CRITICAL TORSIONAL OSCILLATIONS OF A ROTATING ACCELERATED SHAFT

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We assume that the forces applied to the shaft have a variable part which is a moment of constant amplitude $M(x)$ per unit length, distributed along the shaft, and varying with a frequency proportional to the angular velocity.

From the solution corresponding to the harmonic steady state vibration, we deduce, by using Heaviside's expansion, the motion due to a sudden application of the moment $M(x)$. This enables us to compute the effect of the moment when applied with a linearly increasing frequency.

In this analysis the damping will be neglected. In case of a viscous damping the linear character of the equations is not affected and the same method might be used. We did not carry this calculation for two reasons:

1. The exact result will be in general complicated, and involve viscous friction coefficients which will not be very accurately known. Besides, the effect of friction might be roughly taken into account by considering the steady state solution.
2. In most cases the damping is not viscous but due to the hysteresis or internal friction of the material. This is proved by the experimental fact that the energy absorbed in the vibration of elastic bodies is proportional to the frequency and not to its square. This effect might be taken roughly into account by energetic considerations.

The equation of vibration of the shaft may be written,

$$M(x, t) + \frac{\partial}{\partial x}\left[K(x)\frac{\partial \theta}{\partial x}\right] = I(x)\frac{\partial^2 \theta}{\partial t^2}, \tag{1}$$

where θ is the angular coördinate of a cross-section, $K(x)$ the torsional rigidity and $I(x)$ the moment of inertia per unit length.

In order to find a solution of this equation, we will first study the behavior of the free oscillations given by the equation,

$$\frac{\partial}{\partial x}\left[K(x)\frac{\partial \theta}{\partial x}\right] = I(x)\frac{\partial^2 \theta}{\partial t^2}.$$

This equation has an infinite number of solutions of the type $\theta = \Theta_i(x)e^{i\omega_i t}$ corresponding to the free harmonic oscillations of the shaft with the given boundary conditions.

Each of the functions $\Theta_i(x)$ is a characteristic function which satisfies the Sturm-Liouville differential equation,

$$\frac{d}{dx}\left[K(x)\frac{d}{dx}\,\Theta_i(x)\right] + \omega_i^2 I(x)\Theta_i(x) = 0 \tag{2}$$

and also the given boundary conditions. These functions are related by the orthogonality condition,

$$\int_0^a \Theta_m(x)\,\Theta_n(x)\,dx = 0 \quad m \neq n. \tag{3}$$

This analysis is valid when the inertia moments are concentrated instead of distributed, for we can always approximate concentrated loads by a continuous distribution.

We now assume that the applied moment is harmonic of the form, $M(x, t) = M_0(x)e^{i\omega t}$. The solution of equation (1) may then be written

$$\theta(x, t) = \theta_0(x)e^{i\omega t},$$

and the equation becomes an ordinary differential equation,

$$M_0(x) + \frac{d}{dx}\left[K(x)\frac{d\theta_0}{dx}\right] + \omega^2 I(x)\,\theta_0 = 0. \tag{4}$$

For our purpose the best way of solving this equation is to expand $\theta_0(x)$ in a series of orthogonal functions $\Theta_i(x)$,

$$\theta_0(x) = \sum_{i=1}^{\infty} A_i\Theta_i(x).$$

Substituting this expression into equation (4) and taking into account the identity (2), we find,

$$M_0(x) = I(x)\sum_{i=1}^{\infty} A_i(\omega_i^2 - \omega^2)\Theta_i(x).$$

If we multiply both sides by $\Theta_k(x)$ and integrate along the shaft between the limits (o, a), due to the orthogonality condition (3), all the unknown constants A, are eliminated except A_k; we finally get

$$A_i = \frac{C_i}{\omega_i^2 - \omega^2}, \qquad C_i = \frac{\int_0^a M_0(x)\,\Theta_i(x)dx}{\int_0^a I(x)\,\Theta_i^2(x)dx}.$$

The required solution for the forced harmonic vibration is

$$\theta_h(x, t) = e^{i\omega t}\sum_{i=1}^{\infty}\frac{C_i\Theta_i(x)}{\omega_i^2 - \omega^2}. \tag{5}$$

We will now calculate the motion of the shaft due to a sudden applied distributed moment $M_0(x)$. Consider the following well-known integral taken in the complex plane

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega = \begin{cases}1 & t > 0\\ 0 & t < 0\end{cases}.$$

