

HOLOGRAPHIC CODES ON BRUHAT–TITS BUILDINGS AND DRINFELD SYMMETRIC SPACES

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ABSTRACT. This paper is based on the author’s talk at the Arbeitstagung 2017. It discusses some general approaches to the construction of classical and quantum holographic codes on Bruhat–Tits trees and buildings and on Drinfeld symmetric spaces, in the context of the p -adic AdS/CFT correspondence.

Dedicated to Yuri Manin on the occasion of his 80th birthday

1. INTRODUCTION

This paper is based on the talk given by the author at the Arbeitstagung 2017 “Physical Mathematics” in honor of Yuri Manin’s 80th birthday. It is an introduction to an ongoing joint project with Matthew Heydeman, Sarthak Parikh and Ingmar Saberi, on the construction of holographic classical and quantum codes on Bruhat–Tits trees and higher rank Bruhat–Tits buildings and on Drinfeld symmetric spaces, and associated entanglement entropy formulae. A discussion of the entanglement entropy and the relation to other holographic codes constructions, such as [42], will be presented in a forthcoming joint paper in preparation. The present paper should be regarded as covering some background material on the question of constructing holographic codes on p -adic symmetric spaces, based on algebro-geometric properties.

In [32], [33], Manin gave a compelling view of the idea of “Arithmetical Physics”, according to which physics in the usual Archimedean setting or real and complex numbers would cast non-Archimedean shadows that live over the finite primes and arithmetic properties associated to these non-Archimedean models can be used to better understand the physics that we experience at the Archimedean “prime at infinity”. According to this general philosophy $\text{Spec}(\mathbb{Z})$ is the “arithmetic coordinate” of physics and geometry. A famous example where this principle manifests itself is given by the description of the Polyakov measure for the bosonic string in terms of the Faltings height function at algebraic points of the moduli space of curves, which leads naturally to the question of whether the Polyakov measure is in fact an adelic object and whether there is an overall arithmetic expression for the string partition function, [34], [35]. More generally, one can ask to what extent are the fundamental laws of physics adelic. Does physics in the Archimedean setting (partition functions, action functionals, real and complex variables) have p -adic manifestations? Can these be used to provide convenient “discretized models” of physics, powerful enough to determine their Archimedean counterpart?

Various forms of p -adic and adelic phenomena in physics and their relation to the usual Archimedean formulation were developed over the years. We refer the readers to [7], [10], [46], [47] for some references relevant to the point of view discussed in this paper.

Here we focus in particular on the holographic AdS/CFT correspondence and on the recent viewpoint relating information (entanglement entropy) of quantum states on the boundary to geometry (classical gravity) on the bulk, [41] and the tensor networks and holographic codes approach of [42]. The existence of a p -adic version of the holographic AdS/CFT correspondence was already proposed in [40], based on earlier results of Manin [36], [37] expressing the Green function on a compact Riemann surface with Schottky uniformization to configurations of geodesics in the bulk hyperbolic handlebody (which are higher genus generalizations of Euclidean BTZ black holes [28]) and results of Drinfeld and Manin [39] on periods of Mumford curves uniformized by p -adic Schottky groups.

In [22] we developed a non-Archimedean version of AdS/CFT holography, based on the approach originally proposed in [40], which would be compatible with the more recent viewpoint on the holographic correspondence based on the ideas of tensor networks and holographic codes and the correspondence between entanglement entropy and bulk geometry. Versions of p -adic AdS/CFT correspondence were also developed in [18], and in subsequent work [3], [19], [16], [15], [17] and others. The theme of non-archimedean versions of holography has clearly become a very active area of current research.

In this paper, we return to the point of view of tensor networks and holographic codes discussed in [22] and we present some new constructions which are based on the geometry of Bruhat–Tits trees and buildings and of Drinfeld symmetric spaces.

The main difference between the approach we propose here and other constructions of holographic codes such as [42], or for instance [3], [5], [21], lies in the fact that we rely on well known techniques for the construction of classical codes associated to algebro-geometric objects [45] and on algorithms relating classical to quantum codes [6]. The construction of algebro-geometric codes played a crucial role in the study of asymptotic problems in coding theory, as shown by Manin in [38].

We first present here a construction of holographic codes that is based on the geometry of the Bruhat–Tits trees and algebro-geometric Reed–Solomon codes associated to projective lines over a finite field, together with an application of the CRSS algorithm that associates quantum codes to classical q -ary codes.

We then revisit the approach to holographic codes via tessellations of the hyperbolic plane, as in [42]. Instead of relating such constructions to the Bruhat–Tits trees via a non-canonical planar embedding of the tree, as in [22], we use here a purely p -adic viewpoint, working with the Drinfeld p -adic upper half plane as a replacement of the real hyperbolic plane, and its (canonical) map to the Bruhat–Tits tree. Instead of tessellations of the real hyperbolic plane we use actions of p -adic Fuchsian groups on the Drinfeld plane and associated surface codes. We show that this approach is

restricted by the strong constraints that exist on p -adic Fuchsian groups. For example, we show that a p -adic analog of the holographic pentagon code of [42] constructed with this method can only exist when $p = 2$.

We then propose an extension of this approach via holographic codes to higher rank buildings, based on algebro-geometric codes associated to higher dimensional algebraic varieties, as constructed in [45].

2. ALGEBRO-GEOMETRIC CODES ON THE BRUHAT–TITS TREE

In this section we describe a construction of holographic codes on the Bruhat–Tits trees that are obtained via Reed–Solomon algebro-geometric codes on projective lines over finite fields.

2.1. Reed–Solomon codes and classical codes on the Bruhat–Tits tree. The set of algebraic points $X(\mathbb{F}_q)$ of a curve X over a finite field \mathbb{F}_q can be used to construct algebro-geometric error-correcting codes, see [44]. Algebro-geometric codes associated to a curve X over a finite field \mathbb{F}_q consists of a choice of a set A of algebraic points $A \subset X(\mathbb{F}_q)$ and a divisor D on X with support disjoint from A . The linear code $C = C_X(A, D)$ is obtained by considering rational functions $f \in \mathbb{F}_q(X)$ with poles at D and evaluating them at the points of A . A bound on the order of pole of f at D determines the dimension of the linear code.

We are interested here in the simplest case of algebro-geometric codes, the Reed–Solomon codes constructed using the points of $\mathbb{P}^1(\mathbb{F}_q)$. Given a set of points $A \subset \mathbb{P}^1(\mathbb{F}_q)$ with $\#A = n \leq q + 1$ we consider two types of Reed–Solomon codes, one constructed using the point $\infty \in \mathbb{P}^1(\mathbb{F}_q)$ as divisor, that is, using polynomials $f \in \mathbb{F}_q[x]$, and using a set A of $n \leq q$ points in $\mathbb{A}^1(\mathbb{F}_q) = \mathbb{F}_q$ for evaluation. The corresponding Reed–Solomon code $C = \{(f(x_1), \dots, f(x_n)) : f \in \mathbb{F}_q[x], \deg(f) < k\}$ gives an $[n, k, n - k + 1]_q$ classical code, where $n \leq q$. The other type of Reed–Solomon codes are obtained using homogeneous polynomials and a set A of $n \leq q + 1$ points in $\mathbb{P}^1(\mathbb{F}_q)$. The resulting code $\hat{C} = \{(f(u_1, v_1), \dots, f(u_n, v_n)) : f \in \mathbb{F}_q[u, v], \text{homogeneous with } \deg(f) < k\}$, with $x_i = (u_i : v_i) \in \mathbb{P}^1(\mathbb{F}_q)$. We also consider generalized Reed–Solomon codes of these two types, where for a vector $w = (w_1, \dots, w_n) \in \mathbb{F}_q^n$ one defines

$$C_{w,k} = \{(w_1 f(x_1), \dots, w_n f(x_n)) : f \in \mathbb{F}_q[x], \deg(f) < k\}$$

$$\hat{C}_{w,k} = \{(w_1 f(u_1, v_1), \dots, w_n f(u_n, v_n)) : f \in \mathbb{F}_q[u, v], \text{homogeneous, } \deg(f) < k\}.$$

For \mathbb{K} a finite extension of \mathbb{Q}_p with residue field \mathbb{F}_q , with $q = p^r$, the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$ is a homogeneous tree with valence $q + 1 = \#\mathbb{P}^1(\mathbb{F}_q)$ and with ends $\partial\mathcal{T}_{\mathbb{K}} = \mathbb{P}^1(\mathbb{K})$. The choice of a projective coordinate on $\mathbb{P}^1(\mathbb{K})$ fixes three points $\{0, 1, \infty\} \in \mathbb{P}^1(\mathbb{K})$, hence it fixes a unique root vertex $\nu_0 \in V(\mathcal{T}_{\mathbb{K}})$. The star of vertices surrounding ν_0 can then be identified with a copy of $\mathbb{P}^1(\mathbb{F}_q)$, which in algebro-geometric terms corresponds to the reduction modulo the maximal ideal \mathfrak{m} in $\mathcal{O}_{\mathbb{K}}$.

The root vertex ν_0 is therefore associated to the reduction curve \mathbb{P}^1 . We can construct a holographic *classical code* to the Bruhat–Tits tree by assigning to the root

vertex ν_0 and its star of $q + 1$ edges a Reed–Solomon code with an assigned number k of logical inputs (q -ary bits) located at ν_0 and outputs at each of the $q + 1$ legs. This can be done by a (generalized) Reed–Solomon code $\hat{C}_{w,k}$ of maximal length $n = q + 1$, seen as an encoding $\hat{C}_{w,k} : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^{q+1}$, which inputs a k -tuple of q -ary bits $a = (a_0, \dots, a_{k-1}) \in \mathbb{F}_q^k$, uses the homogeneous polynomial $f_a(u, v) = \sum_{i=0}^k a_i u^i v^{k-1-i}$, and outputs a q -ary bit $f(u_j, v_j) \in \mathbb{F}_q$ at each point $x_j = (u_j : v_j) \in \mathbb{P}^1(\mathbb{F}_q)$ identified with a leg of the vertex ν_0 in the Bruhat–Tits tree.

The choice of the projective coordinate on $\mathbb{P}^1(\mathbb{K})$, hence of the root vertex ν_0 in $\mathcal{T}_{\mathbb{K}}$, determines a choice of a leg at each other vertex $\nu \neq \nu_0$, given by the unique direction out of ν towards the root ν_0 . We can identify this choice with a choice of the point $\{\infty\}$ in each copy of $\mathbb{P}^1(\mathbb{F}_q)$ at each vertex $\nu \neq \nu_0$ of the tree. Proceeding from the center, if we assign a Reed–Solomon code at each vertex of $\mathcal{T}_{\mathbb{K}}$, and by homogeneity we expect all of them to have the same number k of inputs, we see that at each successive steps the leg of the star of edges at ν has already one value assigned at the leg labelled by the point $\infty \in \mathbb{P}^1(\mathbb{F}_q)$, which corresponds to the output coming from the matching leg in the star of the previous vertex coming from the root ν_0 . Thus, in projective coordinates $(u : v)$ where $(0 : 1)$ is the point at infinity, the Reed–Solomon code $\hat{C}_{w,k}$ associated to the vertex ν takes $k - 1$ new inputs $a = (a_1, \dots, a_{k-1}) \in \mathbb{F}_q^{k-1}$ and one additional input a_0 given by the value at ∞ assigned by the previous code, and deposits a new q -ary bit $f_a(u, v) = \sum_{i=0}^{k-1} a_i u^i v^{k-1-i}$ at each of the remaining legs at the vertex ν pointing away from the root, labelled by the points $x \in \mathbb{A}^1(\mathbb{F}_q) = \mathbb{F}_q$.

