can be computed completely explicitly. Indeed, it is shown in [7], [8] that, for a quasi-unipotent self map $f : T^n_{\mathbb{R}} \to T^n_{\mathbb{R}}$, the Lefschetz zeta function has an explicit form that is completely determined by the map on the first homology. Under the quasi-unipotent assumption all the eigenvalues of the induced map on H_1 are roots of unity, hence the characteristic polynomial $\det(1 - t\, f_*|H_1(X))$ is a product of cyclotomic polynomials $\Phi_{m_1}(t)\cdots\Phi_{m_N}(t)$ where

$$\Phi_m(t) = \prod_{d|m} (1-t^d)^{\mu(m/d)},$$

with $\mu(n)$ the Möbius function. It is shown in [8] that the Lefschetz zeta function has the form

$$\zeta_{\mathcal{L},f}(t) = \prod_{d|m} (1-t^d)^{-s_d}, \tag{6.7}$$

where $m = \mathrm{lcm}\{m_1, \ldots, m_N\}$ and

$$s_d = \frac{1}{d} \sum_{k|d} F_k \, \mu(d/k)$$

$$F_k = \prod_{i=1}^{N} (\Phi_{m_i/(k,m_i)}(1))^{\varphi(m_i)/\varphi(m_i/(k,m_i))}$$

where the Euler function

$$\varphi(m) = m \prod_{p|m,\, p \text{ prime}} (1-p^{-1})$$

is the degree of $\Phi_m(t)$.

Remark 6.4. The properties of dynamical Artin–Mazur zeta functions change significantly when, instead of considering varieties over $\mathbb{C}$ one considers varieties in positive characteristic, [16], [19]. The prototype model of this phenomenon is illustrated by considering the Bost–Connes endomorphisms $\sigma_n : \lambda \mapsto \lambda^n$ of $\mathbb{G}_m(\bar{\mathbb{F}}_p)$. In this case, the dynamical zeta function of σ_n is rational or transcendental depending on whether p divides n (Theorem 1.2 and 1.3 and §3 of [16] and Theorem 1 of [17]). Similar phenomena in the more general case of endomorphisms of Abelian varieties in positive characteristic have been investigated in [19]. In the positive characteristic setting, where one is considering the characteristic p version of the Bost–Connes system of [26], one should then replace the dynamical zeta function by the tame zeta function considered in [19].

7. Spectra and zeta functions

A motivic measure is a ring homomorphism $\mu : K_0(\mathcal{V}) \to R$, from the Grothendieck ring of varieties $K_0(\mathcal{V})$ to a commutative ring R. Examples include the counting measure, for varieties defined over finite fields, which counts the number of algebraic points over $\mathbb{F}_q$, the topological Euler characteristic or the Hodge–Deligne polynomials for complex algebraic varieties.

The Kapranov motivic zeta function [43] is defined as $\zeta(X,t) = \sum_{n=0}^{\infty} [S^n(X)] t^n$, where $S^n(X) = X^n/S_n$ are the symmetric products of X and $[S^n(X)]$ are the classes in $K_0(\mathcal{V})$. Similarly, the zeta function of a motivic measured is defined as

$$\zeta_\mu(X,t) = \sum_{n=0}^{\infty} \mu(S^n(X))\, t^n. \tag{7.1}$$

It is viewed as an element in the Witt ring $W(R)$. The addition in $K_0(\mathcal{V})$ is mapped by the zeta function to the addition in $W(R)$, which is the usual product of the power series,

$$\zeta_\mu(X \sqcup Y, t) = \zeta_\mu(X,t) \cdot \zeta_\mu(Y,t) = \zeta_\mu(X,t) +_{W(R)} \zeta_\mu(Y,t). \tag{7.2}$$

The motivic measure $\mu : K_0(\mathcal{V}) \to R$ is said to be exponentiable (see [59], [60]) if the zeta function (7.1) defines a ring homomorphism

$$\zeta_\mu : K_0(\mathcal{V}) \to W(R),$$

that is, if in addition to (7.2) one also has

$$\zeta_\mu(X \times Y, t) = \zeta_\mu(X,t) \star_{W(R)} \zeta_\mu(Y,t). \tag{7.3}$$

We investigate here how to lift the zeta functions of exponentiable motivic measures to the level of spectra. To this purpose, we first investigate how to construct a spectrum whose π_0 is a dense subring $W_0(R)$ of the Witt ring $W(R)$ and then we consider how to lift the ring homomorphisms given by zeta functions ζ_μ of exponentiable measures with a rationality and a factorization condition.

7.1. **The Endomorphism Category.** Let R be a commutative ring. We denote by $\mathcal{E}_R$ the endomorphism category of R, which is defined as follows (see [1], [2], [31]).

Definition 7.1. *The category $\mathcal{E}_R$ has objects given by the pairs (E, f) of a finite projective module E over R and an endomorphism $f \in \mathrm{End}_R(E)$, and morphisms given by morphisms $\phi : E \to E'$ of finite projective modules that commute with the endomorphisms, $f' \circ \phi = \phi \circ f$. The endomorphism category has direct sum $(E,f) \oplus (E', f') = (E \oplus E', f \oplus f')$ and tensor product $(E,f) \otimes (E',f') = (E \otimes E', f \otimes f')$.*

The category of finite projective modules over R is identified with the subcategory corresponding to the objects $(E, 0)$ with trivial endomorphism.

An exact sequence in $\mathcal{E}_R$ is a sequence of objects and morphisms in $\mathcal{E}_R$ which is exact as a sequence of finite projective modules over R (forgetting the endomorphisms). This determines a collection of admissible short exact sequence (and of admissible monomorphisms and epimorphisms). The endomorphism category $\mathcal{E}_R$ is then an exact category, hence it has an associated K-theory defined via the Quillen Q-construction, [58]. This assigns to the exact category $\mathcal{E}_R$ the category $\mathcal{Q}\mathcal{E}_R$ with the same objects and morphisms $\mathrm{Hom}_{\mathcal{Q}\mathcal{E}_R}((E,f),(E',f'))$ given by diagrams

$$\begin{tikzcd} & (E'', f'') \arrow[dl] \arrow[dr] & \\ (E, f) & & (E', f'), \end{tikzcd}$$

where the first arrow is an admissible epimorphism and the second an admissible monomorphism, with composition given by pullback. By the Quillen construction K-theory of $\mathcal{E}_R$ is then $K_{n-1}(\mathcal{E}_R) = \pi_n(\mathcal{N}(\mathcal{Q}\mathcal{E}_R))$, with $\mathcal{N}(\mathcal{Q}\mathcal{E}_R)$ the nerve of $\mathcal{Q}\mathcal{E}_R$.

The forgetful functor $(E, f) \mapsto E$ induces a map on K-theory

$$K_n(\mathcal{E}_R) \to K_n(\mathcal{P}_R) = K_n(R),$$

which is a split surjection. Let

$$\mathcal{E}_n(R) := \mathrm{Ker}(K_n(\mathcal{E}_R) \to K_n(R)).$$

In the case of K_0 an explicit description is given by the following, [1], [2]. Let $K_0(\mathcal{E}_R)$ denote the K_0 of the endomorphism category $\mathcal{E}_R$. It is a ring with the product structure induced by the tensor product. It is proved in [1], [2] that the quotient

$$W_0(R) = K_0(\mathcal{E}_R)/K_0(R) \tag{7.4}$$

embeds as a dense subring of the big Witt ring $W(R)$ via the map

$$L : (E, f) \mapsto \det(1 - t\, M(f))^{-1}, \tag{7.5}$$

with $M(f)$ the matrix associated to $f \in \mathrm{End}_R(E)$, where $\det(1-t\, M(f))^{-1}$ is viewed as an element in $\Lambda(R) = 1 + tR[[t]]$. As a subring $W_0(R) \hookrightarrow W(R)$ of the big Witt ring, $W_0(R)$ consists of the rational Witt vectors

$$W_0(R) = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} \mid a_i, b_i \in R,\ n, m \geq 0 \right\}.$$

Equivalently, one can consider the ring $\mathcal{R} = (1 + tR[t])^{-1}R[t]$ and identify the above with $1+t\mathcal{R}$, where the multiplication in $1+t\mathcal{R}$ corresponds to the addition in the Witt ring, and the Witt product is determined by the identity $(1-at)\star(1-bt) = (1-abt)$.

This description of Witt rings in terms of endomorphism categories was applied to investigate the arithmetic structures of the Bost–Connes quantum statistical mechanical system, see [23], [53], [54].

This relation between the Grothendieck ring and Witt vectors was extended to the higher K-theory in [37], where an explicit description for the kernels $\mathcal{E}_n(R)$ is obtained, by showing that

$$\mathcal{E}_{n-1}(R) = \mathrm{Coker}(K_n(R) \to K_n(\mathcal{R})),$$

where $\mathcal{R} = (1 + tR[t])^{-1}R[t]$ and $K_n(R) \to K_n(\mathcal{R})$ is a split injection. The identification above is obtained in [37] by showing that there is an exact sequence

$$0 \to K_n(R) \to K_n(\mathcal{R}) \to K_{n-1}(\mathcal{E}_R) \to K_{n-1}(R) \to 0. \tag{7.6}$$

The identification (7.4) for K_0 is then recovered as the case with $n = 0$ that gives an identification $\mathcal{E}_0(R) \simeq 1 + t\mathcal{R}$.