The path of integration is supposed to pass below the real axis. The applied moment may hence be represented as a sum of harmonic forces with a continuous set of frequencies by the expression

$$\frac{M_0(x)}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega.$$

This integral can be expressed as a contour integral, if we note that for $t > 0$ we may add to the path of integration the infinite half circle passing through $(-\infty, +i\infty, +\infty)$. Hence for $t > 0$

$$\frac{M_0(x)}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega = \frac{M_0(x)}{2\pi i}\oint\frac{e^{i\omega t}}{\omega}\,d\omega.$$

To each of the harmonic moments composing this expression corresponds a motion given by equation (5); the total motion due to the sudden moment will be the sum of all these elementary motions and expressed by

$$\theta_2(x, t) = \sum_{i=1}^{\infty}\frac{C_i\Theta_i(x)}{2\pi i}\oint\frac{e^{i\omega t}}{\omega(\omega_i^2 - \omega^2)}.$$

This integral is easily calculated by the method of residues,

$$\theta_1(x, t) = \sum_{i=1}^{\infty}\frac{C_i\Theta_i}{\omega_i^2}\,[1 - \cos\omega_i t]. \qquad (6)$$

This expression gives the motion due to a sudden applied moment $M_0(x)$ as a sum of free oscillations.

In case of concentrated loads, the formula is similar. Suppose that the shaft carries n discs of moments of inertia I_k. The distributed moment $M_0(x)$ is replaced by the moments M_{0k} applied to those discs, and the distributed amplitude $\theta(x)$ by the amplitude θ_k of each disc. There will be, in general, n periods of free oscillations. The coefficients C_i are given by the following expressions

$$C_i = \frac{\sum_{k=1}^{n} M_{0k}\,\Theta_{ik}}{\sum_{k=1}^{n} I_k\,\Theta_{ik}^2}.$$

We call Θ_{ik} the amplitude of oscillation of the "k th" disc due to the free oscillation of order "i." If all the moments are suddenly applied at the same time the angular displacement of each disc is given by

$$\theta_k(t) = \sum_{i=1}^{n} \frac{C_i\, \Theta_{ik}}{\omega_i^2}\, [1 - \cos \omega_i t]. \tag{7}$$

Using a well-known method, we will now deduce from this analysis the motion due to a moment of the form $M_0(x)\psi(t)$, where $\psi(t)$ is an arbitrary function of time. It may be considered as composed of an infinite number of small increments $M_0(x)\,\frac{d\psi}{dt}dt$. Each of these increments produces an oscillation given by (6) or (7) so that the total motion in case of distributed parameters is given by the integral

$$\theta(x, t) = \int_0^t \theta_1[x, t - \tau]\, \frac{d}{d\tau}\, \psi(\tau) d\tau.$$

Integrating by parts and assuming $\psi(o) = 0$,

$$\theta(x, t) = \int_0^t \frac{\partial}{\partial t}\, [\theta_1(x, t - \tau)]\psi(\tau) d\tau,$$

$$\theta(x, t) = \sum_{i=1}^{\infty} \frac{C_i}{\omega_i}\, \Theta_i(x)\ [\sin \omega_i t] \int^t \psi(\tau) \cos \omega_i \tau d\tau - \cos \omega_i t \int_0^t \psi(\tau) \sin \omega_i \tau d\tau.$$

$$\text{Put } f_1(\omega_i, t) = \int^t \psi(\tau) \cos \omega_i \tau d\tau,$$

$$f_2(\omega_i, t) = \int_0^t \psi(\tau) \sin \omega_i \tau d\tau.$$

The expression (7) for $\theta(x,t)$ shows that the motion due to an applied moment of the form $M_0(x)\psi(t)$ is composed of a series of free oscillations each of which has an amplitude

$$\frac{C_i}{\omega_i}\, \Theta_i(x)\, \sqrt{f_1^2(\omega_i, t) + f_2^2(\omega_i, t)}.$$

This principle leads to the solution of the announced problem. Consider a shaft rotating with a constant acceleration. In most cases the amplitude of variation of the torque will be constant. We call v the angular acceleration and put $\beta = \frac{v}{2}$. The variable part of the moment is, per unit length,

$$M_0(x)\psi(t) = M_0(x) \sin (\beta t^2 + \varphi).$$

Put

$$F(\omega_i, t) = \sin \omega_i t \int_0^t \cos \omega_i \tau \sin (\beta\tau^2 + \varphi)d\tau - \cos \omega_i t \int_0^t \sin \omega_i \tau \sin (\beta\tau^2 + \varphi)d\tau.$$

The motion is then given by formula (7) in the form

$$\theta(x, t) = \sum_{i=1}^{\infty} \frac{C_i}{\omega_i}\Theta(x)F(\omega_i, t). \tag{8}$$

In order to calculate F we shall introduce the quantities

$$\alpha = t\sqrt{\frac{v}{\pi}} = 4\sqrt{N_i}$$

$$\xi = t\sqrt{\frac{v}{\pi}} = \alpha\frac{N}{N_i}$$

where N_i is the number of revolutions performed when the critical speed is reached at the moment t_i, and N the number of revolutions performed after any time t.