Note how the construction considered here has one root vertex play a special role with an \mathbb{F}_q^k logical input and a Reed–Solomon code of length $q + 1$, while all the other vertices have a further logical input of \mathbb{F}_q^{k-1} . This asymmetry is inevitable if we want to use the algebro-geometric structure underlying the Bruhat–Tits tree to construct a classical code, since the root vertex plays the special role of the algebraic curve given by the reduction modulo \mathfrak{m} , while the sets of vertices in the tree at distance m from the root correspond to reducing modulo powers \mathfrak{m}^m . Thus, the asymmetric role of the root vertex and the other vertices is built into the relation between $\mathbb{P}^1(\mathbb{K})$ and its reduction curves.

The construction described here determines a classical code associated to the Bruhat–Tits tree with logical inputs at the vertices and outputs at the forward pointing legs. In the limit where one considers the whole tree, the outputs consist of a q -ary bit deposited at each point of the boundary $\mathbb{P}^1(\mathbb{K})$. We want to transform this classical code built using the algebro-geometric properties of the Bruhat–Tits tree, into a quantum error correcting code that generates a holographic code for the Bruhat–Tits tree and its boundary at infinity.

2.2. Classical Algebro-Geometric Codes for Mumford Curves. The construction above can be generalized in the case of Mumford curves. Let Γ be a p -adic Schottky group and $\Omega_{\Gamma} = \mathbb{P}^1(\mathbb{K}) \setminus \Lambda_{\Gamma}$ the domain of discontinuity of Γ acting on the boundary $\mathbb{P}^1(\mathbb{K})$, the complement of the limit set Λ_{Γ} . The quotient $X = \Omega_{\Gamma}/\Gamma$ is a Mumford curve of genus g equal to the number of generators of the Schottky group.

Unlike complex Riemann surfaces, which always admit a Schottky uniformization, only very special p -adic curves admit a Mumford curve uniformization. Indeed, these curves must have the property that their reduction mod \mathfrak{m} is totally split: as a curve over \mathbb{F}_q it consists of a collection of \mathbb{P}^1 's with incidence relations described by the dual graph G . This is the finite graph at the center of the quotient $\mathcal{T}_{\mathbb{K}}/\Gamma$, obtained as the quotient $G = \mathcal{T}_{\Gamma}/\Gamma$, where \mathcal{T}_{Γ} is the subtree of $\mathcal{T}_{\mathbb{K}}$ spanned by the geodesic axes of the hyperbolic elements $\gamma \neq 1$ of Γ , with $\partial\mathcal{T}_{\Gamma} = \Lambda_{\Gamma}$.

Using the identification between the finite graph G and the dual graph of the reduction curve, we can again associate to each vertex in G a copy of $\mathbb{P}^1(\mathbb{F}_q)$ (the corresponding component in the curve), and to each of these projective lines a Reed–Solomon code as in the previous construction. Now, however, we need to impose compatibility conditions between these codes at the incidence points between different components of the curve, that is, along the edges of the finite graph G . Thus, we associate to the finite graph G a classical code $C(G)$ constructed as follows. Start with a code $\hat{C}_{w,k}$ associated to each vertex v , which inputs $a = (a_0, \dots, a_{k-1})$ and outputs $f_a(u, v) = \sum_i a_i u^i v^{k-1-i}$ at each point $x = (u : v)$ of the associated $\mathbb{P}^1(\mathbb{F}_q)$. Consider the set \mathcal{F} of functions $f = (f_1, \dots, f_N)$, with $N = \#V(G)$, and f_i a homogeneous polynomial of degree $\deg(f_i) < k$ on the i -th component $\mathbb{P}^1(\mathbb{F}_q)$, with the property that if $x = (u_i : v_i) = (u_j : v_j)$ is an intersection point between the i -th and the j -th components of the reduction curve, then $f_i(u_i : v_i) = f_j(u_j : v_j)$. Thus, each edges $e \in E(G)$ imposes a relation between f_i and f_j which requires the value that the codes $\hat{C}_{w,k}$ at the vertices v_i and v_j deposit at the point x to be the same. Thus, the resulting code $C(G)$ is an \mathbb{F}_q -linear code with input \mathbb{F}_q^{kN-M} where $N = \#V(G)$ and $M = \#E(G)$. We have $kN - M = (k - 1)N + 1 - b_1(G)$ hence we need to assume $k > 1 + (b_1(G) - 1)/N$.

The free legs of the graph G are all the legs that point towards the infinite trees in $\mathcal{T}_{\mathbb{K}}/\Gamma$ that extend from the vertices of G to the boundary Mumford curve $X(\mathbb{K}) = \partial\mathcal{T}_{\mathbb{K}}/\Gamma$. At each vertex along these trees we consider Reed–Solomon codes as in the case of $\mathbb{P}^1(\mathbb{K})$, with one input coming from the previous vertex closer to G and $k - 1$ new inputs and outputs at the q forward pointing legs. This determines a classical code associated to the infinite graph $\mathcal{T}_{\mathbb{K}}/\Gamma$, with logical inputs at the vertices and outputs at the points of the Mumford curve $X(\mathbb{K})$. The finite graph G and the infinite graph $\mathcal{T}_{\mathbb{K}}/\Gamma$ containing it are a genus g generalization of the p -adic BTZ black hole, which corresponds to the $g = 1$ case of Mumford–Tate elliptic curves.

2.3. Reed–Solomon Codes and Quantum Algebro-Geometric Codes. There is a general procedure for passing from classical codes to quantum codes, based on the Calderbank–Rains–Shor–Sloane algorithm [6], see also [1]. It can be applied to certain classes of algebro-geometric codes and in particular to generalized Reed–Solomon codes.

Let $\mathcal{H} = \mathbb{C}^q$ be the Hilbert space of a single q -ary qubit and $\mathcal{H}_n = (\mathbb{C}^q)^{\otimes n}$ the space of n q -ary qubits. We label an orthonormal basis of \mathcal{H} by $|a\rangle$ with $a \in \mathbb{F}_q$. Thus, a q -ary qubit is a vector $\psi = \sum_{a \in \mathbb{F}_q} \lambda_a |a\rangle$ with $\lambda_a \in \mathbb{C}$, and an n -tuple of q -ary qubits

is given by a vector $\psi = \sum_{a=(a_1 \dots a_n) \in \mathbb{F}_q^n} \lambda_a |a\rangle$ where $|a\rangle = |a_1\rangle \otimes \dots \otimes |a_n\rangle$. Quantum error correcting codes are subspaces \mathcal{C} of \mathcal{H}_n that are error correcting for a certain number of “ q -ary bit flip” and “phase flip” errors. More precisely, an error operator E is detectable by a quantum code \mathcal{C} if $P_{\mathcal{C}} E P_{\mathcal{C}} = \lambda_E P_{\mathcal{C}}$, where $P_{\mathcal{C}}$ is the orthogonal projection onto the code subspace and λ_E is a scalar. In particular, one considers error operators that affect up to a certain number of qubits in an n -qubits state, namely error operators of the form $E = E_1 \otimes \dots \otimes E_n$, of weight $\omega(E) = \#\{i : E_i \neq I\}$. The minimum distance $d_Q(\mathcal{C})$ of the quantum code is the largest d such that all errors with $\omega(E) < d$ are detectable.

The bit and phase flip error operators are defined on a single q -ary qubit as

$$T_b |a\rangle = |a + b\rangle, \quad R_b |a\rangle = \xi^{\text{Tr}(\langle a, b \rangle)} |a\rangle,$$

where ξ is a p -th primitive root of unity, and $\text{Tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ is the trace function, $\text{Tr}(a) = \sum_{i=0}^{r-1} a^{p^i}$, with $\langle a, b \rangle = \sum_{i=1}^r a_i b_i$ and with $R^{b_i} |a_j\rangle = \xi^{\text{Tr}(a_j b_i)} |a_j\rangle$. Let $\{\gamma_i\}_{i=1}^r$ be a basis of \mathbb{F}_q as an \mathbb{F}_p -vector space, so that $a = \sum_i a_i \gamma_i$ and $b = \sum_i b_i \gamma_i$. Then the error operators T_b and R_b can be written respectively as

$$T_b = T^{b_1} \otimes \dots \otimes T^{b_r}, \quad R_b = R^{b_1} \otimes \dots \otimes R^{b_r}$$

with T and R given by the operators acting on \mathbb{C}^p of matrix form

$$T = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & & & & \\ & \xi & & & \\ & & \xi^2 & & \\ & & & \ddots & \\ & & & & \xi^{p-1} \end{pmatrix}$$

satisfying the commutation relation $TR = \xi RT$. The operators $T_a R_b$ with $a, b \in \mathbb{F}_q$ form an orthonormal basis for $M_{q \times q}(\mathbb{C})$ under the inner product $\langle A, B \rangle = q^{-1} \text{Tr}(A^* B)$, hence these operators generate all possible quantum errors on the space \mathbb{C}^q of a single q -ary qubit. The action of error operators on a state of n q -ary qubits can similarly be written in terms of operators $T_a R_b$ with

$$E_{a,b} = T_a R_b = (T_{a_1} \otimes \dots \otimes T_{a_n})(R_{b_1} \otimes \dots \otimes R_{b_n}),$$

for $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{F}_q^n$. The operators $E_{a,b}$ satisfy $E_{a,b}^p = I$ and the commutation and composition rules

$$E_{a,b} E_{a',b'} = \xi^{\langle a, b' \rangle - \langle b, a' \rangle} E_{a',b'} E_{a,b}, \quad E_{a,b} E_{a',b'} = \xi^{-\langle b, a' \rangle} E_{a+a', b+b'},$$

where $\langle a, b \rangle = \sum_i \langle a_i, b_i \rangle = \sum_{i,j} a_{i,j} b_{i,j}$, with $a_i, b_i \in \mathbb{F}_q$, written as $a_i = \sum_j a_{i,j} \gamma_j$ and $b_i = \sum_j b_{i,j} \gamma_j$, after identifying \mathbb{F}_q as a vector space with \mathbb{F}_p^r . Thus, we can consider the group $\mathcal{G}_n = \{\xi^i E_{a,b}, a, b \in \mathbb{F}_q^n, 0 \leq i \leq p-1\}$ of order pq^{2n} . A quantum stabilizer error-correcting code \mathcal{C} is a subspace $\mathcal{C} \subset \mathcal{H}_n$ that is a joint eigenspace of operators $E_{a,b}$ in an abelian subgroup $\mathcal{S} \subset \mathcal{G}_n$.

Let $\varphi \in \text{Aut}_{\mathbb{F}_p}(\mathbb{F}_p^r)$ be an automorphism. In particular, we consider φ given by the trace as in [1], so that we the associated pairing is

$$\langle (a, b), (a', b') \rangle = \langle a, \varphi(b') \rangle - \langle a', \varphi(b) \rangle = \text{Tr}(\langle a, b' \rangle_* - \langle a', b \rangle_*),$$

where, for $a, b \in \mathbb{F}_q^n$, the inner product $\langle a, b \rangle \neq \langle a, b \rangle_*$, since $\langle a, b \rangle_* = \sum_{i=1}^n a_i b_i$, while $\langle a, b \rangle = \sum_{i=1}^n \langle a_i, b_i \rangle = \sum_{i=1}^n \sum_{j=1}^r a_{i,j} b_{i,j}$. If $C \subset \mathbb{F}_q^{2n}$ is a classical self-orthogonal code with respect to this pairing, then the subgroup $\mathcal{S} \subset \mathcal{G}_n$ given by the elements $\xi^i E_{a, \varphi(b)}$ with $(a, b) \in C$ is an abelian subgroup of \mathcal{G}_n , because of the commutation rule above. This construction is the CRSS algorithm that associates to a self-orthogonal classical $[2n, k, d]_q$ code a stabilizer quantum $[[n, n - k, d_Q]]_q$ -code, where $d_Q = \min\{\omega(a, b) : (a, b) \in C^\perp \setminus C\}$, where the weight $\omega(a, b) = \#\{i : a_i \neq 0 \text{ or } b_i \neq 0\}$, and $C^\perp = \{(v, w) \in \mathbb{F}_q^{2n} : \langle (a, b), (v, w) \rangle = 0, \forall (a, b) \in C\}$.