7.2. **From categories to Γ-spaces.** The Segal construction [62] associates a Γ-space (hence a spectrum) to a category $\mathcal{C}$ with a zero object and a categorical sum. Let Γ^0 be the category of finite pointed sets with objects $\underline{n} = \{0, 1, \dots, n\}$ and morphisms $f : \underline{n} \to \underline{m}$ the functions with $f(0) = 0$. Let Δ_* denote the category of pointed simplicial sets. The construction can be generalized to symmetric monoidal categories, [65]. The associated Γ-space $F_{\mathcal{C}} : \Gamma^0 \to \Delta_*$ is constructed as follows. First assign to a finite pointed set X the category $P(X)$ with objects all the pointed subsets of X with morphisms given by inclusions. A functor $\Phi_X : P(X) \to \mathcal{C}$ is summing if it maps $\emptyset \in P(X)$ to the zero object of $\mathcal{C}$ and given $S, S' \in P(X)$ with $S \cap S' = \{\star\}$ the base point of X, the morphism $\Phi_X(S) \oplus \Phi_X(S') \to \Phi_X(S \cup S')$ is an isomorphism. Let $\Sigma_{\mathcal{C}}(X)$ be the category whose objects are the summing functors Φ_X with morphisms the natural transformations that are isomorphisms on all objects of $P(X)$. Setting

$$\Sigma_{\mathcal{C}}(f)(\Phi_X)(S) = \Phi_X(f^{-1}(S)),$$

for a morphisms $f : X \to Y$ of pointed sets and $S \in P(Y)$ gives a functor $\Sigma_{\mathcal{C}} : \Gamma^0 \to$ Cat to the category of small categories. Composing with the nerve $\mathcal{N}$ gives a functor

$$F_{\mathcal{C}} = \mathcal{N} \circ \Sigma_{\mathcal{C}} : \Gamma^0 \to \Delta_*$$

which is the Γ-space associated to the category $\mathcal{C}$. The functor $F_{\mathcal{C}} : \Gamma^0 \to \Delta_*$ obtained in this way is extended to an endofunctor $F_{\mathcal{C}} : \Delta_* \to \Delta_*$ via the coend

$$F_{\mathcal{C}}(K) = \int^{\underline{n}} K^n \wedge F_{\mathcal{C}}(\underline{n}).$$

One obtains the spectrum $\mathbb{X} = F_{\mathcal{C}}(\mathbb{S})$ associated to the Γ-space $F_{\mathcal{C}}$ by setting $\mathbb{X}_n = F_{\mathcal{C}}(S^n)$ with maps $S^1 \wedge F_{\mathcal{C}}(S^n) \to F_{\mathcal{C}}(S^{n+1})$. The construction is functorial in $\mathcal{C}$, with respect to functors preserving sums and the zero object.

When $\mathcal{C}$ is the category of finite sets, $F_{\mathcal{C}}(\mathbb{S})$ is the sphere spectrum $\mathbb{S}$, and when $\mathcal{C} = \mathcal{P}_R$ is the category of finite projective modules over a commutative ring R, the spectrum $F_{\mathcal{P}_R}(\mathbb{S}) = K(R)$ is the K-theory spectrum of R.

The Segal construction determines a functor from the category of small symmetric monoidal categories to the category of -1-connective spectra. It is shown in [65] that this functor determines an equivalence of categories between the stable homotopy category of -1-connective spectra and a localization of the category of small symmetric monoidal categories, obtained by inverting morphisms sent to weak homotopy equivalences by the functor.

For another occurrence of Γ-spaces in the context of $\mathbb{F}_1$-geometry, see [25].

7.3. **Spectrum of the Endomorphism Category and Witt vectors.** Let R be a commutative ring with unit and let $\mathcal{P}_R$ denote the category of finite projective modules over R. Also let $\mathcal{E}_R$ be the endomorphism category recalled above. By the Segal construction above we obtain associated Γ-spaces $F_{\mathcal{P}_R}$ and $F_{\mathcal{E}_R}$ and spectra $F_{\mathcal{P}_R}(\mathbb{S}) = K(R)$, the K-theory spectrum of R, and $F_{\mathcal{E}_R}(\mathbb{S})$, the spectrum of the endomorphism category.

We obtain in the following way a functorial "spectrification" of the Witt ring $W_0(R)$, namely a spectrum $\mathbb{W}(R)$ with $\pi_0\mathbb{W}(R) = W_0(R)$.

Lemma 7.2. *For a commutative ring R, the inclusion of the category $\mathcal{P}_R$ of finite projective modules as the subcategory of the endomorphism category $\mathcal{E}_R$ determines a long exact sequence*

$$\cdots \to \pi_n(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_{n-1}(F_{\mathcal{P}_R}(\mathbb{S})) \to \cdots$$
$$\cdots \to \pi_0(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S}))$$

of the homotopy groups of the spectra $F_{\mathcal{P}_R}(\mathbb{S})$, $F_{\mathcal{E}_R}(\mathbb{S})$ and the cofiber $\mathbb{W}(R) := F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})$ obtained from the Γ-spaces $F_{\mathcal{P}_R} : \Gamma^0 \to \Delta_*$ and $F_{\mathcal{E}_R} : \Gamma^0 \to \Delta_*$ associated to the categories $\mathcal{P}_R$ and $\mathcal{E}_R$. The spectrum $\mathbb{W}(R)$ satisfies $\pi_0\mathbb{W}(R) = W_0(R)$.

Proof. The functoriality of the Segal construction implies that the inclusion of $\mathcal{P}_R$ as the subcategory of $\mathcal{E}_R$ given by objects $(E, 0)$ with trivial endomorphism determines a map of Γ-spaces $F_{\mathcal{P}_R} \to F_{\mathcal{E}_R}$, which is a natural transformation of the functors $F_{\mathcal{P}_R} : \Gamma^0 \to \Delta_*$ and $F_{\mathcal{E}_R} : \Gamma^0 \to \Delta_*$. After passing to endofunctors $F_{\mathcal{P}_R} : \Delta_* \to \Delta_*$ and $F_{\mathcal{E}_R} : \Delta_* \to \Delta_*$ we obtain a map of spectra $K(R) \to F_{\mathcal{E}_R}(\mathbb{S})$, induced by the inclusion of $\mathcal{P}_R$ as subcategory of $\mathcal{E}_R$. The category Δ_* of simplicial sets has products and equalizers, hence pullbacks. Thus, given two functors $F, F' : \Gamma^0 \to \Delta_*$, a natural transformation $\alpha : F \to F'$ is mono if and only if for all objects $X \in \Gamma^0$ the morphism $\alpha_X : F(X) \to F'(X)$ is a monomorphism in Δ_*. An embedding $\mathcal{C} \hookrightarrow \mathcal{C}'$ determines by composition an embedding $\Sigma_{\mathcal{C}}(X) \hookrightarrow \Sigma_{\mathcal{C}'}(X)$ of the categories of summing functors, for each object $X \in \Gamma^0$. This gives a monomorphism $F_{\mathcal{C}}(X) = \mathcal{N}\Sigma_{\mathcal{C}}(X) \to F_{\mathcal{C}'}(X) = \mathcal{N}\Sigma_{\mathcal{C}'}(X)$, hence a monomorphism $F_{\mathcal{C}} \to F_{\mathcal{C}'}$ of Γ-spaces. Arguing as in Lemma 1.3 of [61] we then obtain from such a map $F_{\mathcal{C}} \to F_{\mathcal{C}'}$ of Γ-spaces a long exact sequence of homotopy groups of the associated spectra

$$\cdots \to \pi_n(F_{\mathcal{C}}(\mathbb{S})) \to \pi_n(F_{\mathcal{C}'}(\mathbb{S})) \to \pi_n(F_{\mathcal{C}'}(\mathbb{S})/F_{\mathcal{C}}(\mathbb{S})) \to \pi_{n-1}(F_{\mathcal{C}}(\mathbb{S})) \to \cdots$$
$$\cdots \to \pi_0(F_{\mathcal{C}}(\mathbb{S})) \to \pi_0(F_{\mathcal{C}'}(\mathbb{S})) \to \pi_0(F_{\mathcal{C}'}(\mathbb{S})/F_{\mathcal{C}}(\mathbb{S})),$$

where $F_{\mathcal{C}'}(\mathbb{S})/F_{\mathcal{C}}(\mathbb{S})$ is the cofiber. When applied to the subcategory $\mathcal{P}_R \hookrightarrow \mathcal{E}_R$ this gives the long exact sequence

$$\cdots \to \pi_n(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_{n-1}(F_{\mathcal{P}_R}(\mathbb{S})) \to \cdots$$
$$\cdots \to \pi_0(F_{\mathcal{P}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})).$$

Here we have $\pi_n(F_{\mathcal{P}_R}(\mathbb{S})) = K_n(R)$. Moreover, by construction we have $\pi_0(F_{\mathcal{E}_R}(\mathbb{S})) = K_0(\mathcal{E}_R)$ so that we identify

$$\pi_0(F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})) = W_0(R) = K_0(\mathcal{E}_R)/K_0(R).$$

Thus, the spectrum $\mathbb{W}(R) := F_{\mathcal{E}_R}(\mathbb{S})/F_{\mathcal{P}_R}(\mathbb{S})$ given by the cofiber of $F_{\mathcal{P}_R}(\mathbb{S}) \to F_{\mathcal{E}_R}(\mathbb{S})$ provides a spectrum whose zeroth homotopy group is the Witt ring $W_0(R)$. $\square$

The forgetful functor $\mathcal{E}_R \to \mathcal{P}_R$ also induces a corresponding map of Γ-spaces $F_{\mathcal{E}_R} \to F_{\mathcal{P}_R}$. Moreover, one can also construct a spectrum with π_0 equal to $W_0(R)$ using the characterization given in [37], that we recalled above, in terms of the map on K-theory (and on K-theory spectra) $K(R) \to K(\mathcal{R})$ with $\mathcal{R} = (1 + rR[t])^{-1}R[t]$.

