

# Feasible Joint Posterior Beliefs\*

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## Abstract

We study the set of possible joint posterior belief distributions of a group of agents who share a common prior regarding a binary state and who observe some information structure. Our main result is that, for the two agent case, a quantitative version of Aumann’s Agreement Theorem provides a necessary and sufficient condition for feasibility. For any number of agents, a related “no trade” condition likewise provides a characterization of feasibility. We use our characterization to construct joint belief distributions in which agents are informed regarding the state, and yet receive no information regarding the other’s posterior. We study a related class of Bayesian persuasion problems with a single sender and multiple receivers, and explore the extreme points of the set of feasible distributions.

## 1 Introduction

The question of whether agents’ observed behavior is compatible with rationality is fundamental and pervasive in microeconomics. A particularly interesting case is that of beliefs: when are agents’ beliefs compatible with Bayes’ Law?

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Consider a single agent who has a prior belief regarding an uncertain event, and who updates to a posterior belief after receiving some information. An analyst observes the prior and the distribution of posteriors, but not the information structure. When is the distribution of posterior beliefs feasible? That is, when is it compatible with Bayesian updating according to some information structure? Feasibility in this case corresponds to what is known as the martingale condition: the expectation of the posterior must equal the prior. This fundamental result is usually attributed to Aumann and Maschler and is known as the Splitting Lemma. It is a key tool in the theories of Bayesian persuasion (Kamenica and Gentzkow, 2011) and of games with incomplete information (Aumann and Maschler, 1995).

Thus, in the case of a single agent, the analyst can readily determine feasibility. We ask the same question, but regarding groups of more than one agent. In particular, we consider a group of agents for which one can observe a common prior, as well as a joint distribution of posteriors. This distribution is said to be feasible if it is the result of Bayesian updating induced by some joint information structure observed by the agents. Is there also in this case a simple characterization of feasibility, that does not require the analyst to test the infinitely many possible information structures?

Clearly, feasibility implies that each agent’s posterior distribution must satisfy the martingale condition. An additional important obstruction to feasibility is given by Aumann’s seminal Agreement Theorem (Aumann, 1976). Aumann showed that rational agents cannot agree to disagree on posteriors: when posteriors are common knowledge then they cannot be unequal. This implies that feasibility is precluded if, for example, agent 1 has posterior  $1/5$  whenever agent 2 has posterior  $2/5$ , and likewise agent 2 has posterior  $2/5$  whenever agent 1 has posterior  $1/5$ .

There are, as we show, examples of distributions that are not feasible, even though they do not involve agents who agree to disagree as above. Thus the Agreement Theorem does not provide a necessary and sufficient condition for feasibility. Our first main result is a quantitative version of the Agreement Theorem; a relaxation that constrains beliefs to be approximately equal when they are approximately common knowledge. In our second main result we show that for two agents these constraints provide a necessary and sufficient condition for feasibility. This condition is not as simple as the martingale condition, and yet is an intrinsic property of the distribution that can be checked directly. For example, this condition can be used to easily verify or refute feasibility for finitely supported distributions.

For more than two agents, we show in our third main result that a “no trade” condition gives a necessary and sufficient condition for feasibility. To derive this condition we introduce—as an analytic tool—a mediator who trades a good of uncertain value with the agents. When an agent trades, it is at a price that reflects her expected value, and thus the agents do not gain or lose in expectation. This implies that the mediator likewise does not turn a profit, and so by bounding

the mediator’s profit by zero we attain an obviously necessary condition for feasibility. We show that this condition is also necessary.

We apply these characterizations to study *independent* joint posterior belief distributions: these are induced by information structures in which each agent receives non-trivial information regarding the state, and yet gains no information about the other’s posterior.

A natural related question is what we call *social Bayesian persuasion*. This is a class of Bayesian persuasion problems that includes a single sender and multiple receivers. First, the sender chooses an information structure to be revealed to the agents. Then, each receiver chooses an action. Finally, both the sender and the receivers receive payoffs. The key assumption is that each receiver’s utility depends only on the unknown state and her own action, as is common in the social learning literature; we therefore describe these persuasion problems as social. Thus, we can assume that each receiver’s action depends only on her own posterior. The sender’s utility depends on the actions of the receivers, and so the sender’s problem is to choose an optimal feasible joint posterior belief distribution.

As an example of a social Bayesian persuasion problem, we study the problem of a “polarizing” sender who wishes to maximize the difference between two receivers’ posteriors. This example highlights the limitations imposed by feasibility, as the sender cannot hope to always achieve complete polarization in which one receiver has posterior 0 and the other posterior 1; such joint posterior distributions are precluded by the Agreement Theorem. We solve this problem in some particular regimes, and show that non-trivial solutions exist in others, depending on the details of how the difference between posteriors is quantified.

A related question is how anti-correlated feasible posteriors can be. While they can clearly be perfectly correlated, by the Agreement Theorem they cannot be perfectly anti-correlated. This question can be phrased as a social Bayesian persuasion problem, which we solve. For the uniform prior we show that the covariance between posterior has to be at least  $-1/32$ , and construct a feasible distribution, supported on four points, that achieves this bound.