We can view the CRSS algorithm assigning the quantum stabilizer code \mathcal{C} to the classical code C as an encoding process that takes the q^k input vectors $(v, w) \in \mathbb{F}_q^{2n}$ of the classical code C and encodes the states $|(v, w)\rangle$ using the vectors $\psi \in (\mathbb{C}^q)^{\otimes n}$ satisfying $E_{v, \varphi(w)}\psi = \lambda\psi$ in a common eigenspace of the $E_{v, \varphi(w)}$.

A slightly more general version of the CRSS algorithm starts with two classical linear q -ary codes $C_1 \subseteq C_2$ of length n and dimensions k_1 and k_2 and associates to them a quantum code $\mathcal{C} = \mathcal{C}(C_1, C_2)$ with parameters $[[n, k_2 - k_1, \min\{d(C_2 \setminus C_1), d(C_1^\perp \setminus C_2)\}]]_q$, see [13], [26]. The procedure for the construction of the quantum code is similar to the version of the CRSS algorithm recalled above. One constructs a code $C = \gamma C_1 + \bar{\gamma} C_2$ in $\mathbb{F}_{q^2}^n$ with γ a primitive element of \mathbb{F}_{q^2} and $\{\gamma, \bar{\gamma}\}$ a linear basis of \mathbb{F}_{q^2} as an \mathbb{F}_q -vector space. By identifying $\mathbb{F}_{q^2}^n$ as an \mathbb{F}_q -vector space with \mathbb{F}_q^{2n} , we obtain a self orthogonal $C \subset \mathbb{F}_q^{2n}$, to which the CRSS algorithm discussed before can be applied.

Conditions under which Reed-Solomon codes satisfy a self-dual condition, and the corresponding quantum Reed-Solomon codes obtained via a CRSS type algorithm are analyzed, for instance, in [12] and [14]. We use here a construction of [1], which shows that, if C is a q^2 -ary classical $[n, k, d]_{q^2}$ -code, which is Hermitian-self-dual, then there exists an associated q -ary $[[n, n - 2k, d_Q]]_q$ -quantum code, with $d_Q \geq d$. Here Hermitian-self-dual means that the classical code C is self dual with respect to the ‘‘Hermitian’’ pairing

$$\langle v, w \rangle_H = \sum_{i=1}^n v_i w_i^q, \quad \text{for } v, w \in \mathbb{F}_{q^2}^n.$$

This is a variant of the CRSS algorithm described above, where a Hermitian-self-dual code of length n over the field extension \mathbb{F}_{q^2} is used to construct a self-dual code \tilde{C} of length $2n$ over \mathbb{F}_q to which the CRSS algorithm can be applied, obtained by expanding the code words $v \in C$ using a basis $\{1, \gamma\}$ of \mathbb{F}_{q^2} as a \mathbb{F}_q -vector space, where γ is an element in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$ satisfying $\gamma^q = -\gamma + \gamma_0$ for some fixed $\gamma_0 \in \mathbb{F}_q$. Using this approach, it suffices to construct generalized Reed–Solomon codes $\hat{C}_{w,k}$ of length $n < q + 1$ over \mathbb{F}_{q^2} that are Hermitian self-dual, in order to obtain associated quantum codes $\hat{C}_{w, 2n-k}$

as a code subspace of the n q -ary qubits space $\mathcal{H}_n = (\mathbb{C}^q)^{\otimes n}$. It is possible to ensure the Hermitian self-duality condition for generalized Reed–Solomon codes by taking the weights vector $w = (w_1, \dots, w_n) \in (\mathbb{F}_{q^2}^*)^n$ to satisfy $\sum_{i=1}^n w_i^{q+1} x_i^{qj+\ell} = 0$ for all $0 \leq j, \ell \leq k-1$, where $x = (x_1, \dots, x_n) \in \mathbb{F}_{q^2}^n$ are the n chosen points (excluding ∞) of $\mathbb{P}^1(\mathbb{F}_{q^2})$. Using this method, it is proved in [29] that the choice $w_i = 1$, with $n = q^2 = \#\mathbb{F}_{q^2}$ and $k = q$, produces a Reed–Solomon code $C = C_{1,q}$ that is Hermitian-self-dual, and an associated $[[q^2 + 1, q^2 - 2q + 1, q + 1]]_q$ -quantum Reed–Solomon code \hat{C} . Moreover, it is also shown in [29] that for w_i satisfying $w_i^{q+1} = (\prod_{j \neq i} (x_i - x_j))^{-1}$, with $n \leq q$ and $x = (x_1, \dots, x_n) \in \mathbb{F}_q$, and for $k \leq \lfloor n/2 \rfloor$, the generalized Reed–Solomon codes satisfy $C_{w,k} \subseteq C_{w,n-k}$ and $C_{w,k}$ is hermitian self dual to $C_{w,n-k}$. Thus, hence the CRSS algorithm can be applied to obtain an $[[n, n - 2k, k + 1]]_q$ -quantum Reed–Solomon code $\mathcal{C}_{w,k} = \mathcal{C}(C_{w,k}, C_{w,n-k})$.

2.4. The case of the perfect tensors. In particular, from the construction described above we see that we obtain the case of perfect tensors as the special case where $n = q$ and $k = (q - 1)/2$. We obtain this using the generalized Reed–Solomon codes as in Theorem 6 of [29], for the case $n \leq q$ and $k \leq \lfloor \frac{n}{2} \rfloor$, with a choice of the weights w_i satisfying $w_i^{q+1} = (\prod_{j \neq i} (x_i - x_j))^{-1}$, with $n = q$ and $x_i \in \mathbb{F}_q$. As shown in Theorem 6 of [29], this produces two classical generalized Reed–Solomon codes $C_{w, \frac{q-1}{2}} \subseteq C_{w, \frac{q+1}{2}}$ that are hermitian self-dual. The associated quantum generalized Reed–Solomon code is then obtained via the general construction of Ashikhmin–Knill (Theorem 4 and Corollary 1 of [1]) that associates to a classical $[n, k, d]_{q^2}$ code contained in its hermitian dual a quantum $[[n, n - 2k, d]]_q$ code. One can see directly that, in the case of perfect tensors when $n = q$, the weights are constant and given by $w_i^{q+1} = p - 1$ for all $i = 1, \dots, q$.

Thus, we can regard the construction described above with generalized Reed–Solomon codes as a generalization of the usual construction of perfect tensors, which recovers the perfect tensor case for a particular choice of (constant) weights of the classical Reed–Solomon codes.

The more general cases with non-constant weights assign different weights to different directions in the Bruhat–Tits tree. These may be useful in view of holographic models where the bulk geometry is dynamical, as in [15], and also described by different weights in different directions in the tree.

2.5. Holographic quantum codes on Bruhat–Tits trees and Mumford curves.

We use the procedure described above to pass from classical algebro-geometric codes, in particular generalized Reed–Solomon codes, to associated quantum stabilizer codes, to construct a holographic code on the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$ associated to the classical codes constructed above.

We have seen above that, in order to apply the CRSS algorithm, we pass to a quadratic extension \mathbb{F}_{q^2} of \mathbb{F}_q and consider Reed–Solomon codes over \mathbb{F}_{q^2} . In terms of the Bruhat–Tits tree, we can pass to an unramified quadratic extension \mathbb{L} of the field \mathbb{K} , so that the Bruhat–Tits tree $\mathcal{T}_{\mathbb{L}}$ is obtained from the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$

simply by adding new branches at each vertex, so as to obtain a homogeneous tree of valence $q^2 + 1$. Since the extension is unramified, it is not necessary to insert new vertices along the edges, and we can view the tree $\mathcal{T}_{\mathbb{K}}$ as a subtree of $\mathcal{T}_{\mathbb{L}}$. We then proceed to construct a classical code associated to $\mathcal{T}_{\mathbb{L}}$ using Reed-Solomon codes $\hat{C}_{w,k}$ placed at the vertices according to the procedure described in the previous sections. Using the construction above, with $w_i = 1$ and $k = q$, we associate to each vertex a quantum Reed-Solomon code \hat{C} with code parameters $[[q^2 + 1, q^2 - 2q + 1, q + 1]]_q$. This corresponds to considering a state space $\mathcal{H}_{q^2+1} = (\mathbb{C}^q)^{\otimes q^2+1}$ associated to each vertex, which we can think of as a state with a q -ary qubit sitting at each of the $q^2 + 1$ points of $\mathbb{P}^1(\mathbb{F}_{q^2})$, or equivalently at each of the legs surrounding that vertex in $\mathcal{T}_{\mathbb{L}}$. The quantum code \hat{C} detects quantum errors of weight up to $q + 1 = \#\mathbb{P}^1(\mathbb{F}_q)$. Thus, by identifying $\mathcal{T}_{\mathbb{K}} \subset \mathcal{T}_{\mathbb{L}}$ and $\mathbb{P}^1(\mathbb{F}_q) \subset \mathbb{P}^1(\mathbb{F}_{q^2})$ as the set of directions along the subtree $\mathcal{T}_{\mathbb{K}}$, we can arrange that the code \hat{C} corrects quantum errors along the $\mathcal{T}_{\mathbb{K}}$ directions. One can also use a bipartition $A \cup A^c$ of the edges at each vertex of $\mathcal{T}_{\mathbb{L}}$, with $\#A = k$ and associate to the bipartition a pair of codes $\hat{C}_{w,k}$ and $\hat{C}_{w,n-k}$ with associated quantum Reed-Solomon codes $\hat{C}_{w,k}$ as above at the vertices of $\mathcal{T}_{\mathbb{K}}$.

One can think of the classical codes \hat{C} associated to the vertices of the Bruhat–Tits tree in this way as performing an encoding of $q + 1$ classical q^2 -ary bits associated to the points of $\mathbb{P}^1(\mathbb{K})$ into $q^2 + 1$ classical q^2 -ary bits associated to the points of $\mathbb{P}^1(\mathbb{L})$. Thus, the whole classical code associated to this quadratic extension can be seen as a way of encoding a state consisting of classical q^2 -ary bits associated to the edges of $\mathcal{T}_{\mathbb{K}}$ into a set of classical q^2 -ary bits associated to the edges of $\mathcal{T}_{\mathbb{L}}$, and the latter into a state of q^2 -ary bits associated to the set of boundary points $\mathbb{P}^1(\mathbb{L})$. The corresponding CRSS quantum codes \hat{C} at the vertices of the Bruhat–Tits tree $\mathcal{T}_{\mathbb{L}}$ encode the input given by the common eigenspace of the error operators associated to the code words of the classical code \hat{C} into a state consisting of a q -ary qubit placed at each leg around the vertex.