One can obtain in this way a reformulation in terms of spectra of the result of [37]. However, for our purposes here, it is preferable to work with the spectrum constructed in Lemma 7.2 above.

We give a variant of Lemma 7.2 that will be useful in the following. We denote by $\mathcal{P}_R^{\pm}$ and $\mathcal{E}_R^{\pm}$, respectively, the categories of $\mathbb{Z}/2\mathbb{Z}$-graded finite projective R-modules and the $\mathbb{Z}/2\mathbb{Z}$-graded endomorphism category with objects given by pairs $\{(E_+, f_+), (E_-, f_-)\}$, which we write simply as $(E_\pm, f_\pm)$ and with morphisms $\phi : E_\pm \to E'_\pm$ of $\mathbb{Z}/2\mathbb{Z}$-graded finite projective modules that commute with $f_\pm$. The sum in $\mathcal{E}_R^{\pm}$ is given by

$$(E_\pm, f_\pm) \oplus (E'_\pm, f'_\pm) = ((E_+ \oplus E'_+, E_- \oplus E'_-), (f_+ \oplus f'_+, f_- \oplus f'_-))$$

while the tensor product $(E_\pm, f_\pm) \otimes (E'_\pm, f'_\pm)$ is given by

$$((E_+ \otimes E'_+ \oplus E_- \otimes E'_-, f_+ \otimes f'_+ \oplus f_- \otimes f'_-), (E_+ \otimes E'_- \oplus E_- \otimes E'_+, f_+ \otimes f'_- \oplus f_- \otimes f'_+)).$$

Again we consider $\mathcal{P}_R^{\pm}$ as a subcategory of $\mathcal{E}_R^{\pm}$ with trivial endomorphisms.

Lemma 7.3. *The map $\delta : K_0(\mathcal{E}_R^{\pm}) \to K_0(\mathcal{E}_R)$ given by $[E_\pm, f_\pm] \mapsto [E_+, f_+] - [E_-, f_-]$ is a ring homomorphism and it descends to a ring homomorphism*

$$K_0(\mathcal{E}_R^{\pm})/K_0(\mathcal{P}_R^{\pm}) \to K_0(\mathcal{E}_R)/K_0(R) \simeq W_0(R).$$

Proof. The map is clearly compatible with sums. Compatibility with product also holds since $[E_\pm, f_\pm] \cdot [E'_\pm, f'_\pm] \mapsto ([E_+, f_+] - [E_-, f_-]) \cdot ([E'_+, f'_+] - [E'_-, f'_-])$. Moreover, it maps $K_0(\mathcal{P}_R^{\pm})$ to $K_0(\mathcal{P}_R)$. $\square$

As before, the categories $\mathcal{P}_R^{\pm}$ and $\mathcal{E}_R^{\pm}$ have associated Γ-spaces $F_{\mathcal{P}_R^{\pm}} : \Gamma^0 \to \Delta_*$ and $F_{\mathcal{E}_R^{\pm}} : \Gamma^0 \to \Delta_*$ and spectra $F_{\mathcal{P}_R^{\pm}}(\mathbb{S})$ and $F_{\mathcal{E}_R^{\pm}}(\mathbb{S})$. The following result follows as in Lemma 7.2.

Lemma 7.4. *The inclusion of $\mathcal{P}_R^{\pm}$ as a subcategory of $\mathcal{E}_R^{\pm}$ induces a long exact sequence*

$$\cdots \to \pi_n(F_{\mathcal{P}_R^{\pm}}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R^{\pm}}(\mathbb{S})) \to \pi_n(F_{\mathcal{E}_R^{\pm}}(\mathbb{S})/F_{\mathcal{P}_R^{\pm}}(\mathbb{S})) \to \pi_{n-1}(F_{\mathcal{P}_R^{\pm}}(\mathbb{S})) \to \cdots$$
$$\cdots \to \pi_0(F_{\mathcal{P}_R^{\pm}}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R^{\pm}}(\mathbb{S})) \to \pi_0(F_{\mathcal{E}_R^{\pm}}(\mathbb{S})/F_{\mathcal{P}_R^{\pm}}(\mathbb{S}))$$

of the homotopy groups of the spectra $F_{\mathcal{P}_R^{\pm}}(\mathbb{S})$ and $F_{\mathcal{E}_R^{\pm}}(\mathbb{S})$, which at the level of π_0 gives $K_0(\mathcal{P}_R^{\pm}) \to K_0(\mathcal{E}_R^{\pm}) \to K_0(\mathcal{E}_R^{\pm})/K_0(\mathcal{P}_R^{\pm})$.

We denote by $\mathbb{W}^{\pm}(R) = F_{\mathcal{E}_R^{\pm}}(\mathbb{S})/F_{\mathcal{P}_R^{\pm}}(\mathbb{S})$ the cofiber of $F_{\mathcal{P}_R^{\pm}}(\mathbb{S}) \to F_{\mathcal{E}_R^{\pm}}(\mathbb{S})$.

7.4. **Exponentiable measures and maps of Γ-spaces.** The problem of lifting to the level of spectra the Hasse–Weil zeta function associated to the counting motivic measure for varieties over finite fields was discussed in [20]. We consider here a very similar setting and procedure, where we want to lift a zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W(R)$ associated to an exponentiable motivic measure to the level of spectra. To this purpose, we make some assumptions of rationality and the existence of a factorization for our zeta functions of exponentiable motivic measures. We then consider the spectrum $K(\mathcal{V})$ of [67], [69] with $\pi_0 K(\mathcal{V}) = K_0(\mathcal{V})$ and a spectrum, obtained from a Γ-space, associated to the subring $W_0(R)$ of the big Witt ring $W(R)$.

Suppose given a motivic measure, that is, a ring homomorphism $\mu : K_0(\mathcal{V}) \to R$ of the Grothendieck ring of varieties to a commutative ring R that satisfies the following properties:

(1) **exponentiability**: the associated zeta function $\zeta_\mu(X,t)$ is a ring homomorphism $\zeta_\mu : K_0(\mathcal{V}) \to W(R)$ to the Witt ring of R;
(2) **rationality**: the zeta function is a ring homomorphism $\zeta_\mu : K_0(\mathcal{V}) \to W_0(R) \subset W(R)$ with values in the subring $W_0(R)$ of the Witt ring;
(3) **factorization**: the rational functions $\zeta_\mu(X,t)$ admit a factorization into linear factors
$$\zeta_\mu(X,t) = \frac{\prod_i (1-\alpha_i t)}{\prod_j (1-\beta_j t)} = \zeta_{\mu,+}(X,t) \ -_W \ \zeta_{\mu,-}(X,t)$$
where $\zeta_{\mu,+}(X,t) = \prod_j (1-\beta_j t)^{-1}$ and $\zeta_{\mu,-}(X,t) = \prod_i (1-\alpha_i t)^{-1}$ and $-_W$ is the difference in the Witt ring, that is the ratio of the two polynomials.

