# PRACTICAL CONSIDERATIONS FOR THE GENERATION OF LARGE-ORDER SPHERICAL HARMONICS

(*Research Note*)

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**Abstract.** Techniques for generating large-order $Y_l^m(\theta, \varphi)$ are discussed.

Observations of the free oscillations of the Sun and Earth are usually expanded in spherical harmonics $Y_l^m(\theta, \varphi)$, as these are the normal modes of a perfect sphere. Current trends in helioseismology have included full-disk measurements of the Sun with spatial resolution as high as a few arc sec per pixel element. Such measurements allow in principle the determination of oscillation modes with spherical harmonic orders as high as $l \approx 1000$. Although a thorough examination of all possible modes with $l \leq 1000$ ($10^6$ of them) is not feasible on a finite computer budget, one can imagine more limited studies – e.g. looking at all degrees $m$ for $l = 100$, or all modes with $l \leq 40$ – that are interesting and are best done by fitting data directly to spherical harmonics of fairly large order. The question arises as to how to generate these.

The standard method of generating spherical harmonics is to use one of many possible recursion relations to produce the necessary Associated Legendre Polynomials. However I have found that many of these recursion relations are not terribly stable, especially near the endpoints, and that problems can arise even for $l$ as small as 20. I note here one simple recursion relation than can be used to generate all $Y_l^m(\theta, \varphi)$ up to $l = 200$, and also a slightly more involved procedure that allows $Y_l^m(\theta, \varphi)$ of much higher orders to be obtained.

Using the notation of Jackson (1975) the spherical harmonic of order $l$ and degree $m$ is

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta) e^{im\varphi},$$

with the Associated Legendre Polynomial

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l.$$

Defining a normalized Legendre Polynomial by

$$W_l^m(x) = (1-x^2)^{-m/2} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(x),$$

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we have

$$Y_l^m(\theta, \varphi) = W_l^m(\cos\theta)\sin^m\theta\, e^{im\varphi},$$

and

$$W_l^l(x) = \frac{(-1)^l}{2^l l!}\left[\frac{2l+1}{4\pi}(2l)!\right]^{1/2}$$

$$= (-1)^l\left[\frac{2l+1}{4\pi}\prod_{k=1}^{l}\left(1-\frac{1}{2k}\right)\right]^{1/2}, \qquad (1)$$

a constant near unity, and

$$W_l^{l-1}(x) = \sqrt{2l}\, x\, W_l^l(x). \qquad (2)$$

The $W_l^m(x)$ are more convenient to work with than the $P_l^m(x)$ since the former have values of order unity for $l$, $m$, and $x$.

The recursion relation for changing $m$ only (No. 1 in Magnus *et al.*, 1966),

$$(1-x^2)[(l+m+2)(l-m-1)]^{1/2}\, W_l^{m+2}(x) + 2(m+1)x\, W_l^{m+1}(x) +$$

$$+ [(l-m)(l+m+1)]^{1/2}\, W_l^m(x) = 0, \qquad (3)$$

is fairly stable when run in the direction of decreasing $m$, using Equations (1) and (2) for input, and its stability increases near the endpoints in $x$. Unfortunately it is not stable everywhere; near $m = l$ errors are multiplied by $\sim\sqrt{l}$ for each pass through the recursion relation. An estimate of the error propogation in using this relation to calculate $W_l^m(x)$ for the worst possible $m$ and $x$ gives a maximum error of $5 \times 10^4\,\varepsilon$ for $l = 100$, $5 \times 10^{10}\,\varepsilon$ for $l = 200$, and $> 10^{25}\,\varepsilon$ for $l = 500$, where $\varepsilon$ is the initial error. For double precision real numbers $\varepsilon \approx 10^{-14}$, hence the above recursion relation should provide satisfactory results for all $l$, $m < 200$.

$Y_l^m(\theta, \varphi)$ with larger $l$ can also be generated using recursion relation No. 6 from Magnus *et al.*,

$$x(l-m+1)^{1/2}\, W_l^m(x) - \left[\frac{2l+1}{2l+3}(l+m+1)\right]^{1/2} W_{l+1}^m(x) -$$

$$-(l+m)^{1/2}\, W_l^{m-1}(x) = 0, \qquad (4)$$

which is stable when $W_{l+1}^m(x)$ is the endpoint. Using a rapidly converging trigonometric expansion for $W_l^0(x)$ (adapted from Equation (8.7.1) in Abramowitz and Stegun (1964)), and Equation (1) for $W_l^l(x)$, one has sufficient input to use this relation to generate a table of $W_l^m(x)$ for all $l$ and $m$. Since now errors are multiplied by of order unity during each pass through recursion relation, one would have to go up to very high $l$ and $m$ before roundoff errors become a problem.

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