In order to combine these quantum codes placed at the vertices of the Bruhat–Tits tree $\mathcal{T}_{\mathbb{L}}$ into a holographic code over the whole tree, with logical inputs in the bulk and physical outputs at the boundary $\mathbb{P}^1(\mathbb{L})$, notice that at each vertex v we have the same subspace \mathcal{H}_v given by the common eigenspace $\mathcal{H}_v = \{\psi : E_{v,\varphi(w)}\psi = \lambda\psi\}$ for all words (v, w) in the classical code. We encode states $\psi_v \in \mathcal{H}_v$ as $\psi_v = (\psi_{v,x})_{x \in \mathbb{P}^1(\mathbb{F}_{q^2})}$, where the points $x \in \mathbb{P}^1(\mathbb{F}_{q^2})$ label the legs around the vertex v , so that we think of $\psi_{v,x} \in \mathbb{C}^q$ as the q -ary qubit deposited on the leg x by the quantum code \hat{C} sitting at the vertex v . Starting at the root vertex and proceeding towards the outside of the tree, at each next step, the leg $\infty \in \mathbb{P}^1(\mathbb{F}_{q^2})$ around the new vertex v is the one connected to a leg $x_i \in \mathbb{F}_{q^2} \subset \mathbb{P}^1(\mathbb{F}_{q^2})$ of the previous vertex v' , which receives an output $\psi_{v',i}$. Thus, the q -ary qubit $\psi_{v,\infty}$ is determined as it has to match the output $\psi_{v',i}$ of the previous code, while the remaining possible inputs correspond to the choices of $\psi \in \mathcal{H}_v$ with that fixed $\psi_{v,\infty}$ component. Proceeding towards the boundary of the tree determines a holographic code on $\mathcal{T}_{\mathbb{L}}$ that outputs q -ary qubits at the points of $\mathbb{P}^1(\mathbb{L})$. As mentioned above, the quantum code detects errors along

the subtree $\mathcal{T}_{\mathbb{K}}$. As in the case of the classical codes, there is an asymmetry in this construction of the holographic code between the roles of the root vertex and of the remaining vertices of the Bruhat–Tits tree.

3. DISCRETE AND CONTINUOUS BULK SPACES: BRUHAT-TITS BUILDINGS AND DRINFELD SYMMETRIC SPACES

Unlike its Archimedean counterparts, either Euclidean $\text{AdS}_2/\text{CFT}_1$ with bulk \mathbb{H}^2 and boundary $\mathbb{P}^1(\mathbb{R})$ or Euclidean $\text{AdS}_3/\text{CFT}_2$ with bulk \mathbb{H}^3 and boundary $\mathbb{P}^1(\mathbb{C})$, the p -adic AdS/CFT correspondence has two different choices of bulk spaces (one discrete and one continuous) which share the same conformal boundary at infinity. The discrete version of the bulk space is given by the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$ of $\text{PGL}(2, \mathbb{K})$, with \mathbb{K} a finite extension of \mathbb{Q}_p , while the continuous form of the bulk space is given by Drinfeld’s p -adic upper half plane Ω . Both have the same boundary $\mathbb{P}^1(\mathbb{K})$. We argue here that the full picture of the p -adic AdS/CFT correspondence should take into account both of these bulk spaces and the relation between them induced by the norm map.

The rank-two case can be generalized to higher rank, with the Bruhat–Tits buildings of $\text{PGL}(n, \mathbb{K})$ generalizing the Bruhat–Tits tree and the higher dimensional Drinfeld symmetric spaces generalizing the Drinfeld upper half plane, see [23].

The geometry of the Drinfeld plane. We review quickly the geometry of the Drinfeld upper half plane, see [4]. We denote by \mathbb{K} a finite extension of \mathbb{Q}_p and by \mathbb{C}_p the completion of the algebraic closure of \mathbb{K} . Drinfeld’s p -adic upper half plane is the space

$$\Omega = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{K}).$$

We also denote by $\mathcal{T}_{\mathbb{K}}$ the Bruhat–Tits tree of $\text{PGL}(2, \mathbb{K})$, with boundary at infinity $\mathbb{P}^1(\mathbb{K})$. It is convenient to think of $\mathbb{P}^1(\mathbb{C}_p)$ as the set of classes, up to homotheties in \mathbb{C}_p^* , of non-zero \mathbb{K} -linear maps $\varphi : \mathbb{K}^2 \rightarrow \mathbb{C}_p$, with $\mathbb{P}^1(\mathbb{K})$ the set of classes of maps as above with \mathbb{K} -rank equal to one. This can be seen by identifying points $(\alpha : \beta)$ of \mathbb{P}^1 with homogeneous ideals $\langle y\alpha - x\beta \rangle$ in the polynomial ring in the variables (x, y) . The \mathbb{K} -linear map $\varphi : \mathbb{K}^2 \rightarrow \mathbb{C}_p$ given by $\varphi(x, y) = y\alpha - x\beta$ has a non-trivial kernel when $\alpha/\beta \in \mathbb{K}$ (assuming $\beta \neq 0$) and is invertible if $\alpha/\beta \in \mathbb{C}_p \setminus \mathbb{K}$. Thus, $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{K})$ can be identified with the set of homothety classes of invertible \mathbb{K} -linear maps $\varphi : \mathbb{K}^2 \rightarrow \mathbb{C}_p$.

Given such an injective linear map, one can then compose it with the norm on \mathbb{C}_p . Recall that the Bruhat–Tits tree can be defined in terms of equivalence classes of norms. Namely, vertices of the Bruhat–Tits tree correspond to classes of lattices M in \mathbb{K}^2 up to similarity, namely $M_1 \sim M_2$ if $M_1 = \lambda M_2$ for some $\lambda \in \mathbb{K}^*$. To a lattice M one associates a norm $|\cdot|_M$, namely a real valued function on \mathbb{K}^2 which is positive on non-zero elements, satisfies $|a \cdot \underline{x}|_M = |a| \cdot |\underline{x}|_M$ for all $a \in \mathbb{K}$ and $\underline{x} \in \mathbb{K}^2$, with $|a|$ the p -adic norm on \mathbb{K} , and $|\underline{x} + \underline{y}|_M \leq \max\{|\underline{x}|_M, |\underline{y}|_M\}$. The norm $|\underline{x}|_M$ is defined as follows. Let π be a uniformizer in $\mathcal{O}_{\mathbb{K}}$ such that $k = \mathcal{O}_{\mathbb{K}}/\pi\mathcal{O}_{\mathbb{K}}$ is the residue field $k = \mathbb{F}_q$. The fractional ideal $\{\lambda \in \mathbb{K} : \lambda \underline{x} \in M\}$ is generated by a power π^m . The norm is then

defined as $|\underline{x}|_M = q^m$ on non-zero vectors. Equivalent norms $|\cdot|_{M_1} = \gamma|\cdot|_{M_2}$ for $\gamma \in \mathbb{R}_+^*$ correspond to equivalent lattices. Two vertices in the Bruhat–Tits tree are adjacent iff the corresponding equivalence classes of lattices have representatives satisfying $\pi M \subset M' \subset M$. To see this in terms of norms, we can choose an $\mathcal{O}_{\mathbb{K}}$ -basis $\{e_1, e_2\}$ for M and $\{e_1, \pi e_2\}$ for M' . Then $|xe_1 + ye_2|_M = \max\{|x|, |y|\}$ and $|xe_1 + ye_2|_{M'} = \max\{|x|, |\pi|^{-1} \cdot |y|\}$. The edge e between the vertices $v = [M]$ and $v' = [M']$ is then parameterized by the classes of norms $|xe_1 + ye_2|_t = \max\{|x|, |\pi|^{-t} \cdot |y|\}$ for $0 \leq t \leq 1$, see [23]. This description of the Bruhat–Tits tree in terms of equivalence classes of norms on \mathbb{K}^2 determines a map from the Drinfeld upper half plane to the Bruhat–Tits tree, directly induced by the norm. Namely, given a point in Ω , which we identify as above with an invertible \mathbb{K} -linear map $\varphi : \mathbb{K}^2 \rightarrow \mathbb{C}_p$, we obtain a surjective map

$$\Upsilon : \Omega \rightarrow \mathcal{T}_{\mathbb{K}}$$

by setting $\Upsilon(\varphi) = |\cdot|_{\varphi}$, where $|\cdot|_{\varphi}$ is the norm on \mathbb{K}^2 defined by $|\underline{x}|_{\varphi} = |\varphi(\underline{x})|$, where the norm on the right-hand-side is the p -adic norm on \mathbb{C}_p . The explicit form of this map is discussed in [4]. We identify a point $(\zeta_0 : \zeta_1) \in \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{K})$ with the map $\varphi : \mathbb{K}^2 \rightarrow \mathbb{C}_p$ that maps $xe_1 + ye_2 \mapsto x\zeta_0 + y\zeta_1 \in \mathbb{P}^1(\mathbb{C}_p)$. In an affine patch (say with $\zeta_1 \neq 0$) we can write the homotety class of φ as $xe_1 + ye_2 \mapsto x\zeta + y \in \mathbb{C}_p \setminus \mathbb{K}$. Then the preimages under the map Υ of two adjacent vertices v, v' of $\mathcal{T}_{\mathbb{K}}$ and the edge e connecting them are given, respectively, by

$$\Upsilon^{-1}(v) = \{\zeta \in \mathbb{C}_p : |\zeta| \leq 1\} \setminus \bigcup_{a \in \mathcal{O}_{\mathbb{K}}/\pi\mathcal{O}_{\mathbb{K}}} \{\zeta \in \mathbb{C}_p : |\zeta - a| < 1\}$$

$$\Upsilon^{-1}(v') = \{\zeta \in \mathbb{C}_p : |\zeta| \leq q^{-1}\} \setminus \bigcup_{b \in \pi\mathcal{O}_{\mathbb{K}}/\pi^2\mathcal{O}_{\mathbb{K}}} \{\zeta \in \mathbb{C}_p : |\zeta - b| < q^{-1}\}$$

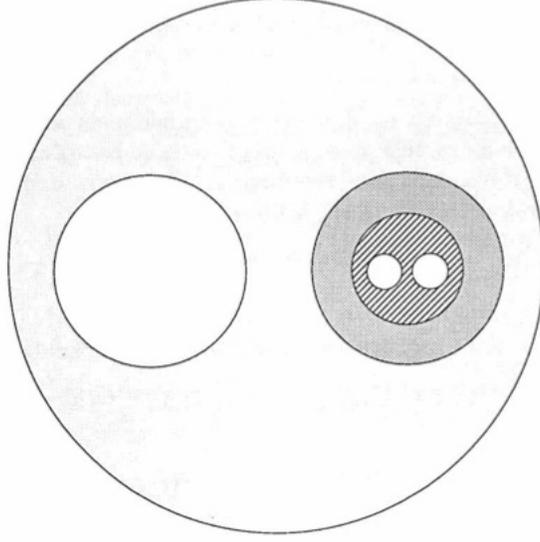
where $v = [M]$, $v' = [M']$ with $\pi M \subset M' \subset M$, and for $e_t = (1 - t)v + tv'$, for $0 < t < 1$, along the edge e

$$\Upsilon^{-1}(e_t) = \{\zeta \in \mathbb{C}_p : |\zeta| \leq q^{-t}\},$$

while

$$\begin{aligned} \Upsilon^{-1}(e) &= \{\zeta \in \mathbb{C}_p : |\zeta| \leq 1\} \setminus \bigcup_{a \in (\mathcal{O}_{\mathbb{K}} \setminus \pi\mathcal{O}_{\mathbb{K}})/\pi\mathcal{O}_{\mathbb{K}}} \{\zeta \in \mathbb{C}_p : |\zeta - a| < 1\} \\ &\setminus \bigcup_{b \in \pi\mathcal{O}_{\mathbb{K}}/\pi^2\mathcal{O}_{\mathbb{K}}} \{\zeta \in \mathbb{C}_p : |\zeta - b| < q^{-1}\}. \end{aligned}$$

For a detailed proof of this fact we refer to §2 of [4]. A part of the Drinfeld plane corresponding to the regions $\Upsilon^{-1}(v)$, $\Upsilon^{-1}(v')$ and $\Upsilon^{-1}(e)$ with $\partial e = \{v, v'\}$ in the Bruhat–Tits tree can be illustrated as follows (from [4]):



where the light colored region is $\Upsilon^{-1}(v)$, the striped shaded region is $\Upsilon^{-1}(v')$, and the dark shaded cylinder connecting them is $\Upsilon^{-1}(e)$. Thus, one can visualize the Drinfeld plane as a continuum that is a “tubular neighborhood” of the discrete Bruhat–Tits tree, with the regions $\Upsilon^{-1}(v)$ viewed as the p -adic analog of pair-of-pants decompositions for complex Riemann surfaces. A lift of the projection map Υ to the Bruhat–Tits tree realizes the tree as a skeleton of the Drinfeld plane.