Lemma 7.5. *A motivic measure $\mu : K_0(\mathcal{V}) \to R$ with the three properties listed above determines a functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \to \mathcal{E}_R^\pm$ where $\mathcal{C}_\mathcal{V}$ is the assembler category encoding the scissor-congruence relations of the Grothendieck ring $K_0(\mathcal{V})$ and $\mathcal{E}_R^\pm$ is the $\mathbb{Z}/2\mathbb{Z}$-graded endomorphism category.*

Proof. The objects of $\mathcal{C}_\mathcal{V}$ are varieties X and the morphisms are locally closed embeddings, [67], [69]. To an object X we assign an object of $\mathcal{E}_R$ obtained in the following way. Consider a factorization
$$\zeta_\mu(X,t) = \frac{\prod_{i=1}^n (1-\alpha_i t)}{\prod_{j=1}^m (1-\beta_j t)} = \zeta_{\mu,+}(X,t) \ -_W \ \zeta_{\mu,-}(X,t)$$
as above of the zeta function of X. Let $E_+^{X,\mu} = R^{\oplus m}$ and $E_-^{X,\mu} = R^{\oplus n}$ with endomorphisms $f_\pm^{X,\mu}$ respectively given in matrix form by $M(f_+^{X,\mu}) = \mathrm{diag}(\beta_j)_{j=1}^m$ and $M(f_-^{X,\mu}) = \mathrm{diag}(\alpha_i)_{i=1}^n$. The pair $(E_\pm^{X,\mu}, f_\pm^{X,\mu})$ is an object of the endomorphism category $\mathcal{E}_R^\pm$. Given an embedding $Y \hookrightarrow X$, the zeta function satisfies
$$\zeta_\mu(X,t) = \zeta_\mu(Y,t) \cdot \zeta_\mu(X \smallsetminus Y, t) = \zeta_\mu(Y,t) \ +_W \ \zeta_\mu(X \smallsetminus Y, t).$$
Using the factorizations of each term, this gives
$$(E_\pm^{X,\mu}, f_\pm^{X,\mu}) = (E_\pm^{Y,\mu}, f_\pm^{Y,\mu}) \oplus (E_\pm^{X\smallsetminus Y,\mu}, f_\pm^{X\smallsetminus Y,\mu}),$$
hence a morphism in $\mathcal{E}_R^\pm$ given by the canonical morphism to the direct sum
$$(E_\pm^{Y,\mu}, f_\pm^{Y,\mu}) \to (E_\pm^{X,\mu}, f_\pm^{X,\mu}).$$
∎

Proposition 7.6. *The functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \to \mathcal{E}_R^\pm$ of Lemma 7.5 induces a map of Γ-spaces and of the associated spectra $\Phi_\mu : K(\mathcal{V}) \to F_{\mathcal{E}_R^\pm}(\mathbb{S})$. The induced maps on the homotopy groups has the property that the composition*

$$K_0(\mathcal{V}) \stackrel{\Phi_\mu}{\to} K_0(\mathcal{E}_R^\pm) \stackrel{\delta}{\to} K_0(\mathcal{E}_R) \to K_0(\mathcal{E}_R)/K_0(R) = W_0(R) \tag{7.7}$$

with δ as in Lemma 7.3, is given by the zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W_0(R)$.

Proof. The Γ-space associated to the assembler category $\mathcal{C}_\mathcal{V}$ is obtained in the following way, [67], [69]. One first associates to the assembler category $\mathcal{C}_\mathcal{V}$ another category $\mathcal{W}(\mathcal{C}_\mathcal{V})$ whose objects are finite collections $\{X_i\}_{i\in I}$ of non-initial objects of $\mathcal{C}_\mathcal{V}$ with morphisms $\varphi = (f, f_i) : \{X_i\}_{i\in I} \to \{X'_j\}_{j\in J}$ given by a map of the indexing sets $f : I \to J$ and morphisms $f_i : X_i \to X'_{f(i)}$ in $\mathcal{C}_\mathcal{V}$, such that, for every fixed $j \in J$ the collection $\{f_i : X_i \to X'_j \,:\, i \in f^{-1}(j)\}$ is a disjoint covering family of the assembler $\mathcal{C}_\mathcal{V}$. This means, in the case of the assembler $\mathcal{C}_\mathcal{V}$ underlying the Grothendieck ring of varieties, that the f_i are closed embeddings of the varieties X_i in the given X'_j with disjoint images. We first show that the functor $\Phi_\mu : \mathcal{C}_\mathcal{V} \to \mathcal{E}_R^\pm$ of Lemma 7.5 extends to a functor (for which we still use the same notation) $\Phi_\mu : \mathcal{W}(\mathcal{C}_\mathcal{V}) \to \mathcal{E}_R^\pm$. We define $\Phi_\mu(\{X_i\}_{i\in I}) = \oplus_{i\in I}\Phi_\mu(X_i) = \oplus_{i\in I}(E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu})$. Given a covering family $\{f_i : X_i \to X'_j \,:\, i \in f^{-1}(j)\}$ as above, each morphism $f_i : X_i \to X'_j$ determines a morphism $\Phi_\mu(f_i) : (E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \to (E_\pm^{X'_j,\mu}, f_\pm^{X'_j,\mu})$ given by the canonical morphism to the direct sum $(E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \to (E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \oplus (E_\pm^{X'_j\smallsetminus X_i,\mu}, f_\pm^{X'_j\smallsetminus X_i,\mu})$. This determines a morphism $\Phi_\mu(\varphi) : \oplus_{i\in I}(E_\pm^{X_i,\mu}, f_\pm^{X_i,\mu}) \to \oplus_{j\in J}(E_\pm^{X'_j,\mu}, f_\pm^{X'_j,\mu})$. We then show that the functor $\Phi_\mu : \mathcal{W}(\mathcal{C}_\mathcal{V}) \to \mathcal{E}_R^\pm$ constructed in this way determines a map of the associated Γ-spaces. The Γ-space associated to $\mathcal{W}(\mathcal{C}_\mathcal{V})$ is constructed in [67], [69] as the functor that assigns to a finite pointed set $S \in \Gamma^0$ the simplicial set given by the nerve $\mathcal{N}\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V})$, where the coproduct of assemblers $S \wedge \mathcal{C}_\mathcal{V} = \wedge_{s\in S\smallsetminus\{s_0\}}\mathcal{C}_\mathcal{V}$ has an initial object and a copy of the non-initial objects of $\mathcal{C}_\mathcal{V}$ for each point $s \in S \smallsetminus \{s_0\}$ and morphisms induced by those of $\mathcal{C}_\mathcal{V}$. This means that we can regard objects of $\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V})$ as collections $\{X_{s,i}\}_{i\in I}$, for some $s \in S \smallsetminus \{s_0\}$ and morphisms $\varphi_s = (f_s, f_{s,i}) : \{X_{s,i}\}_{i\in I} \to \{X'_{s,j}\}_{j\in J}$ as above. In order to obtain a map of Γ-spaces between $F_\mathcal{V} : S \mapsto \mathcal{N}\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V})$ and $F_{\mathcal{E}_R^\pm} : S \mapsto \mathcal{N}\Sigma_{\mathcal{E}_R^\pm}(S)$, we construct a functor $\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V}) \to \Sigma_{\mathcal{E}_R^\pm}(S)$ from the category $\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V})$ described above to the category of summing functors $\Sigma_{\mathcal{E}_R^\pm}(S)$. To an object $X_{S,I} := \{X_{s,i}\}_{i\in I}$ in $\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V})$ we associate a functor $\Phi_{X_{S,I}} : \mathcal{P}(S) \to \mathcal{E}_R^\pm$ that maps a subset $A_+ = \{s_0\} \sqcup A \in \mathcal{P}(X)$ to $\Phi_{X_{S,I}}(A_+) = \oplus_{a\in A}\Phi_\mu(\{X_{a,i}\}_{i\in I})$ where $\Phi_\mu : \mathcal{W}(\mathcal{C}_\mathcal{V}) \to \mathcal{E}_R^\pm$ is the functor constructed above. It is a summing functor since $\Phi_{X_{S,I}}(A_+ \cup B_+) = \Phi_{X_{S,I}}(A_+) \oplus \Phi_{X_{S,I}}(B_+)$ for $A_+ \cap B_+ = \{s_0\}$. This induces a map of simplicial sets $\mathcal{N}\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V}) \to \mathcal{N}\Sigma_{\mathcal{E}_R^\pm}(S)$ which determines a natural transformation of the functors $F_\mathcal{V} : S \mapsto \mathcal{N}\mathcal{W}(S \wedge \mathcal{C}_\mathcal{V})$ and $F_{\mathcal{E}_R^\pm} : S \mapsto \mathcal{N}\Sigma_{\mathcal{E}_R^\pm}(S)$. This map of Γ-spaces in turn determines a map of the associated spectra and an induced map of their homotopy groups. It remains to check that the induced map at the level of π_0 agrees with the expected map of Grothendieck rings $K_0(\mathcal{V}) \to K_0(\mathcal{E}_R^\pm)$, hence with the zeta function when further mapped to $K_0(\mathcal{E}_R)$

and to the quotient $K_0(\mathcal{E}_R)/K_0(R)$. This is the case since by construction the induced map $\pi_0 K(\mathcal{V}) = K_0(\mathcal{V}) \to K_0(\mathcal{E}_R^{\pm}) = \pi_0 F_{\mathcal{E}_R^{\pm}}(\mathbb{S})$ is given by the assignment $[X] \mapsto [E_{\pm}^{X,\mu}, f_{\pm}^{X,\mu}]$. $\square$

Corollary 7.7. *The map of Grothendieck rings given by the composition* (7.7) *also lifts to a map of spectra.*