The question of social Bayesian persuasion is closely tied to the study of the extreme points of the set of feasible distributions, as the sender’s optimum is always achieved at an extreme point. In the single agent case, the well known concavification argument of [Aumann and Maschler \(1995\)](#) and [Kamenica and Gentzkow \(2011\)](#) shows that every extreme point of the set of feasible posterior distributions has support of size at most two. In contrast, we show that for two or more agents there exist extreme points with countably infinite support. In the other direction, we show that every extreme point is supported on a set that is small, in the sense of having zero Lebesgue measure. To this end, we prove an extension of a classical result of [Lindenstrauss \(1965\)](#) regarding extreme points of the set of measures with given marginals.

This work leaves open a number of interesting questions. In particular, the study of the extreme

feasible distributions is far from complete. For example, we do not have a simple characterization of extreme points. We also do not know if every extreme point has countable support.

**Related literature.** [Mathevet et al. \(2019\)](#) study Bayesian persuasion with multiple receivers and finite signals. They find an implicit characterization of feasibility: considering the entire belief hierarchy, they show that feasibility is equivalent to consistency of the hierarchy. Our condition is explicit—i.e., formulated directly in terms of the joint distribution of first-order beliefs. Certain compatibility questions for belief hierarchies (without application to feasibility of joint belief distribution or Bayesian persuasion) were recently addressed by [Brooks et al. \(2019\)](#).

Very recently and independently, [Ziegler \(2020\)](#) considered an information design setting with two receivers, and derived a necessary condition for feasibility. his condition is sufficient for the case of binary signals. We further discuss Ziegler’s condition in §B.

Our characterization of feasible distributions in Theorem 2 is based on the classic results of [Kellerer \(1961\)](#) regarding the existence of dominated measures with given marginals. In the economic literature, such tools were applied by [Gershkov et al. \(2013\)](#) and [Gershkov et al. \(2018\)](#). See also the discussion in [Hart and Reny \(2015\)](#). Our feasibility condition shares some similarity with the condition of feasibility for reduced-form auctions ([Border, 1991](#)).

The remainder of the paper is organized as follows. In §2 we introduce the formal problem. In §3 we present our main results. Applications are presented in §4. The proof of our main result appears in §5. In §6 we study social Bayesian persuasion and the extreme feasible beliefs. Further proofs are provided in the Appendix.

## 2 Model

**Information structures and posterior beliefs.** We consider a binary state space  $\Omega = \{\ell, h\}$ , and a set of  $n$  agents  $N = \{1, 2, \dots, n\}$ . An *information structure*  $I = ((S_i)_{i \in N}, \mathbb{P})$  consists of signal sets  $S_i$  (each equipped with a sigma-algebra  $\Sigma_i$ , which we suppress) for each agent  $i$ , and a distribution  $\mathbb{P} \in \Delta(\Omega \times S_1 \times \dots \times S_n)$ . Let  $\omega, s_1, \dots, s_n$  be the random variables corresponding to the  $n + 1$  coordinates of the underlying space  $\Omega \times S_1 \times \dots \times S_n$ . When it is unambiguous, we also use  $s_i$  to denote a generic element of  $S_i$ .

The *prior probability* of the high state is denoted by  $p = \mathbb{P}(\omega = h)$ . Throughout the paper we assume that  $p \in (0, 1)$ , i.e., we exclude the degenerate cases of  $p = 0$  and  $p = 1$ .

All  $n$  agents initially have prior  $p$  regarding the state  $\omega$ . Then, each agent  $i$  observes the signal  $s_i$ . The *posterior belief*  $x_i$  attributed to the high state by agent  $i$  after receiving the signal  $s_i$  is

$$x_i = \mathbb{P}(\omega = h \mid s_i).$$

We denote by  $P_I$  the *joint distribution of posterior beliefs* induced by the information structure  $I$ . This probability measure on  $[0, 1]^N$  is the joint distribution of  $(x_1, x_2, \dots, x_n)$ . I.e., for each measurable  $B \subset [0, 1]^N$ ,

$$P_I(B) = \mathbb{P}\left(\left(x_1, \dots, x_n\right) \in B\right).$$

We similarly denote the *conditional joint distributions of posterior beliefs* by  $P_I^\ell$  and  $P_I^h$ ; these are the joint distributions of  $(x_1, \dots, x_n)$ , conditioned on the state  $\omega$ .

For a probability measure  $P \in \Delta([0, 1]^N)$  and for  $i \in N$ , we denote by  $P_i$  the marginal distribution, i.e., the distribution of the projection on the  $i^{\text{th}}$  coordinate.

**Feasible joint posterior beliefs.** When is a given distribution on  $[0, 1]^N$  equal to the distribution of posterior beliefs induced by some information structure? This is the main question we study in this paper. The following definition captures this notion formally.

**Definition 1.** *Given  $p \in (0, 1)$ , we say that a distribution  $P \in \Delta([0, 1]^N)$  is  $p$ -feasible if there exists some information structure  $I$  with prior  $p$  such that  $P = P_I$ .*

When  $p$  is understood from the context we will simply use the term “feasible” rather than  $p$ -feasible.