Higher rank buildings and Drinfeld symmetric spaces. An analogous description holds relating the Bruhat–Tits buildings $\mathcal{T}_{n,\mathbb{K}}$ of $\mathrm{PGL}_{n+1}(\mathbb{K})$, with \mathbb{K} a finite extension of \mathbb{Q}_p and the associated Drinfeld symmetric space

$$\Omega_n = \mathbb{P}^n(\mathbb{C}_p) \setminus \cup_{H \in \mathcal{H}_{\mathbb{K}}} H,$$

where $\mathcal{H}_{\mathbb{K}}$ is the set of all \mathbb{K} -rational hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$. There is again a map $\Upsilon_n : \Omega_n \rightarrow \mathcal{T}_{n,\mathbb{K}}$ where the preimages of simplices in the Bruhat–Tits building is described in terms of norm conditions, [23].

The Bruhat–Tits building $\mathcal{T}_{n,\mathbb{K}}$ of $\mathrm{PGL}_{n+1}(\mathbb{K})$ is a simplicial complex with vertex set $V(\mathcal{T}_{n,\mathbb{K}}) = \mathcal{T}_{n,\mathbb{K}}^0$ given by the similarity classes $M_1 \sim M_2$ if $M_1 = \lambda M_2$ for $\lambda \in \mathbb{K}^*$ of lattices in an $n + 1$ dimensional vector space V over \mathbb{K} . A set $\{[M_0], \dots, [M_\ell]\}$ of such classes defines an ℓ -simplex in $\mathcal{T}_{n,\mathbb{K}}^\ell$ in the Bruhat–Tits building iff $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \dots \supsetneq M_\ell \supsetneq \pi M_0$, with $\pi \in \mathcal{O}_{\mathbb{K}}$ a prime element with $\mathbb{F}_q = \mathcal{O}_{\mathbb{K}}/\pi\mathcal{O}_{\mathbb{K}}$ the residue field. Such a sequence determines a flag $\bar{M}_0 \supsetneq \bar{M}_1 \supsetneq \dots \supsetneq \bar{M}_\ell \supsetneq 0$ of subspaces $\bar{M}_i = M_i/\pi M_i$ of an $n + 1$ -dimensional \mathbb{F}_q -vector space. The ℓ -simplices in $\mathcal{T}_{n,\mathbb{K}}^\ell$ containing a given vertex $[M]$ are in one-to-one correspondence with such flags with $[M_0] = [M]$. As before, we consider norms on $V \simeq \mathbb{K}^{n+1}$ and similarity classes of norms. There is a $\mathrm{PGL}_{n+1}(\mathbb{K})$ -equivariant homeomorphism between the resulting space of equivalence classes of norms and the geometric realization of the simplicial complex $\mathcal{T}_{n,\mathbb{K}}$.

Consider then points $\zeta = (\zeta_0 : \cdots : \zeta_n) \in \mathbb{P}^n(\mathbb{C}_p)$ and the map $\varphi : V \rightarrow \mathbb{C}_p$ given by $\sum_{i=0}^n a_i e_i \mapsto \sum_{i=0}^n a_i \zeta_i$. The map $|\sum_{i=0}^n a_i e_i|_\varphi = |\sum_{i=0}^n a_i \zeta_i|$ determines an equivalence class of norms iff the point $\zeta \in \mathbb{P}^1(\mathbb{C}_p)$ does not lie in any \mathbb{K} -rational hyperplane. This determines the map $\Upsilon : \Omega_n \rightarrow \mathcal{T}_{n,\mathbb{K}}$ that generalizes in higher rank the map from the Drinfeld plane to the Bruhat-Tits tree. As in the previous case, one can describe the preimages under this map. For example, the preimage of a vertex $v = [M]$ is given by

$$\Upsilon^{-1}(v) = \{|\zeta_0| = \cdots = |\zeta_n| = 1\} \setminus \cup_H \{\zeta \pmod{\pi} \in H\}$$

with the union over hyperplanes and

$$\Upsilon^{-1}(e_t) = \{|\zeta_0| = \cdots = |\zeta_{n-1}| = 1, |\zeta_n| = q^{-t}\}$$

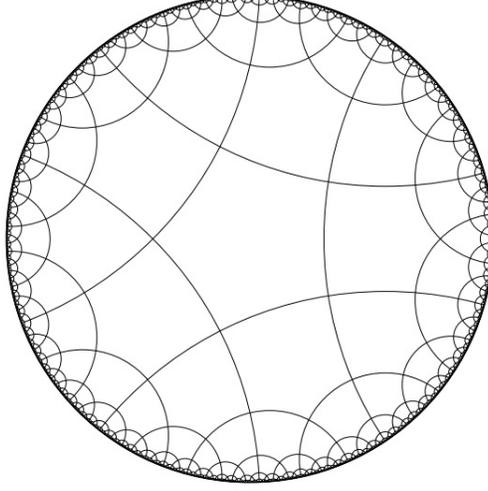
for e_t point along an edge e , with $0 < t < 1$, see §2 of [23] for more details.

4. TENSOR NETWORKS ON THE DRINFELD PLANE

Because the p -adic AdS/CFT correspondence has two different choices of bulk space, in addition to considering classical and quantum codes associated to the Bruhat-Tits tree in constructing a version of tensor networks, we can also work with the Drinfeld p -adic upper half plane. Because this is a continuous rather than a discrete space, the type of construction we can consider there will be more similar to the type of construction of tensor networks on the ordinary upper half plane (the 2-dimensional real hyperbolic plane \mathbb{H}^2) described in [42]. The map Υ from the Drinfeld upper half plane to the Bruhat-Tits tree will then make it possible to relate the construction of tensor networks on the first to the latter. To this purpose, we start by reviewing the construction of the pentagon holographic code from [42].

4.1. Pentagon Code on the Real Hyperbolic Plane. In [42] a holographic code is constructed using a tessellation of the real hyperbolic plane \mathbb{H}^2 by pentagons, with quantum codes given by a six leg perfect tensor placed at each tile. Unlike the codes discussed in the previous section on Bruhat-Tits trees, this code has no preferred base point in the tiling and all tiles are treated equally, and the codes are symmetric with respect to permutations of the five legs places across the edges of the tiles, thus preserving the full symmetry group of the tiling. We discuss briefly some aspects of this pentagon code here before turning to analogous constructions on the Drinfeld p -adic upper half plane.

The real hyperbolic plane \mathbb{H}^2 (which we can conveniently represent as the Poincaré disk) has a regular periodic tessellation by right-angle pentagons.



The corresponding symmetry group is the Fuchsian group $\Gamma \subset \text{PSL}(2, \mathbb{R})$ of signature $(2, 2, 2, 2, 2)$ generated by the reflections about the sides of a single right-angled hyperbolic pentagon. An interesting property of this Fuchsian group, from the algebro-geometric perspective is the fact that, if one subdivides an equilateral right-angled hyperbolic pentagon into 10 triangles with angles $\pi/2, \pi/4, \pi/5$, then one can realize the group Γ as a finite index subgroup of a triangle Fuchsian group Γ' of signature $(2, 4, 5)$. These Fuchsian groups have the property that the quotient Riemann surfaces $\mathbb{H}/\Gamma = X$ is arithmetic as an algebraic curve (that is, it is defined over a number field), [8].

The construction of the pentagon holographic code in [42] places over each tile of this right-angled pentagon tiling a quantum code given by the six leg perfect tensor determined by a 5-qubit $[[5, 1, 3]]_2$ -quantum code

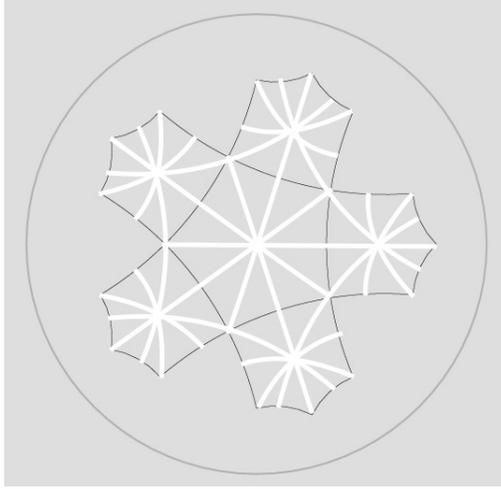
$$\mathcal{C} \subset \mathcal{H}^{\otimes 5}, \quad \mathcal{C} = \{\psi \in \mathcal{H}^{\otimes 5} : S_j \psi = \psi\}$$

where $S_1 = X \otimes Z \otimes Z \otimes X \otimes I$, with X, Y, Z the Pauli gates and $S_2, S_3, S_4, S_5 = S_1 S_2 S_3 S_4$ the cyclic permutations of S_1 , and with $\mathcal{H} = \mathbb{C}^2$ the 1-qbit Hilbert space. This is visualized as a code over \mathbb{H}^2 that has one logical input at each tiles of the pentagon tessellation and physical outputs across each edge of the tile, which are contracted with the legs of the nearby tiles, so that the resulting holographic code has one logical input at each tiles and outputs at the points at the boundary $\mathbb{P}^1(\mathbb{R})$ that correspond to infinite sequences of tiles.

4.2. Triangle Fuchsian Groups and Holographic Codes. In view of adapting this construction to the p -adic setting, it is better to first consider a modification that will allow us to work directly with the triangle Fuchsian group $\Gamma(2, 4, 5)$ rather than with its index 10 subgroup Γ of signature $(2, 2, 2, 2, 2)$ which is the symmetry group of the regular right-angled pentagon tiling.

This means replacing each pentagons in the tiling with its subdivision into a triangulation of 10 hyperbolic triangles with a vertex at the center of the pentagon tile

and the other vertices in the middle of the edges and at the original vertices of the pentagon.



We then consider holographic codes constructed by quantum codes associated to the triangle tiles. To this purpose, we do not necessarily require the group to be $\Gamma(2, 4, 5)$. We can work directly with the more general case of an arbitrary triangle Fuchsian group $\Gamma(a, b, c) \subset \text{PSL}_2(\mathbb{R})$ of hyperbolic type, $a^{-1} + b^{-1} + c^{-1} < 1$.

A simple way to construct a holographic code based on a tiling of the hyperbolic plane realized by a hyperbolic triangle group is to use a quantum error correcting code described in [42] that encodes a single 3-ary qubit (qutritt) into a space of three 3-ary qubits by

$$\begin{aligned} |0\rangle &\mapsto |000\rangle + |111\rangle + |222\rangle \\ |1\rangle &\mapsto |012\rangle + |120\rangle + |201\rangle \\ |2\rangle &\mapsto |021\rangle + |102\rangle + |210\rangle. \end{aligned}$$

This code can be represented as a perfect tensor $|a\rangle \mapsto T_{abcd}|bcd\rangle$ in the sense of [42]. By placing a copy of this code (thought of as a copy of the tensor T_{abcd} at each triangle tile of the tiling specified by the Fuchsian triangle group, one obtains a holographic code with a logical input qutritt at each tile and physical output qutritts at points of the boundary $\mathbb{P}^1(\mathbb{R})$ corresponding to limit points of infinite sequences of tiles, from a specified base point in the bulk.