Proof. It is possible to realize the map $\delta : K_0(\mathcal{E}_R^{\pm}) \to K_0(\mathcal{E}_R)$ of Lemma 7.3 at the level of spectra. The K-theory spectrum of an abelian category $\mathcal{A}$ is weakly equivalent to the K-theory spectrum of the category of bounded chain complexes over $\mathcal{A}$. Thus, in the case of the category $\mathcal{E}_R$, there is a weak equivalence $K(\mathrm{Ch}^\flat(\mathcal{E}_R)) \tilde{\to} K(\mathcal{E}_R)$ which descends on the level π_0 to the map $K_0(\mathrm{Ch}^\flat(\mathcal{E}_R)) \tilde{\to} K_0(\mathcal{E}_R)$ given by $[E^{\cdot}, f^{\cdot}] \mapsto \sum_k (-1)^k [E^k, f^k]$. To an object $(E^{\pm}, f^{\pm})$ of $\mathcal{E}_R^{\pm}$ we can assign a chain complex in $\mathrm{Ch}^\flat(\mathcal{E}_R)$ of the form $0 \to (E^-, f^-) \xrightarrow{0} (E^+, f^+) \to 0$, where (E^+, f^+) sits in degree 0. This descends on the level of K-theory to a map $K(\mathcal{E}_R^{\pm}) \to K(\mathrm{Ch}^\flat(\mathcal{E}_R))$, which at the level of π_0 gives the map $[E^{\pm}, f^{\pm}] \mapsto [E^+, f^+] - [E^-, f^-]$. The functor $\mathcal{E}_R^{\pm} \to \mathrm{Ch}^\flat(\mathcal{E}_R)$ used here does not respect tensor products, although the induced map $\delta : K_0(\mathcal{E}_R^{\pm}) \to K_0(\mathcal{E}_R)$ at the level of K_0 is compatible with products. Thus, the composition (7.7) can also be lifted at the level of spectra. $\square$

It should be noted that the construction of a derived motivic zeta function outlined above is not the first to appear in the literature. In [20], the authors describe a derived motivic measure $\zeta : K(\mathcal{V}_k) \to K(\mathrm{Rep}_{cts}(\mathrm{Gal}(k^s/k); \mathbb{Z}_\ell))$ from the Grothendieck spectrum of varieties to the K-theory spectrum of the category of continuous ℓ-adic Galois representations. This map corresponds to the assignment $X \mapsto H^*_{\mathrm{et},c}(X \times_k k^s, \mathbb{Z}_\ell)$. In particular, they show that when $k = \mathbb{F}_q$ for ℓ coprime to q, on the level of π_0, ζ corresponds to the Hasse-Weil zeta function. They then use ζ to prove that $K_1(\mathcal{V}_{\mathbb{F}_q})$ is not only nontrivial, but contains interesting algebro-geometric data.

Essentially, the approach in [20] was to start with a Weil Cohomology theory (in this case, ℓ-adic cohomology) and then to construct a derived motivic measure realizing on the level of K-theory the assignment to a variety X of its corresponding cohomology groups. The methods used in the case of ℓ-adic cohomology may not immediately generalize to other Weil cohomology theories. This method has yielded deep insight into the world of algebraic geometry. Our approach here, in contrast, is to take an interesting class of motivic measures, namely Kapranov motivic zeta functions (exponentiable motivic measures, [43], [59], [60]), and to determine reasonable conditions under which such a motivic measure can be derived directly. This method still needs to be studied further to yield additional insights into what it captures about the geometry of varieties.

7.5. **Bost–Connes type systems via motivic measures.** The lifting of the integral Bost–Connes algebra to various Grothendieck rings, their assembler categories, and the associated spectra, that we discussed in [51] and in the earlier sections of this paper, can be viewed as an instance of a more general kind of operation. As discussed in [24], there is a close relation between the endomorphisms σ_n and the maps $\tilde{\rho}_n$ of the integral Bost–Connes algebra and the operation of Frobenius and Verschiebung in the Witt ring. Thus, we can formulate a more general form of the question investigated above, of lifting of the integral Bost–Connes algebra to a Grothendieck ring through an Euler characteristic map, in terms of lifting the Frobenius and Verschiebung operations of a Witt ring to a Grothendieck ring through the zeta function ζ_μ of an exponentiable motivic measure. A prototype example of this more general setting is provided by the Hasse–Weil zeta function $Z : K_0(\mathcal{V}_{\mathbb{F}_q}) \to W(\mathbb{Z})$, which has the properties that the action of the Frobenius F_n on the Witt ring $W(\mathbb{Z})$ corresponds to passing to a field extension, $F_n Z(X_{\mathbb{F}_q}, t) = Z(X_{\mathbb{F}_{q^n}}, t)$ and the action of the Verschiebung V_n on the Witt ring $W(\mathbb{Z})$ is related to the Weil restriction of scalars from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$ (see [59] for a precise statement).

Recall that, if one denotes by $[a]$ the elements $[a] = (1 - at)^{-1}$ in the Witt ring $W(R)$, for $a \in R$, then the Frobenius ring homomorphisms $F_n : W(R) \to W(R)$ of the Witt ring are determined by $F_n([a]) = [a^n]$ and the Verschiebung group homomorphisms $V_n : W(R) \to W(R)$ are defined on an arbitrary $P(t) \in W(R)$ as $F_n : P(t) \mapsto P(t^n)$. These operations satisfy an analog of the Bost–Connes relations

$$F_n \circ F_m = F_{nm}, \quad V_n \circ V_m = V_{nm}, \quad F_n \circ V_n = n \cdot \mathrm{id}, \quad F_n \circ V_m = V_m F_n \text{ if } (n, m) = 1. \tag{7.8}$$

These correspond, respectively, to the semigroup structure of the σ_n and $\tilde{\rho}_n$ of the integral Bost–Connes algebra and the relations $\sigma_n \circ \tilde{\rho}_n = n \cdot \mathrm{id}$, while the last relation is determined in the Bost–Connes case by the commutation of the generators $\tilde{\mu}_n$ and μ^*_m for $(n, m) = 1$.

Definition 7.8. *A motivic measure $\mu : K_0(\mathcal{V}) \to R$ that satisfies the properties listed in §7.4 is of* Bost–Connes type *if there is a lift to $K_0(\mathcal{V})$ of the Frobenius F_n and Verschiebung V_n of the Witt ring $W(R)$ to $K_0(\mathcal{V})$ such that the diagrams commute*

$$\begin{tikzcd} K_0(\mathcal{V}) \arrow[r, "\zeta_\mu"] \arrow[d, "\sigma_n"] & W(R) \arrow[d, "F_n"] \\ K_0(\mathcal{V}) \arrow[r, "\zeta_\mu"] & W(R) \end{tikzcd} \qquad \begin{tikzcd} K_0(\mathcal{V}) \arrow[r, "\zeta_\mu"] \arrow[d, "\tilde{\rho}_n"] & W(R) \arrow[d, "V_n"] \\ K_0(\mathcal{V}) \arrow[r, "\zeta_\mu"] & W(R) \end{tikzcd}$$

Such a motivic measure $\mu : K_0(\mathcal{V}) \to R$ is of homotopic Bost–Connes type *if the maps σ_n and $\tilde{\rho}_n$ in the diagrams above also lift to endofunctors of the assembler category $\mathcal{C}_\mathcal{V}$ of the Grothendieck ring $K_0(\mathcal{V})$ with the endofunctors σ_n compatible with the monoidal structure.*

Lemma 7.9. *There are endofunctors F_n and V_n of the category $\mathcal{E}^\pm_R$ such that the maps they induce on $W_0(R) = K_0(\mathcal{E}_R)/K_0(R)$ agree with the restrictions to $W_0(R) \subset W(R)$ of the Frobenius and Verschiebung maps. These endofunctors determine natural transformations (still denoted F_n and V_n) of the Γ-space $F_{\mathcal{E}^\pm_R} : \Gamma^0 \to \Delta_*$.*

Proof. The homomorphism $K_0(\mathcal{E}_R) \to W_0(R)$ given by

$$(E, f) \mapsto L(E, f) = \det(1 - tM(f))^{-1}$$

sends the pair (R, f_a) with f_a acting on R as multiplication by $a \in R$ to the element $[a] = (1 - at)^{-1}$ in the Witt ring. The action of the Frobenius $F_n([a]) = [a^n]$ is induced from the Frobenius $F_m(E, f) = (E, f^n)$ which is an endofunctor of $\mathcal{E}_R$. This extends to a compatible endofunctor of $\mathcal{E}_R^{\pm}$ by $F_n(E_\pm, f_\pm) = (E_\pm, f_\pm^n)$. Similarly, the Verschiebung map that sends $\det(1-tM(f))^{-1} \mapsto \det(1-t^n M(f))^{-1}$ is induced from the Verschiebung on $\mathcal{E}_R$ given by

$$V_n : (E, f) \mapsto (E^{\oplus n}, V_n(f)), \qquad V_n(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & f \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

since we have $L(E^{\oplus n}, V_n(f)) = \det(1 - t^n M(f))^{-1}$, with compatible endofunctors $V_n(E_\pm, f_\pm) = (E_\pm^{\oplus n}, V_n(f_\pm))$ on $\mathcal{E}_R^{\pm}$. The Frobeniius and Verschiebung on $\mathcal{E}_R^{\pm}$ induce natural transformations of the Γ-space $F_{\mathcal{E}_R^{\pm}} : \Gamma^0 \to \Delta_*$ by composition of the summing functors $\Phi : \mathcal{P}(X) \to \mathcal{E}_R^{\pm}$ in $\Sigma_{\mathcal{E}_R^{\pm}}(X)$ with the endofunctors F_n and V_n of $\mathcal{E}_R^{\pm}$. $\square$