By an elementary transformation we may write

$$F(\omega_i, t) = {}^1/_2\sin(\omega_i t + \varphi) \int_0^t \sin(\beta\tau^2 + \omega_i\tau)d\tau + {}^1/_2 \sin(\omega_i t - \varphi) \int_0^t \sin (\beta\tau^2 - \omega_i\tau)\bar{d}\tau + {}^1/_2 \cos(\omega_i t + \varphi) \int_0^t \cos(\beta\tau^2 + \omega_i\tau)d\tau + {}^1/_2 \cos(\omega_i t - \varphi) \int_0^t \cos (\beta\tau^2 - \omega_i\tau)\bar{d}\tau.$$

Introducing now the quantities α and ξ,

$$\Psi(\xi, \alpha, \varphi) = 2\sqrt{\frac{v}{\pi}}\, F(\omega_i, t) = \sin\left(\omega_i t - \varphi - \frac{\pi}{2}\alpha^2\right)\int_{-\alpha}^{\xi-\alpha} \sin\frac{\pi}{2}y^2dy - \cos\left(\omega_i t - \varphi - \frac{\pi}{2}\alpha^2\right)\int_{-\alpha}^{\xi-\alpha} \cos\frac{\pi}{2}y^2.\, dy + \sin\left(\omega_i t + \varphi + \frac{\pi}{2}\alpha^2\right)\int_{\alpha}^{\xi-\alpha} \sin\frac{\pi}{2}y^2dy + \cos\left(\omega_i t + \varphi + \frac{\pi}{2}\alpha^2\right)\int_{\alpha}^{\xi+\alpha} \cos\frac{\pi}{2}y^2dy. \tag{9}$$

This expression involves the well-known Fresnel integrals. By substitution in equation (8) we get for each normal mode an amplitude given by

$$\theta_i(x, t) = \frac{\sqrt{\pi}}{2}\cdot\frac{C_i\,\Theta_i(x)}{\omega_i\sqrt{v}}\cdot\Psi(\xi, \alpha, \varphi).$$

We are interested in the behavior of this amplitude as a function of time. Practically the critical speed will be reached in more than ten revolutions, so that,

$$\alpha > 4\sqrt{10}$$

take $\alpha > 12$.

Write
$$\int_0^u \cos\frac{\pi}{2}y^2dy = C(u),$$
$$\int_0^u \sin\frac{\pi}{2}y^2dy = S(u).$$

These functions have been tabulated.[1] In our assumption that $\alpha > 12$, the terms written on the second line of equation (9) are small and may be neglected. This equation then reduces to

$$\Psi(\xi, \alpha, \varphi) = \sin\left(\omega_i t - \varphi - \frac{\pi}{2}\alpha^2\right)\int_{-\alpha}^{\xi-\alpha}\sin\frac{\pi}{2}y^2dy - \cos\left(\omega_i t - \varphi - \frac{\pi}{2}\alpha^2\right)\int_{-\alpha}^{\xi-\alpha}\cos\frac{\pi}{2}y^2dy.$$

This is the projection of a rotating vector whose rectangular components are

$$C(\xi - \alpha) + C(\alpha),$$
$$S(\xi - \alpha) + S(\alpha).$$

When ξ varies, the extremity of this vector moves on a curve plotted in the figure. The length of the curve taken as 0 for $\xi = \alpha$ has the value $\xi - \alpha$, and is a linear function of the time.

Consider the case where a great number of revolutions have to be performed before the critical speed is reached. This means that the value of α is great. The conclusions that we will draw from this assumption shall be practically true if $N_i > 12$.

The amplitude Ψ at any moment is represented by a vector having its origin in $0'$ and its end at the point of the curve corresponding to $t\sqrt{\frac{v}{\pi}} = \xi$. We see that the amplitude increases at first slowly, then very rapidly near the point of resonance 0 ($\xi = \alpha$) where the amplitude is $\frac{1}{\sqrt{2}}$. The amplitude then reaches a maximum $1.165\sqrt{2}$ for $\xi - \alpha = 1.25$ and decreases afterward in an oscillating way, down to $\sqrt{2}$.