A possible drawback of this simple construction is the fact that the quantum code we are using does not contain any information about the specific triangle group that determines the tessellation. This should be corrected by taking into consideration the stabilizer subgroups of edges and vertices, and incorporating them into the structure of the quantum code.

This can be done by considering quantum codes placed at the vertices, rather than at the faces, of the tessellation of a hyperbolic triangle group $\Gamma(a, b, c)$. This requires using perfect tensors of different valences, depending on the cardinality of the stabilizer group $G_v \subset \Gamma(a, b, c)$ of the vertex v .

A triangle Fuchsian group $\Gamma(a, b, c)$ in $\mathrm{PSL}_2(\mathbb{R})$ is generated by elements $\gamma_1 = \sigma_1\sigma_2$, $\gamma_2 = \sigma_2\sigma_3$ and $\gamma_3 = \sigma_3\sigma_1$, where the σ_i with $\sigma_i^2 = 1$ are the reflections about the sides of the fundamental domain triangle in \mathbb{H}^2 . The generators γ_i satisfy the relations $\gamma_1^a = \gamma_2^b = \gamma_3^c = \gamma_1\gamma_2\gamma_3 = 1$, that correspond to rotations by angles $2\pi/a$, $2\pi/b$ and $2\pi/c$, respectively, with stabilizer groups $\mathbb{Z}/a\mathbb{Z}$, $\mathbb{Z}/b\mathbb{Z}$, $\mathbb{Z}/c\mathbb{Z}$ associated to the vertices of the tessellation. Let $\ell = \mathrm{lcm}\{a, b, c\}$ and consider the embedding $\mathbb{Z}/a\mathbb{Z} \hookrightarrow \mathbb{Z}/\ell\mathbb{Z}$ by identifying $\mathbb{Z}/\ell\mathbb{Z}$ with ℓ -th roots of unity and mapping the generator of $\mathbb{Z}/a\mathbb{Z}$ to $\zeta^{\ell/a}$, where ζ is a primitive ℓ -th root. Similarly, for the other two groups. We can then consider a construction like the quantum codes described in [22]. At a vertex labelled by a stabilizer $\mathbb{Z}/a\mathbb{Z}$ we consider the polynomial code

$$|\alpha\rangle \mapsto \sum_{\alpha_0, \dots, \alpha_{a-1} \in \mathbb{Z}/\ell\mathbb{Z}} \otimes_{x \in \mathbb{Z}/a\mathbb{Z}} |f_{\underline{\alpha}}(x^{\ell/a})\rangle$$

where $f_{\underline{\alpha}}(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_{a-1} t^{a-1} + \alpha t^a \in \mathbb{Z}/\ell\mathbb{Z}[t]$. This encodes an input in $\ell^2(\mathbb{Z}/\ell\mathbb{Z})$ into an output in $\ell^2(\mathbb{Z}/\ell\mathbb{Z}) \otimes \ell^2(\mathbb{Z}/a\mathbb{Z})$, which we think of as an ℓ -ary qubit deposited at each side of the tessellation around the vertex. We can express this as a tensor $T_{i_0 \dots i_a}$ with $a+1$ legs. By contracting legs along the matching edges of the tessellations we obtain a holographic code that inputs an ℓ -ary qubit at each vertex of the tessellation and outputs at the points in the boundary $\mathbb{P}^1(\mathbb{R})$ that are endpoints of geodesic lines consisting of edges of the tessellation.

4.3. Surface Quantum Codes. There is another interesting construction of quantum stabilizer codes associated to tessellations of the hyperbolic planes, which was developed in [48]. These codes are constructed in general for a tiling defining a 2-dimensional surface (possibly with boundary). In particular, as shown in [48], the construction applies to the case of hyperbolic triangle Fuchsian groups, through the associated Cayley graph and the tessellation defined by it. In particular it applies to the triangle group $\Gamma(2, 4, 5)$ which we use here as a replacement for the right-angled pentagon tile of [42]. The construction of surface codes in [48] arises as a natural generalization of Kitaev's toric code of [27]. They have the advantage that they rely again on the CRSS algorithm that converts classical into quantum codes, hence they can be investigated in terms of classical coding theory techniques.

Consider a tessellation \mathcal{R} of a complex Riemann surface Σ and its dual \mathcal{R}^* that has a vertex for each face of \mathcal{R} with two vertices being adjacent in \mathcal{R}^* if the corresponding faces in \mathcal{R} share a common boundary edge. Let $\mathcal{E} = (\epsilon_{v,e})$ be the vertex-edge incidence matrix of \mathcal{R} and let \mathcal{E}^* be the vertex-edge incidence matrix of the dual graph \mathcal{R}^* . Let V and V^* be the \mathbb{F}_q -vector spaces spanned by the rows of \mathcal{E} and \mathcal{E}^* , respectively. The rows of \mathcal{E} are orthogonal to V^* and the rows of \mathcal{E}^* are orthogonal to V , with respect to the standard pairing $\langle v, v' \rangle = \sum_i v_i v'_i$. The first homology groups of \mathcal{R} and \mathcal{R}^* can be identified with the quotients V^\perp/V^* and $V^{*\perp}/V$. A quantum code can be associated to these data by a version of the CRSS algorithm, using the pair of matrices \mathcal{E} and \mathcal{E}^* . The construction of the quantum code follows the same procedure illustrated above: to pairs (v, w) of vectors $v \in V$, $w \in V^*$, one associates an error operator $E_{(v,w)}$. The condition that the spaces V and V^* are mutually orthogonal

implies that the bilinear pairing $\langle (v, w), (v', w') \rangle = \langle v, w' \rangle - \langle v', w \rangle$ vanishes, hence the group \mathcal{S} formed by these $E_{(v,w)}$ and the ξ^j , $0 \leq j \leq p-1$ is abelian. Thus, one can associate to it a quantum stabilizer code by taking a common eigenspace of the $E_{(v,w)}$. This imposes $\dim V + \dim V^*$ stabilizer conditions on n q -ary qubits, where n is the number of columns of \mathcal{E} and \mathcal{E}^* (number of edges of the graph \mathcal{R}), hence the parameters of the resulting quantum code are $[[n, k, d]]_q$, where $k = n - \dim V - \dim V^*$ and $d = \min\{d_{V^\perp \setminus V^*}, d_{V^{*\perp} \setminus V}\}$ with $d_{V^\perp \setminus V^*} = \min\{\omega(v) : v \neq 0, v \in V^\perp \setminus V^*\}$ with $\omega(v) = \#\{i : v_i \neq 0\}$ and similarly for $d_{V^{*\perp} \setminus V}$.

The Kitaev toric code consists of this construction applied to a graph \mathcal{R} obtained by a tessellation of a torus into squares. Generalizations to other Riemann surfaces and other tessellations were described in [48]. The main idea is to associate quantum surface codes to increasingly large portions of a given tessellation of the hyperbolic plane or to suitable quotients of such regions.

In our case, we can start with the right-angled pentagon tessellation \mathcal{R} and its dual graph \mathcal{R}^* . After choosing a root vertex v_0 of \mathcal{R}^* (the center of a chosen face in the tiling) we denote by \mathcal{R}_N and \mathcal{R}_N^* the finite tessellations obtained by considering only the points that are up to N steps away from v_0 (that is, such that the hyperbolic geodesic to v_0 passes through at most N tiles. Let V_N and E_N be the number of vertices and edges in \mathcal{R}_N and let V_N^* be the number of vertices in \mathcal{R}_N^* . The region \mathcal{R}_N has boundary, so in the construction of the dual graph \mathcal{R}_N^* we assume that the dual graph has $E_N = E_N^*$ where the edges of \mathcal{R}_N^* include an edge cutting through each boundary edge of \mathcal{R}_N and number of vertices V_N^* given by the number of faces of \mathcal{R}_N plus one additional vertex for each boundary edge of \mathcal{R}_N . This will correctly produce, in the limit when $N \rightarrow \infty$ boundary vertices on $\mathbb{P}^1(\mathbb{R})$ at the endpoints of all geodesics of the dual graph \mathcal{R}^* of the tessellation \mathcal{R} , which should be the physical outputs of a holographic quantum code. Note that, starting from the central pentagon as zeroth step, at the first step one adds 10 new pentagons, five of which share an edge with the initial one and five that share a vertex. At the second step, one adds 40 new pentagons, where each of the 5 pentagons of the first step that shared an edge with the central pentagon (we call these tiles of the first kind) will be adjacent to 2 new tiles of the first kind (sharing an edge) and 1 tile of the second kind (sharing a vertex), while each of the 5 tiles of the second kind will be adjacent to 3 new tiles of the first kind and 2 new tiles of the second kind. Thus, if we let F_N be the number of new tiles (faces) added to the tessellation at the N -th step, with $F_N = m_N + n_N$, where m_N and n_N are, respectively, the number of tiles of the first and second kind, namely those that share a full edge or just a vertex with a tile of the $(N-1)$ -st step. We then have the recursion relation

$$m_{N+1} = 2m_N + 3n_N, \quad n_{N+1} = m_N + 2n_N$$

with initial condition $m_1 = n_1 = 5$. This gives

$$\begin{aligned} (m_1, n_1) &= (5, 5), & (m_2, n_2) &= (25, 15), & (m_3, n_3) &= (95, 55), \\ (m_4, n_4) &= (355, 205), & (m_5, n_5) &= (1325, 765), & (m_6, n_6) &= (4945, 2855), \dots \end{aligned}$$

which corresponds to $F_1 = 10, F_2 = 40, F_3 = 150, F_4 = 560, F_5 = 2090, F_6 = 7800 \dots$. Similarly, let V_N denote the number of vertices added to the tessellation at the N -th step in the construction. We count as before the numbers m_N and n_N of faces added at the N -th step, and for each face we count new vertices counterclockwise, counting the leftmost vertex (common to the next adjacent face) and not counting the rightmost vertex (which we include in the counting for the next tile). This gives a number of new vertices equal to $W_N = 2m_N + 3n_N = m_{N+1}$, which is again computed in terms of the recursion above. We have $V_N = \sum_{k=0}^N W_k$ and $V_N^* = \sum_{k=0}^N F_k + E_{\partial,N}$, where $E_{\partial,N}$ is the number of boundary edges at the N -th stage in the construction. This number is also equal to $E_{\partial,N} = 2m_N + 3n_N = m_{N+1}$.

One can also consider closed surfaces (without boundary) and associated quantum codes by passing to Cayley graphs of quotient groups of the triangle Fuchsian group associated to the tessellation. In particular, the case that corresponds to the right-angled pentagon tile of the pentagon code of [42] is $m = 4$ and $\ell = 5$, for which we use the presentation

$$\Gamma(2, 4, 5) = \langle a, b \mid a^2 = 1, b^5 = 1, (ab)^4 = 1 \rangle.$$

The 2-complex used for the construction of the surface code in [48] is built by considering 2-cycles of length $\ell = 5$ and $2m = 8$ of the form $\{x, xb, xb^2, xb^3, xb^4, xb^5 = x\}$ and $\{x, xa, xab, xaba, x(ab)^2, x(ab)^2a, x(ab)^3, a(ab)^3a, x(ab)^4 = x\}$, at every vertex x , where all vertices have valence 3, with two edges $\{x, xb\}$ and $\{x, xb^{-1}\}$ along an $\ell = 5$ -face and the remaining edge $\{x, xa\}$ along a $2m = 8$ -face. By constructing an explicit matrix representation of $\Gamma(2, m, \ell)$ in the matrix group $\text{SL}_3(\mathbb{Z}[\xi])$, with $\xi = 2 \cos(\pi/m\ell)$, and taking reduction of the matrix entries modulo a prime p , one obtains a finite quotient group G , as the image of $\Gamma(2, m, \ell)$ (as a subgroup of $\text{SL}_3(\mathbb{Z}[\xi])$) in the quotient $\text{SL}_3(\mathbb{F}_p[X]/(h(X)))$ where $h(X)$ is a function of the $2m\ell$ -th normalized Chebyshev polynomial. It is shown in [48] that this finite quotient group G has the property that any word in the generators that is the identity in G without being the identity in $\Gamma(2, m, \ell)$ must be of length at least $\log p$. This condition on the finite quotient group ensures that the finite graph given by the Cayley graph of G can be identified with a portion of the infinite Cayley graph of $\Gamma(2, m, \ell)$, given by the neighborhood of size $\log p$ of a vertex. Provided that $\log p$ is sufficiently large, the 2-cycles will then correspond to the ℓ -cycles and $2m$ -cycles in this region, as the only words within that length that are equal to the identity in G are those already equal to the identity in the triangle group. The quantum code associated to the Cayley graph of G and its dual graph then has code parameters $[[n, k, d]]_q$ with $n = E$, dimension $k \geq \frac{E}{3}(1 - 2(\frac{1}{\ell} + \frac{1}{m}))$, where E and V are the number of edges and vertices in the Cayley graph of G . Thus, the dimension grows linearly in the length of the code, while as shown in §3.3 of [48], the minimum distance is proportional to $\log p$.