Proposition 7.10. *Let $\mu : K_0(\mathcal{V}) \to R$ be a motivic measure with the properties of §7.4 that is of homotopical Bost–Connes type. Then the endofunctors σ_n and $\tilde\rho_n$ of the assembler category $\mathcal{C}_{\mathcal{V}}$ determine natural transformations (still denoted by σ_n and $\tilde\rho_n$) of the associated Γ-space $F_{\mathcal{V}} : \Gamma^0 \to \Delta_*$ that fit in the commutative diagrams*

$$\begin{tikzcd} F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] \arrow[d, "\sigma_n"] & F_{\mathcal{E}_R^{\pm}} \arrow[d, "F_n"] \\ F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] & F_{\mathcal{E}_R^{\pm}} \end{tikzcd} \qquad \begin{tikzcd} F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] \arrow[d, "\tilde\rho_n"] & F_{\mathcal{E}_R^{\pm}} \arrow[d, "V_n"] \\ F_{\mathcal{V}} \arrow[r, "\Phi_\mu"] & F_{\mathcal{E}_R^{\pm}} \end{tikzcd}$$

where $\Phi_\mu : F_{\mathcal{V}} \to F_{\mathcal{E}_R^{\pm}}$ is the natural transformation of Γ-spaces of (7.6) *and F_n and V_n are the natural transformations of Lemma 7.9.*

Proof. The natural transformation Φ_ν is determined as in Proposition 7.6 by the functor $\Phi_\mu : \mathcal{C}_{\mathcal{V}} \to \mathcal{E}_R^{\pm}$ that assigns $\Phi_\mu : X \mapsto (E_\pm^X, f_\pm^X)$ constructed as in Lemma 7.5. Suppose we have endofunctors σ_n and $\tilde\rho_n$ of the assembler category $\mathcal{C}_{\mathcal{V}}$ that induce maps σ_n and $\tilde\rho_n$ on $K_0(\mathcal{V})$ that lift the Frobenius and Verschiebung maps of $W(R)$ through the zeta function $\zeta_\mu : K_0(\mathcal{V}) \to W(R)$. This means that $\zeta_\mu(\sigma_n(X), t) = F_n\zeta_\mu(X, t)$ and $\zeta_\mu(\tilde\rho_n(X), t) = V_n\zeta_\mu(X, t) = \zeta_\mu(X, t^n)$. By Lemma 7.9, we have $F_n\zeta_\mu(X, t) = L(F_n(E_\pm^X, f_\pm^X)) = L(E_\pm^X, (f_\pm^X)^n)$ and $V_n\zeta_\mu(X, t) = L(V_n(E_\pm^X, f_\pm^X)) = L((E_\pm^X)^{\oplus n}, V_n(f_\pm^X))$. This shows the compatibilities of the natural transformations in the diagrams above. $\square$

7.6. **Spectra and spectra.** We apply a construction similar to the one discussed in the previous subsections to the case of the map $(X, f) \mapsto \sum_{\lambda \in \mathrm{Spec}(f_*)} m_\lambda \lambda$ that assigns to a variety over $\mathbb{C}$ with a quasi-unipotent map the spectrum of the induced map f_* in homology, seen as an element in $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, as in §6 of [51].

In this section the term spectrum will appear both in its homotopy theoretic sense and in its operator sense. Indeed, we consider here a lift to the level of spectra (in the homotopy theoretic sense) of the construction described in §6 of [51], based on the spectrum (in the operator sense) Euler characteristic.

We consider here a setting as in [33], [38], where (X, f) is a pair of a variety over $\mathbb{C}$ and an endomorphism $f : X \to X$ such that the induced map f_* on $H_*(X, \mathbb{Z})$ has spectrum consisting of roots of unity. As discussed in [51] and in a related form in [33] the spectrum determines a ring homomorphism (an Euler characteristic)

$$\sigma : K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \tag{7.9}$$

where $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ denotes the Grothendieck ring of pairs (X, f) with the operations defined by the disjoint union and the Cartesian product. It is shown in [51] that one can lift the operations σ_n and $\tilde{\rho}_n$ of the integral Bost–Connes algebra from $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ to $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ via the "spectral Euler characteristic" (7.9), and that the operations can further be lifted from $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to a (homotopy theoretic) spectrum with π_0 equal to $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ via the assembler category construction of [67].

In this section we discuss how to lift the right hand side of (7.9), namely the original Bost–Connes algebra $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ with the operations σ_n and $\tilde{\rho}_n$ to the level of a homotopy theoretic spectrum, so that the spectral Euler characteristic (7.9) becomes induced by a map of spectra.

To this purpose, we use the categorification of Bost–Connes systems constructed in [54]. Let $\mathrm{Vect}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ be the category of pairs $(W, \oplus_{r \in \mathbb{Q}/\mathbb{Z}} \bar{W}_r)$ with W a finite dimensional $\mathbb{Q}$-vector space and $\oplus_r \bar{W}_r$ a $\mathbb{Q}/\mathbb{Z}$-graded vector space with $\bar{W} = W \otimes \bar{\mathbb{Q}}$. This is a neutral Tannakian category with fiber functor the forgetful functor $\omega : \mathrm{Vect}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q}) \to \mathrm{Vect}(\mathbb{Q})$ and with $\mathrm{Aut}^{\otimes}(\omega) = \mathrm{Spec}(\bar{\mathbb{Q}}[\mathbb{Q}/\mathbb{Z}]^G)$ and $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, see Theorem 3.2 of [54]. The category $\mathrm{Vect}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ is endowed with additive symmetric monoidal functors $\sigma_n(W) = W$ and $\overline{\sigma_n(W)}_r = \oplus_{r' : \sigma_n(r') = r} \bar{W}_{r'}$ if r is in the range of σ_n and zero otherwise and additive functors $\tilde{\rho}_n(W) = W^{\oplus n}$ and $\overline{\tilde{\rho}_n(W)}_r = \bar{W}_{\sigma_n(r)}$ satisfying $\sigma_n \circ \tilde{\rho}_n = n \cdot \mathrm{id}$ that induce the Bost–Connes maps on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$. As shown in Theorem 3.18 of [54], this category can be equivalently described as a category of automorphisms $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ with objects pairs (W, ϕ) of a $\mathbb{Q}$-vector space V and a G-equivariant diagonalizable automorphism of $\bar{W}$ with eigenvalues that are roots of unity (seen as elements in $\mathbb{Q}/\mathbb{Z}$). There is an equivalence of categories between $\mathrm{Vect}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ and $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ under which the functors σ_n and $\tilde{\rho}_n$ correspond, respectively, to the Frobenius and Verschiebung

$$F_n : (W, \phi) \mapsto (W, \phi^n), \qquad V_n : (W, \phi) \mapsto (W^{\oplus n}, V_n(\phi)).$$

The equivalence is realized by mapping $(W,\phi) \mapsto (W, \oplus_r \bar{W}_r)$ where $\bar{W}_r$ are the eigenspaces of ϕ with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$.

Thus, we can construct a functor from the assmbler category $\mathcal{C}_{\mathbb{C}}^{\mathbb{Z}}$ of §6 of [51], underlying $K_0^{\mathbb{Z}}(\mathcal{V}_{\mathbb{C}})$ to $\mathrm{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ by following along the lines of Lemma 7.5 and Proposition 7.6 above, where we assign $\Phi(X,f) = (H_*(X,\mathbb{Q}), \oplus_r E_r(f_*))$ where $E_r(f_*)$ is the eigenspace with eigenvalue $r \in \mathbb{Q}/\mathbb{Z}$. The Bost–Connes algebra then lifts to the Frobenius and Vershiebung functors on $\mathrm{Aut}_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ and the latter lift to geometric Frobenius and Verschiebung operations on the pairs (X,f) mapping to (X,f^n) and to $(X \times Z_n, \Phi_n(f))$.

This point of view, that replaces the Bost–Connes algebra with it categorification in terms of the Tannakian category $\mathrm{Vect}_{\mathbb{Q}/\mathbb{Z}}^{\bar{\mathbb{Q}}}(\mathbb{Q})$ as in [54] will be useful in the following sections, where we reformulate our categorical setting, by passing from Grothendieck rings, assemblers and spectra, to Tannakian categories of Nori motives.

8. Bost–Connes systems in categories of Nori motives

We now consider a motivic framework, and Bost–Connes type systems that live on Tannakian categories of motives.

Let $D(\mathcal{V})$ the Nori geometric diagrams associated to the category $\mathcal{V}$ of varieties over $\mathbb{Q}$, constructed as described in §2.3.

As in [51] and in § 3 of this paper, we consider here the category $\mathcal{V}^{\hat{\mathbb{Z}}}$ of varieties X with a good action of $\hat{\mathbb{Z}}$ that factors through an action of some finite $\mathbb{Z}/N\mathbb{Z}$. We can view $\mathcal{V}^{\hat{\mathbb{Z}}}$ as an enrichment $\hat{\mathcal{V}}$ of the category $\mathcal{V}$, in the sense described in §2.2.