Hence the maximum amplitude of the normal mode will be

$$\theta_i(x) = 1.165\sqrt{\frac{\pi}{2}}\frac{C_i\,\Theta_i\,(x)}{\omega_i\sqrt{v}}.$$

In the case of concentrated loads, the maximum vibration amplitude of each disc is

$$\theta_{ik} = 1.165\sqrt{\frac{\pi}{2}}\frac{C_i\,\Theta_{ik}}{\omega_i\sqrt{v}}.$$

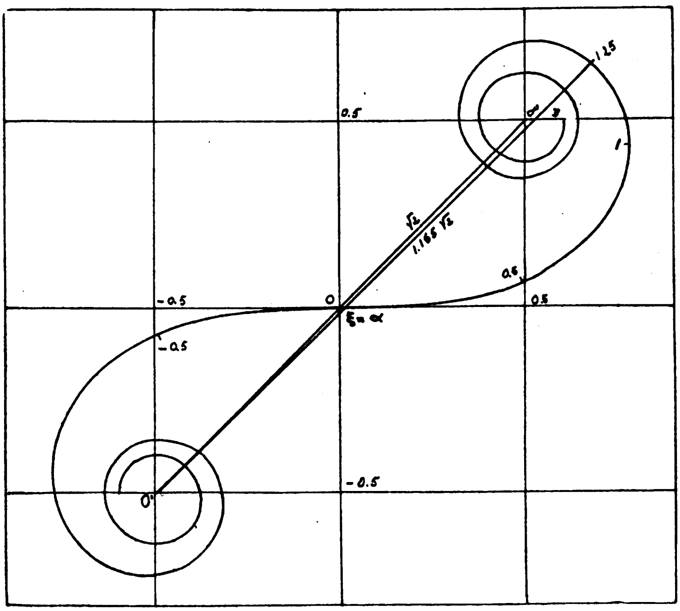
If the moment acting on the shaft is the sum of two moments applied with a phase difference φ,

$$M_{ok} = M'_{ok} \sin \beta t^2 + M''_{ok} \sin(\beta t^2 + \varphi)$$

the maximum amplitudes of the discs will be given by the same formula, provided that C_i is the vectorial sum (with a phase difference φ) of the two coefficients C'_i and C''_i corresponding respectively to the separate moments M'_{ok} and M''_{ok},

$$C_i = \sqrt{C'^2_i + C''^2_i + 2C'_iC''_i \cos \varphi}.$$

Example.—We shall apply this result to the crankshaft of a six cylinder



two cycle engine. The crankshaft is supposed to be free at both ends, and may be considered as a shaft loaded with six discs of moment of inertia I. If M is the mean torque produced by this motor, $\frac{M}{6}$ will be the mean torque applied by the pistons to each of the discs, and at the same time the amplitude of variation of this torque. We assume that the explosions occur at the same moment in pistons equidistant from the center, in the following order,

$$(1, 6) \quad (2, 5) \quad (3, 4).$$

What is the effect of acceleration on the third harmonic of this shaft? The amplitudes of free vibration of the third harmonic are

$$\begin{aligned}\Theta_{31} &= 1\\ \Theta_{32} &= -2\\ \Theta_{33} &= 1\\ \Theta_{34} &= 1\\ \Theta_{35} &= -2\\ \Theta_{36} &= 1\end{aligned}$$

Each piston causes a certain amplitude of vibration of the third harmonic, and this effect is characterized by the corresponding coefficients,

$$C_{3k} = \frac{M}{6I} \cdot \frac{\Theta_{3k}}{\sum\limits_{K=1}^{6} \Theta_{3k}^2} = \frac{M}{F^2 I} \Theta_{3k}.$$

These coefficients have to be added vectorially, due to the fact that there is a phase difference in the applied moments.

Let us first add the vectors which have no phase difference:

$$C_{31} + C_{36} = \frac{M}{F2I} (\Theta_{31} + \Theta_{36}) = 2 \frac{M}{F2I}.$$

$$C_{32} + C_{35} = \frac{M}{F2I} (\Theta_{32} + \Theta_{35}) = -4 \frac{M}{F2I}.$$

$$C_{33} + C_{34} = \frac{M}{F2I} (\Theta_{33} + \Theta_{34}) = 2 \frac{M}{F2I}.$$

These three vectors have a phase difference $\frac{2\pi}{3}$ and their vectorial sum is,

$$C_3 = \frac{M}{KI}.$$

The maximum critical amplitude of vibration of each disc is

$$\theta_{3k} = 1.165 \sqrt{\frac{\pi}{2}} \frac{C_3 \Theta_{3k}}{\omega_3 \sqrt{v}}.$$

The difference between θ_{31} and θ_{32} gives the maximum torsional strain

$$\theta_{31} - \theta_{32} = 1.165 \sqrt{\frac{\pi}{2}} \frac{M}{4I\omega_3 \sqrt{v}}.$$

[1] E. Jahnke and F. Emde, "Funktionentafeln," p. 24.

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