The advantage of thinking in terms of triangle groups rather than pentagon codes is that there is a parallel theory of p -adic hyperbolic triangle groups in $\text{PGL}_2(\mathbb{K})$, for \mathbb{K} a (sufficiently large) finite extension of \mathbb{Q}_p , see [24], [25]. These are much more

severely constrained than the Fuchsian triangle groups in $\mathrm{PSL}_2(\mathbb{R})$ and only exist for small values of p .

4.4. Triangle Groups on the Bruhat-Tits trees. In order to consider analogous constructions in the Drinfeld p -adic upper half plane $\Omega = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{K})$, we first need to consider possible tilings of Ω . As in the case of the real hyperbolic plane \mathbb{H}^2 , we can think of a tessellation of the Drinfeld plane Ω as a fundamental domain \mathcal{F} for the action of a subgroup $\Gamma \subset \mathrm{GL}_2(\mathbb{K})$ and its translates $\gamma(\mathcal{F})$, $\gamma \in \Gamma$, with the property that Ω/Γ is compact. Using the reduction map $\Upsilon : \Omega \rightarrow \mathcal{T}_{\mathbb{K}}$ from the Drinfeld plane to the Bruhat–Tits tree, the property that Ω/Γ is compact translates into the property that $\mathcal{T}_{\mathbb{K}}/\Gamma$ is a finite graph.

Unlike what happens in the case of Fuchsian groups acting on the real hyperbolic plane, the existence of p -adic triangle graphs is much more severely constrained. One is particularly interested in triangle groups of Mumford type. These are triangle groups $\Gamma \subset \mathrm{GL}_2(\mathbb{Q}_p)$ such that $(\mathbb{P}^1(\mathbb{C}_p) \setminus \Lambda_\Gamma)/\Gamma \simeq \mathbb{P}^1(\mathbb{C}_p)$ and the uniformization map $\pi : \mathbb{P}^1(\mathbb{C}_p) \setminus \Lambda_\Gamma \rightarrow \mathbb{P}^1(\mathbb{C}_p)$ is ramified at three points. In particular, by the classification result of [24], [25] a p -adic triangle group of Mumford type, of signature $(2, 4, 5)$ exists only when $p = 2$. No hyperbolic triangle groups of Mumford type exist for $p > 5$. The complete list of hyperbolic p -adic triangle groups $\Gamma(a, b, c)$ of Mumford type that can exist in the cases $p = 2$, $p = 3$, and $p = 5$ is given in [25].

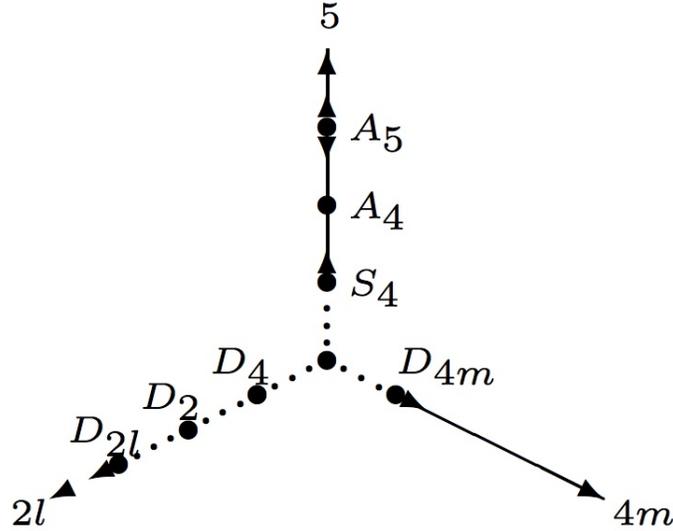
Let \mathcal{F} be a fundamental domain for the action of the triangle group $\Gamma(2, 4, 5)$ on the Drinfeld p -adic upper half plane Ω with $p = 2$ and let T be a fundamental domain for the action of the same group $\Gamma(2, 4, 5)$ on the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$ of a (sufficiently large) finite extension \mathbb{K} of \mathbb{Q}_2 . Since the reduction map $\Upsilon : \Omega \rightarrow \mathcal{T}_{\mathbb{K}}$ is equivariant with respect to the action of $\mathrm{GL}_2(\mathbb{K})$, we can assume that $T = \Upsilon(\mathcal{F})$. More generally, we can consider any choice of one of the possible hyperbolic p -adic triangle groups $\Gamma(a, b, c)$ of Mumford type, with $p \in \{2, 3, 5\}$, acting on the Bruhat–Tits tree of a (sufficiently large) finite extension \mathbb{K} of \mathbb{Q}_p , for one of these three possible values of p , and we proceed in the same way.

A good way of describing the fundamental domain of the action of a finitely generated discrete subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{K})$ on the Bruhat–Tits tree $\mathcal{T}_{\mathbb{K}}$ and the resulting quotient graph is in terms of *graphs of groups*, as shown in [24], [25]. The theory of graphs of groups was developed in [2], [43]. A graph of groups consists of a finite directed graph with groups G_v and G_e associated to the vertices and edges of the graph, with $G_{\bar{e}} = G_e$, together with injective group homomorphisms $\varphi_s : G_e \rightarrow G_{s(e)}$ and $\varphi_t : G_e \rightarrow G_{t(e)}$ from the group associated to an edge to the groups associated to the source and target vertices. The fundamental group of a graph of groups is constructed choosing a spanning tree of the graph: it is generated by the vertex groups G_v together with an element h_e for each edge e , with relations $h_{\bar{e}} = h_e^{-1}$ and

$$h_e^{-1} \varphi_s(g) h_e = \varphi_t(g), \quad \forall g \in G_e$$

and with $h_e = 1$ for all e in the chosen spanning tree. If one denotes by \mathcal{G} the graph and by G_\bullet the collection of groups associated to the vertices and edges, one writes

$\pi_1(\mathcal{G}, G_\bullet) = \varinjlim_{\varphi, \mathcal{G}} G_\bullet$ for the resulting amalgam given by the fundamental group of the graph of groups. In the case where the graph consists of one edge and two vertices, this fundamental group is just the pushforward in the category of groups, namely the amalgamated free product $G_{s(e)} \star_{G_e} G_{t(e)}$. The main idea (see [2], [43]) is to associate to the action of a discrete group on a tree a quotient given not just by a graph but by the richer structure of a graph of groups, which keeps track of the information about the stabilizers of vertices and edges. In the case of a discrete subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{K})$, we consider the tree of groups given by the subtree \mathcal{T}_Γ of the Bruhat–Tits tree $\mathcal{T}_\mathbb{K}$ together with the stabilizers G_v and G_e of vertices and edges, and we obtain a graph of groups as the quotient graph $\mathcal{T}_\Gamma/\Gamma$. It is shown in [25] that p -adic triangle groups of Mumford type are characterized by the property that the quotient graph $T = \mathcal{T}_\Gamma/\Gamma$ is a tree consisting of three lines meeting at a single root vertex v_0 . Such trees are called *tripods*. This tree, decorated with the stabilizer groups of vertices and edges is a tree of groups. The ends of this tree are the three branch points, at 0, 1 and ∞ , of the genus zero curve Ω_Γ/Γ . The group Γ can be reconstructed from the tree of groups (T, G_\bullet) as the associated fundamental group, [24]. Indeed the possible p -adic triangle groups of Mumford type are explicitly constructed using this method. For example, the tripod associated to the p -adic triangle group $\Gamma(2, 4, 5)$ with $p = 2$, seen as a tree of groups, is the case $\ell = m = 1$ of the following family (from [25]):



with subgroups $D_2 \subset D_4 \subset S_4$ and D_2 intersecting $A_4 \subset S_4$ trivially. In the case $\ell = m = 1$ the resulting amalgam agrees with the pushout $S_4 \star_{A_4} A_5$.

4.5. Tessellations of the Drinfeld Plane. A general algorithm exists for computing fundamental domains in Bruhat–Tits trees for the action of certain quaternion groups, see [11]. In these cases the algorithm produces

- (1) a connected subtree \mathcal{D}_Γ of the Bruhat–Tits tree which is a fundamental domain for the group action, in the sense that the edges of \mathcal{D}_Γ form a complete set of coset representatives for $E(\mathcal{T})/\Gamma$;
- (2) the edge and vertex stabilizer groups G_e, G_v for $e \in E(\mathcal{D}_\Gamma)$ and $v \in V(\mathcal{D}_\Gamma)$;
- (3) an explicit form for the quotient map by identifications (v, v', γ) between pairs of boundary vertices v, v' of the fundamental domain \mathcal{D}_Γ , with $\gamma \in \Gamma$ such that $v' = \gamma v$.

This algorithm can be used to produce corresponding tessellations of the Drinfeld p -adic upper half plane. Let $\Gamma, \mathcal{D}_\Gamma, G_e, G_v$, and $\{(v, v', \gamma)\}$ be given as above, through the algorithm of [11]. Using the projection map $\Upsilon : \Omega \rightarrow \mathcal{T}$ from the Drinfeld plane to the Bruhat–Tits tree, we can construct an associated tessellation of the Drinfeld plane, where the tiles are given by γT , with $\gamma \in \Gamma$ and

$$T = \bigcup_{v \in V(\mathcal{D}_\Gamma)} \Upsilon^{-1}(v) \cup \bigcup_{e \in E(\mathcal{D}_\Gamma)} \Upsilon^{-1}(e).$$

The gluing rules for the tiles are prescribed by the data (2) and (3) associated to the fundamental domain on the Bruhat–Tits tree.

4.6. Lifting Holographic Codes from the Bruhat–Tits Tree. Another way to obtain holographic codes on the Drinfeld plane is to lift the construction of the classical and quantum codes on Bruhat–Tits trees described in §2 via the surjection $\Upsilon : \Omega \rightarrow \mathcal{T}_\mathbb{K}$. This means that the “tiles” to which we associate classical and quantum codes in the Drinfeld plane are given, in this case, by the regions $\Upsilon^{-1}(v)$, the preimages in Ω of vertices of the Bruhat–Tits tree, and the outputs of each (classical or quantum) Reed–Solomon code is stored in the connecting regions $\Upsilon^{-1}(e)$. This can be done by choosing a lift of the projection Υ , which realizes the Bruhat–Tits tree as a skeleton of Ω and constructing the holographic code over that skeleton. The choice of a lift of the projection is non-canonical, hence this type of construction has the same kind of drawback of the construction used in [22] to simulate the pentagon code via a choice of a planar embedding of a tree along edges of the pentagon tiling of the real hyperbolic plane. An advantage in this case, however, is that the projection Υ is equivariant with respect to the $\mathrm{GL}_2(\mathbb{Q}_p)$ symmetries so one maintains the symmetries of the tree intact, unlike the case of the planar embedding used in [22].