Define the Nori diagram of *effective pairs* $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ as we recalled earlier in §2.3:

a) One vertex of $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ is a triple $((X,\alpha_X),(Y,\alpha_Y),i)$, of varieties X and Y with good $\hat{\mathbb{Z}}$ actions factoring through finite levels, $\alpha_X : \hat{\mathbb{Z}} \times X \to X$ and $\alpha_Y : \hat{\mathbb{Z}} \times Y \to Y$, and an integer i, together with a closed embedding $j : Y \hookrightarrow X$ that is equivariant with respect to the $\hat{\mathbb{Z}}$ actions. For brevity, we will denote such a triple $(\hat{X},\hat{Y},i)$ and call it a closed embedding in the enrichment $\hat{\mathcal{V}}$.

b) Identity edges, functoriality edges, and coboundary edges are obvious enrichments of the respective edges defined in §2.3, with the requirement that all these maps are $\hat{\mathbb{Z}}$-equivariant.

b1) Let $(\hat{X},\hat{Y})$ and $(\hat{X}',\hat{Y}')$ be two pairs of closed embeddings in $\hat{\mathcal{V}}$. Every morphism $f : X \to X'$ such that $f(Y) \subset Y'$ and $f \circ \alpha_X = \alpha_{X'} \circ f$ produces functoriality edges f^* (or rather (f^*,i)) going from $((X',\alpha_{X'}),(Y',\alpha_{Y'|}),i)$ to (X,Y,i).

b2) Let $(Z \subset Y \subset X)$ be a stair of closed embeddings compatible with enrichments (equivariant with respect tot the $\hat{\mathbb{Z}}$-actions). Then it defines coboundary edges ∂

$$((Y,\alpha_Y),(Z,\alpha_Z),i) \to ((X,\alpha_X),(Y,\alpha_Y),i+1).$$

We have thus defined he Nori geometric diagram of enriched effective pairs, which we denote equivalently by $D(\hat{\mathcal{V}})$ or $D(\mathcal{V}^{\hat{\mathbb{Z}}})$.

Notice that forgetting in this diagram all enrichments, we obtain the map $D(\hat{\mathcal{V}}) \to D(\mathcal{V})$ which is *injective* both on vertices and edges.

8.1. **Bost–Connes system on Nori motives.** We now construct a Bost–Connes system on a category of Nori motives obtained from the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ described above, which lifts to the level of motives the categorification of the Bost–Connes algebra constructed in [54].

As we already mentioned in —S 7.6 above, we can describe the categorification of the Bost–Connes algebra of [54] in terms of the Tannakian category $\mathrm{Vec}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ with suitable functors σ_n and $\tilde{\rho}_n$ constructed as in Theorem 3.7 of [54] or in terms of an equivalent Tannakian category $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ endowed with Frobenius and Verschiebung functors. We use here the second description. For the equivalence of these structures see Theorem 3.18 of [54].

Lemma 8.1. *The assignment $T : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ determines a representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \to \mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ of the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ constructed above.*

Proof. As discussed in the previous subsection, we view elements $((X, \alpha_X), (Y, \alpha_Y), i)$ of $D(\mathcal{V}^{\hat{\mathbb{Z}}})$ in terms of an enhancement $\hat{\mathcal{V}}$ of the category $\mathcal{V}$ defined as in §2.2, by choosing a primitive root of unity that generates the cyclic group $\mathbb{Z}/N\mathbb{Z}$, so that the actions α_X and α_Y are determined by self maps v_X and v_Y as in §2.2. We identify the element above with $((X, v_X), (Y, v_Y), i)$, which we also denoted by $(\hat{X}, \hat{Y}, i)$ in the previous subsection. Since the embedding $Y \hookrightarrow X$ is $\hat{\mathbb{Z}}$-equivariant, the map v_Y is the restriction to Y of the map v_X under this embedding. We denote by ϕ^i the induced map on the cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$. The eigenspaces of ϕ^i are the subspaces of the decomposition of $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ according to characters of $\hat{\mathbb{Z}}$, that is, elements in $\mathrm{Hom}(\hat{\mathbb{Z}}, \mathbb{C}^*) = \nu^* \simeq \mathbb{Q}/\mathbb{Z}$. Thus, we obtain an object $(H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}), \phi^i)$ in the category $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. Edges in the diagram are $\hat{\mathbb{Z}}$-equivariant maps so they induce morphisms between the corresponding objects in the category $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. □

One can also see in a similar way that the fiber functor $T : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ determines an object in the category $\mathrm{Vec}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. Indeed, the pair (X, Y) with $Y \subset X$ is endowed with compatible good $\hat{\mathbb{Z}}$-actions α_X and α_Y that factor through some finite level $\mathbb{Z}/N\mathbb{Z}$, hence the singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$ carries a resulting $\hat{\mathbb{Z}}$-representation. Thus, the vector space $H^i(X(\mathbb{C}), Y(\mathbb{C}), \bar{\mathbb{Q}})$ can be decomposed into eigenspaces of this representations according to characters $\chi \in \mathrm{Hom}(\hat{\mathbb{Z}}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$. Thus, we obtain a decomposition of $H^i(X(\mathbb{C}), Y(\mathbb{C}), \bar{\mathbb{Q}}) = \oplus_{r \in \mathbb{Q}/\mathbb{Z}} \bar{V}_r$ as a $\mathbb{Q}/\mathbb{Z}$-graded vector space. We choose to work with the category

$\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ because the Bost–Connes structure is more directly expressed in terms of Frobenius and Verschiebung, which will make the lifting of this structure to the resulting category of Nori motives more immediately transparent, as we discuss below.

The representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \to \mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ replaces, at this motivic level, our previous use in [51] of the equivariant Euler characteristics $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ (see [47]) as a way to lift the Bost–Connes algebra. We proceed in the following way to obtain the Bost–Connes structure in this setting.

Proposition 8.2. *There is a Bost–Connes system on the category $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$ of Nori motives associated to the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$. The representation $T : D(\mathcal{V}^{\hat{\mathbb{Z}}}) \to \mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ constructed above has the property that the induced functor*

$$\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T) \to \mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$$

intertwines the endofunctors σ_n and $\tilde{\rho}_n$ of the Bost–Connes system on $\mathcal{C}(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$ and the Frobenius F_n and Verschiebung V_n of the Bost–Connes structure on $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$.

Proof. The Frobenius and Verschiebung on $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ are given by $F_n(W, \phi) = (W, \phi^n)$ and $V_n(W, \phi) = (W^{\oplus n}, V_n(\phi))$ with

$$V_n(\phi) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \phi \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{8.1}$$

Consider the mappings

$$\sigma_n : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto ((X, \alpha_X \circ \sigma_n), (Y, \alpha_Y \circ \sigma_n), i)$$

$$\tilde{\rho}_n : ((X, \alpha_X), (Y, \alpha_Y), i) \mapsto (X \times Z_n, \Phi_n(\alpha_X), (Y \times Z_n, \Phi_n(\alpha_Y)), i),$$

where $Z_n = \mathrm{Spec}(\mathbb{Q}^n)$ and $\Phi_n(\alpha)$ is the geometric Verschiebung defined as in §3.3. The effect of the transformation σ_n, when written in terms of the data $((X, v_X), (Y, v_Y), i)$ is to send $v_X \mapsto v_X^n$ and $v_Y \mapsto v_Y^n$, hence it induces the Frobenius map F_n on $(H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}), \phi^i)$ in $\mathrm{Aut}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$. Similarly, we have $T(X \times Z_n, \Phi_n(\alpha_X), (Y \times Z_n, \Phi_n(\alpha_Y)), i) = H^i(X \times Z_n, Y \times Z_n, \mathbb{Q})$ where by the relative version of the Künneth formula $H^i(X(\mathbb{C}) \times Z_n(\mathbb{C}), Y(\mathbb{C}) \times Z_n(\mathbb{C}), \mathbb{Q}) \simeq H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})^{\oplus n}$ with the induced map $V_n(\phi^i)$. The maps σ_n and $\tilde{\rho}_n$ defined as above determine self maps of the diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}})$. By Lemma 7.2.6 of [40] given a map $F : D_1 \to D_2$ of diagrams and a representation $T : D_2 \to R-\mathrm{Mod}$, there is an R-linear exact functor

$\mathcal{F} : C(D_1, T \circ F) \to C(D_2, T)$ such that the following diagram commutes:

$$\begin{tikzcd} D_1 \arrow[rr, "F"] \arrow[d] & & D_2 \arrow[d] \\ C(D_1, T\circ F) \arrow[rr, "\mathcal{F}"] \arrow[dr] & & C(D_2, T) \arrow[dl] \\ & R-\mathrm{Mod} & \end{tikzcd}$$

We still denote by σ_n and $\tilde{\rho}_n$ the endofunctors induced in this way on $C(D(\mathcal{V}^{\hat{\mathbb{Z}}}), T)$. To check the compatibility of the σ_n functors with the monoidal structure, we use the fact that for Nori motives the product structure is constructed using "good pairs" (see §9.2.1 of [40]), that is, elements (X, Y, i) with the property that $H^j(X, Y, \mathbb{Z}) = 0$ for $j \neq i$. For such elements the product is given by $(X, Y, i) \times (X', Y', j) = (X \times X', X \times Y' \cup Y \times X', i + j)$. The diagram category $C(\mathrm{Good}^{eff}, T)$ obtained by replacing effective pairs Pairs^{eff} with good effective pairs Good^{eff} is equivalent to $C(\mathrm{Pairs}^{eff}, T)$ (Theorem 9.2.22 of [40]), hence the tensor structure defined in this way on $C(\mathrm{Good}^{eff}, T)$ determines the tensor structure of $C(\mathrm{Pairs}^{eff}, T)$ and on the resulting category of Nori motives, see §9.3 of [40]. Thus, to check the compatibility of the functors σ_n with the tensor structure it suffices to see that on a product of good pairs, where indeed we have

$$\sigma_n((X, \alpha_X), (Y, \alpha_Y), i) \times \sigma_n((X', \alpha'_X), (Y', \alpha'_Y), j) =$$
$$((X\times X', (\alpha_X\times\alpha'_X)\circ\Delta\circ\sigma_n), ((X\times Y', (\alpha_X\times\alpha'_Y)\circ\Delta\circ\sigma_n)\cup(Y\times X', (\alpha_Y\times\alpha'_X)\circ\Delta\circ\sigma_n)), i+j)$$
$$= \sigma_n(((X, \alpha_X), (Y, \alpha_Y), i) \times ((X', \alpha'_X), (Y', \alpha'_Y), j)).$$

The functors $\tilde{\rho}_n$ are not compatible with the tensor product structure, as expected. $\square$

Remark 8.3. In [54] a motivic interpretation of the categorification of the Bost–Connes algebra is given by identifying the Tannakian category $\mathrm{Vec}^{\bar{\mathbb{Q}}}_{\mathbb{Q}/\mathbb{Z}}(\mathbb{Q})$ with a limit of orbit categories of Tate motives. Here we presented a different motivic categorification of the Bost–Connes algebra by lifting the Bost–Connes structure to the level of the category of Nori motives. In [54] a motivic Bost–Connes structure was also constructed using the category of motives over finite fields and the larger class of Weil numbers replacing the roots of unity of the Bost–Connes system.

8.2. **Motivic sheaves and the relative case.** The argument presented in Proposition 8.2 lifting the Bost–Connes structure to the category of Nori motives, which provides a Tannakian category version of the list to Grothendieck rings via the equivariant Euler characteristics $K_0^{\hat{\mathbb{Z}}}(\mathcal{V}) \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, can also be generalized to the relative setting, where we considered the Euler characteristic

$$\chi_S^{\hat{\mathbb{Z}}} : K_0^{\hat{\mathbb{Z}}}(\mathcal{V}_S) \to K_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_S)$$

with values in the Grothendieck ring of constructible sheaves, discussed in §3 of this paper. The categorical setting of Nori motives that is appropriate for this relative case is the Nori category of motivic sheaves introduced in [3].

We recall here briefly the construction of the category of motivic sheaves of [3] and we show that the Bost–Connes structure on the category of Nori motives described in Proposition 8.2 extends to this relative setting.

Consider pairs $(X \to S, Y)$ of varieties over a base S with $Y \subset X$ endowed with the restriction $f_Y : Y \to S$. Morphisms $f : (X \to S, Y) \to (X' \to S, Y')$ are morphisms of varieties $h : X \to X'$ satisfying the commutativity of

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

and such that $h(Y) \subset Y'$. As before, we consider varieties endowed with good $\hat{\mathbb{Z}}$-action that factor through some finite level $\mathbb{Z}/N\mathbb{Z}$. We denote by (S, α) the base with its good $\hat{\mathbb{Z}}$-action and by $((X\alpha_X) \to (S, \alpha), (Y, \alpha_Y))$ the pairs as above where we assume that the map $f : X \to S$ and the inclusion $Y \hookrightarrow X$ are $\hat{\mathbb{Z}}$-equivariant.

Following [3], a diagram $D(\mathcal{V}_S)$ is obtained by considering as vertices elements of the form $(X \to S, Y, i, w)$ with $(X \to S, Y)$ a pair as above, $i \in \mathbb{N}$ and $w \in \mathbb{Z}$. The edges are given by the three types of edges

(1) geometric morphisms $h : (X \to S, Y) \to (X' \to S, Y')$ as above determine edges $h^* : (X' \to S, Y', i, w) \to (X \to S, Y, i, w)$;
(2) connecting morphisms $\partial : (Y \to S, Z, i, w) \to (X \to S, Y, i + 1, w)$ for a chain of inclusions $Z \subset Y \subset X$;
(3) twisted projections: $(X, Y, i, w) \to (X \times \mathbb{P}^1, Y \times \mathbb{P}^1 \cup X \times \{0\}, i + 2, w + 1)$.

For consistency with our previous notation we have here written the morphisms in the contravariant (cohomological) way rather than in the covariant (homological) way used in §3.3 of [3].

Note that in the previous section, following [40] we described the effective Nori motives as $\mathcal{MN}^{eff} = C(\mathrm{Pairs}^{eff}, T)$, with the category of Nori motives $\mathcal{MN}$ being then obtained as the localization of $\mathcal{MN}^{eff}$ at $(\mathbb{G}_m, \{1\}, 1)$ (inverting the Lefschetz motive). Here in the setting of [3] the Tate motives are accounted for in the diagram construction by the presence of the twist w and the last class of edges.

Given $f : X \to S$ and a sheaf $\mathcal{F}$ on X one has $H^i_S(X; \mathcal{F}) = R^i f_* \mathcal{F}$. In the case of a pair $(f : X \to S, Y)$, let $j : X \smallsetminus Y \hookrightarrow X$ be the inclusion and consider $H^i_S(X, Y; \mathcal{F}) = R^i f_* j_! \mathcal{F}|_{X \smallsetminus Y}$. The diagram representation T in this case maps $T(X \to S, Y, i, w) = H^i_S(X, Y, \mathcal{F})(w)$ to the (Tate twisted) constructible sheaf $H^i_S(X, Y; \mathcal{F})$. It is shown in [3] that the Nori formalism of geometric diagrams applies to this setting and gives rise to a Tannakian category of motivic sheaves $\mathcal{MN}_S$. In particular one considers the case where $\mathcal{F}$ is constant with $\mathcal{F} = \mathbb{Q}$, so that the diagram representation $T : D(\mathcal{V}_S) \to \mathbb{Q}_S$

and the induced functor on $\mathcal{MN}_S$ replace at the motivic level the Euler characteristic map on the relative Grothendieck ring $K_0(\mathcal{V}_S) \to K_0(\mathbb{Q}_S)$.

As in the previous cases, we consider an enhancement of this category of motivic sheaves, in the sense of §2.2, by introducing good $\hat{\mathbb{Z}}$-actions that factor that factor through a finite quotient. We modify the construction of [3] in the following way.

We consider a diagram $D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)})$ where the vertices are elements

$$((X, \alpha_X) \to (S, \alpha), (Y, \alpha_Y), i, w)$$

so that the maps $f : X \to S$ and the inclusion $Y \hookrightarrow X$ are $\hat{\mathbb{Z}}$-equivariant, and with morphisms as above, where all the maps are required to be compatible with the $\hat{\mathbb{Z}}$-actions. One obtains by the same procedure as in [3] a category of equivariant motivic sheaves $\mathcal{MN}^{\hat{\mathbb{Z}}}_S$. The representation above maps $D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)})$ to $\hat{\mathbb{Z}}$-equivariant constructible sheaves over (S, α). Then the same argument we used §3 at the level of Grothendieck rings, assemblers and spectra applies to this setting and gives the following result.

Proposition 8.4. *The maps of diagrams*

$$\sigma_n : D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)}) \to D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha\circ\sigma_n)})$$

$$\tilde{\rho}_n : D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S,\alpha)}) \to D(\mathcal{V}^{\hat{\mathbb{Z}}}_{(S\times Z_n,\Phi_n(\alpha))})$$

given by

$$\sigma_n((X, \alpha_X) \to (S, \alpha), (Y, \alpha_Y), i, w) = ((X, \alpha_X \circ \sigma_n) \to (S, \alpha \circ \sigma_n), (Y, \alpha_Y \circ \sigma_n), i, w)$$

$$\tilde{\rho}_n((X, \alpha_X) \to (S, \alpha), (Y, \alpha_Y), i, w) =$$
$$((X \times Z_n, \Phi_n(\alpha_X)) \to (S \times Z_n, \Phi_n(\alpha)), (Y \times Z_n, \Phi_n(\alpha_Y)), i, w)$$

determine functors of the resulting category of motivic sheaves $\mathcal{MN}^{\hat{\mathbb{Z}}}_S$ such that $\sigma_n \circ \tilde{\rho}_n = n\,\mathrm{id}$ and $\tilde{\rho}_n \circ \sigma_n$ is a product with (Z_n, α_n). Thus, one obtains on the category $\mathcal{MN}^{\hat{\mathbb{Z}}}_S$ a Bost–Connes system as in Definition 2.8.

Proof. The argument is as in Proposition 3.1, using again, as in Proposition 8.2 the fact that maps of diagrams induce functors of the resulting categories of Nori motives. $\square$

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