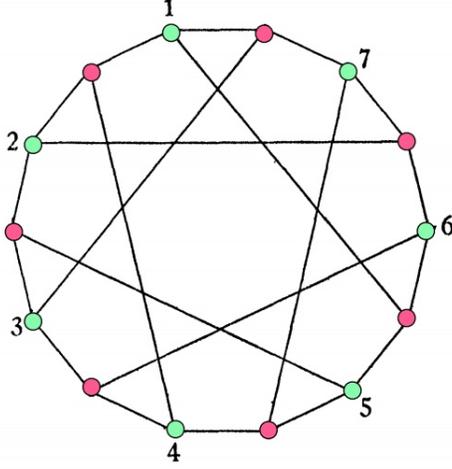
5. HOLOGRAPHIC CODES ON HIGHER RANK BRUHAT–TITS BUILDINGS

As above, we denote by $\mathcal{T}_{n,\mathbb{K}}$ the Bruhat–Tits building of $\mathrm{GL}_{n+1}(\mathbb{K})$ and by Ω_n the Drinfeld symmetric space.

Consider first the case of the Bruhat–Tits building of $\mathrm{GL}_3(\mathbb{K})$, with \mathbb{K} a finite extension of \mathbb{Q}_p with residue field \mathbb{F}_q , $q = p^r$. The set of vertices adjacent to a given vertex $v \in V(\mathcal{T}_{2,\mathbb{K}})$ is a bipartite set, consisting of the set of $q^2 + q + 1$ \mathbb{F}_q -rational points of the projective plane \mathbb{P}^2 over \mathbb{F}_q together with the set of $q^2 + q + 1$ \mathbb{F}_q -rational lines of the projective plane \mathbb{P}^2 over \mathbb{F}_q . The surface X over \mathbb{F}_q obtained by blowing up all the \mathbb{F}_q -rational points of \mathbb{P}^2 contains an exceptional divisor (a line) for each

\mathbb{F}_q -rational points of \mathbb{P}^2 and a proper transform (also a line) for each \mathbb{F}_q -rational line in \mathbb{P}^2 . Thus, to each vertex w adjacent to the given vertex v we associate a line ℓ_w in the blowup surface X . Let u, w be vertices adjacent to v : the set $\{u, v, w\}$ corresponds to a 2-simplex in the 2-dimensional simplicial complex $\mathcal{T}_{2, \mathbb{K}}$ if and only if the lines ℓ_u and ℓ_w intersect nontrivially in X .

In the case of \mathbb{Q}_2 one obtains the well known picture below, with the 7 points and 7 lines of $\mathbb{P}^2(\mathbb{F}_2)$ as vertices and with 21 edges, [9].



In order to extend the construction of holographic codes to higher rank Bruhat–Tits buildings, in a way that reflects the associated geometries over finite fields that determine the local structure of the building, we need to replace the classical Reed–Solomon codes with algebro-geometric codes associated to higher-dimensional algebraic varieties.

5.1. Codes on the Bruhat–Tits buildings of GL_3 from algebro-geometric codes on surfaces. A general procedure for constructing algebro-geometric codes over higher-dimensional algebraic varieties generalizing the Reed–Solomon codes is described in [45], see also [20]. Given a smooth projective variety X over \mathbb{F}_q with an ample line bundle \mathcal{L} , one obtains a linear code $C(X, \mathcal{L}, \mathcal{P})$, where \mathcal{P} is a set of \mathbb{F}_q -algebraic points of X , as the image of the germ map

$$\alpha : \Gamma(X, \mathcal{L}) \rightarrow \bigoplus_{x \in \mathcal{P}} \mathcal{L}_x \simeq \mathbb{F}_q^n,$$

which evaluates sections $s \in \Gamma(X, \mathcal{L})$ at points $x \in \mathcal{P}$, with the last identification given by a choice of an isomorphism $\mathcal{L}_x \simeq \mathbb{F}_q$ of the fibers at $x \in \mathcal{P}$, with $n = \#\mathcal{P}$.

For example, for $X = \mathbb{P}^2$, with $\mathcal{L} = \mathcal{O}(m)$, with $0 < m \leq q$, and \mathcal{P} the set of all \mathbb{F}_q -rational points of \mathbb{P}^2 , one obtains a code $C(\mathbb{P}^2, \mathcal{O}(m), \mathbb{P}^2(\mathbb{F}_q))$ with length $n = q^2 + q + 1$, dimension $k = \frac{1}{2}(m + 1)(m + 2)$, and minimum distance bounded by $d \geq q^2 + q + 1 - m(q + 1)$, see [20].

We focus here on the case of the Bruhat–Tits building of $GL_3(\mathbb{K})$, with \mathbb{K} a finite extension of \mathbb{Q}_p with residue field \mathbb{F}_q , $q = p^r$. As we mentioned above, the link of

a vertex in the Bruhat–Tits building is described in terms of the geometry of an algebraic surface X obtained by blowing up all the \mathbb{F}_q -algebraic points of \mathbb{P}^2 .

We use the example above of algebro-geometric codes $C(\mathbb{P}^2, \mathcal{L}, \mathbb{P}^2(\mathbb{F}_q))$ associated to line bundles \mathcal{L} over \mathbb{P}^2 to construct a classical holographic code on the Bruhat–Tits building of $\mathrm{GL}_3(\mathbb{K})$. We fix a base vertex in the building and assign as logical input the datum of a divisor D on \mathbb{P}^2 so that $\mathcal{L} = \mathcal{L}(D)$. Consider then the surface X over \mathbb{F}_q obtained by blowing up all the \mathbb{F}_q -rational points of \mathbb{P}^2 , and the pullback $\pi^*\mathcal{L}$ under the projection map, and line bundles of the form $\hat{\mathcal{L}} = \pi^*\mathcal{L} \otimes \mathcal{O}(-\sum_i k_i E_i)$ where the E_i are the exceptional divisors of the blowup. Assume that D and the k_i are chosen so that $\hat{\mathcal{L}}$ is represented by an effective divisor on X . We now consider the $q^2 + q + 1$ lines in X determined by the \mathbb{F}_q -lines of \mathbb{P}^2 and the $q^2 + q + 1$ lines that correspond to the \mathbb{F}_q -points of \mathbb{P}^2 and the set \mathcal{P} consisting of the $q + 1$ \mathbb{F}_q -rational points of each of these lines, with $\#\mathcal{P} = 2(q + 1)(q^2 + q + 1)$. The code $C(X, \hat{\mathcal{L}}, \mathcal{P})$ can be viewed as a code that, given the logical input D at the base vertex v , deposits an output given by a vector in \mathbb{F}_q^{q+1} at each adjacent vertex w in the Bruhat–Tits building. These outputs are related by a consistency condition, which is determined by the edges and 2-cells of the building. Namely, whenever w and u are vertices adjacent to v , such that $\{v, w, u\}$ is a 2-cell in the building, we know the corresponding condition on X is that the two lines ℓ_w and ℓ_u intersect. The presence of a point of intersection means that the corresponding vectors in \mathbb{F}_q^{q+1} must agree in one of the $q + 1$ coordinates.

When one propagates the construction to nearby vertices in the Bruhat–Tits building, part of the logical input is reserved for the output \mathbb{F}_q^{q+1} -vector of the nearby vertices already reached by the previous steps from the chosen root vertex. As in the case of the Bruhat–Tits tree, we identify the given \mathbb{F}_q^{q+1} -vector (computed as output by the previous code) with assigned values at one of the lines in X that corresponds to one of the lines in \mathbb{P}^2 (which we can think of as the \mathbb{P}^1 at infinity in \mathbb{P}^2). There is a consistency condition for the output at a new vertex w that is adjacent to a 2-cell where the remaining two vertices v and v' already have outputs $\underline{x}(v), \underline{x}(v') \in \mathbb{F}_q^{q+1}$ assigned by the previous codes: the outputs $\underline{x}(v), \underline{x}(v')$ at the two previous vertices v, v' are two vectors in \mathbb{F}_q^{q+1} that agree in one coordinate, hence they fix the values of the sections at two intersecting lines in X . The resulting output $\underline{x}(w)$ at the new vertex w is then computed by the values at the $q + 1$ points of the line ℓ_w of all sections s that satisfy the constraints given by the assigned values at the points of ℓ_v and $\ell_{v'}$. The construction can in this way be propagated to the rest of the Bruhat–Tits building of $\mathrm{GL}_3(\mathbb{K})$. This illustrates the general approach to constructing classical holographic codes on higher rank Bruhat–Tits buildings.

A construction of quantum holographic codes can be obtained from these classical codes using a version of the CRSS algorithm (possibly by allowing more general types of weighted versions of the classical codes, as we discussed in the case of the Reed–Solomon codes). The details of the corresponding quantum codes for higher rank buildings will be discussed in forthcoming work.

5.2. Codes on Drinfeld symmetric spaces. Another possible approach to the construction of holographic codes for higher-rank p -adic symmetric spaces consists of working with Drinfeld symmetric spaces instead of Bruhat–Tits buildings. This extends the approach discussed in §4 on codes associated to actions of discrete groups on the Drinfeld plane.

In the higher rank setting, we consider two possible viewpoints. The first is based on the projection map from the Drinfeld symmetric space $\Upsilon : \Omega_n \rightarrow \mathcal{T}_{n,\mathbb{K}}$, from the Drinfeld space $\Omega_n = \mathbb{P}^n(\mathbb{C}_p) \setminus \cup_{H \in \mathcal{H}_{\mathbb{K}}} H$ (the complement of the \mathbb{K} -rational hyperplanes in \mathbb{P}^n) to the Bruhat–Tits building of $\mathrm{GL}_n(\mathbb{K})$. The idea here, as in §4.6 above, is to lift via the projection map a construction of holographic classical and quantum codes from the Bruhat–Tits building to the space Ω_n , with logical inputs associated to the regions $\Upsilon^{-1}(v)$, with v the vertices of $\mathcal{T}_{n,\mathbb{K}}$ and outputs and compatibility conditions along the edges, faces, and higher-dimensional cells. Since the projection map Υ is equivariant with respect to the $\mathrm{GL}_n(\mathbb{K})$ action, whatever symmetry the codes constructed on $\mathcal{T}_{n,\mathbb{K}}$ exhibit will be inherited by the resulting codes on Ω_n .

The other possible approach consists of constructing a tensor network directly associated to a given action of a discrete subgroup Γ of $\mathrm{GL}_n(\mathbb{K})$ on the symmetric space Ω_n . Roughly, the main idea in this case is to assign logical inputs to the fundamental domains of the action, while outputs should be associated to the generators of the discrete group with compatibility conditions resulting from the relations. In this way, the codes assigned to each copy of the fundamental domain can be compatibly assembled into a global holographic code on Ω_n , with logical inputs in the bulk and outputs at the boundary. The outputs should live on the points in the limit set of the group action on the rational hyperplanes $H \in \mathcal{H}_{\mathbb{K}}$. We will discuss these constructions of holographic codes on higher rank p -adic symmetric spaces in forthcoming work.

Acknowledgment. The author thanks Matthew Heydeman, Sarthak Parikh, and Ingmar Saberi for many very useful discussions and an ongoing collaboration on several topics discussed in this paper, and especially Sarthak Parikh for suggesting several improvements to the paper. The author is partially supported by NSF grant DMS-1707882 and by the Perimeter Institute for Theoretical Physics.

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