**The single agent case.** Before tackling the question of feasibility for  $n$  agents, it is instructive to review the well-understood case of a single agent.

The so-called *martingale condition* states that the average posterior belief of a feasible posterior distribution is equal to the prior. Formally, given a  $p$ -feasible distribution  $P \in \Delta([0, 1])$ , it must be the case that

$$\int_0^1 x \, dP(x) = p.$$

The necessity of this condition for  $p$ -feasibility follows from the law of iterated expectation. For the single agent case, the martingale condition is necessary and sufficient for  $P$  to be  $p$ -feasible. This result is known as the Splitting Lemma and is attributed to Aumann and Maschler.

### 3 Joint feasible beliefs

**The two agent case and the Agreement Theorem.** In the two agent case the martingale condition is not sufficient for feasibility. An additional obstruction to feasibility is given by Aumann’s celebrated Agreement Theorem (Aumann, 1976). We here provide a rephrasing of this theorem in a form that will be useful for us later.

**Theorem A** (Aumann). *Let  $((S_i)_i, \mathbb{P})$  be an information structure, and let  $B_1 \subseteq S_1$  and  $B_2 \subseteq S_2$  be subsets of possible signal realizations for agents 1 and 2, respectively. If*

$$\mathbb{P}(s_1 \in B_1, s_2 \notin B_2) = \mathbb{P}(s_2 \in B_2, s_1 \notin B_1) = 0 \quad (1)$$

then

$$\mathbb{E}(x_1 \cdot \mathbf{1}_{s_1 \in B_1}) = \mathbb{E}(x_2 \cdot \mathbf{1}_{s_2 \in B_2}).$$

To understand why this is a reformulation of the Agreement Theorem, note that condition (1) implies that

$$\mathbb{P}(s_1 \in B_1, s_2 \in B_2) = \mathbb{P}(s_1 \in B_1) = \mathbb{P}(s_2 \in B_2),$$

and thus that the event  $\{s_1 \in B_1, s_2 \in B_2\}$  is self evident: it is common knowledge whenever it occurs. Thus, this form of the agreement theorem states that if agents have common knowledge of the event  $\{s_1 \in B_1, s_2 \in B_2\}$  then their average beliefs on this event must coincide. The original theorem follows by choosing  $B_1$  and  $B_2$  such that  $x_1$  is constant on  $B_1$  and  $x_2$  is constant on  $B_2$ . The statement of Theorem A is close in form to that of the No Trade Theorem of [Milgrom and Stokey \(1982\)](#), which provides the same obstruction to feasibility; we discuss this further below.

Using Theorem A it is easy to construct examples of distributions that are not feasible, even though they satisfy the martingale condition. For example, consider the prior  $p = \frac{1}{2}$  and the distribution  $P = \frac{1}{2}\delta_{0,1} + \frac{1}{2}\delta_{1,0}$ , where  $\delta_{x_1, x_2}$  denotes the point mass at  $(x_1, x_2)$ . Clearly  $P$  satisfies the martingale condition. Nevertheless, it is not feasible, since under this distribution agents “agree to disagree,” when agent 1 attributes probability 1 to the high state and agent 2 considers the probability of this state to be 0. Indeed, assume towards a contradiction that it is equal to  $P_I$ , for some information structure  $I = ((S_i)_i, \mathbb{P})$ . Then the events  $B_1 = \{s \in S_1 : x_1(s) = 1\}$  and  $B_2 = \{s \in S_2 : x_2(s) = 0\}$  satisfy the common knowledge condition (1). But

$$\mathbb{P}(\omega = h \mid s_1 \in B_1) = 1 \neq 0 = \mathbb{P}(\omega = h \mid s_2 \in B_2),$$

in contradiction to Theorem A.

This example can be extended to a more general necessary condition for feasibility of a joint posterior distribution.

**Corollary A.** *Let  $P \in \Delta([0, 1]^2)$  be feasible for some  $p$ , and let  $A_1$  and  $A_2$  be measurable subsets of  $[0, 1]$ . Denote complements by  $\overline{A_i} = [0, 1] \setminus A_i$ . If*

$$P(A_1 \times \overline{A_2}) = P(\overline{A_1} \times A_2) = 0 \quad (2)$$

then

$$\int_{A_1} x \, dP_1(x) = \int_{A_2} x \, dP_2(x). \quad (3)$$

Corollary A is a recasting of Theorem A into a direct condition for feasibility: condition (2) states that the event  $A_1 \times A_2$  is self evident: it has the same probability as  $A_1$  and the same probability as  $A_2$ . And each of the two integrals in (3) is equal to the average belief of agent  $i$  conditioned on  $A_i$ , times the probability of  $A_i$ . Hence Corollary A follows immediately from Theorem A, by setting  $B_i = \{s_i : x_i(s(i)) \in A_i\}$  for  $i = 1, 2$ .

As an example of an application, let  $A_1 = \{q_1\}$  and  $A_2 = \{q_2\}$  for some  $q_1 \neq q_2$ , and assume that  $A_1 \times A_2$  has positive probability and is self evident. Then it is impossible that  $P$  is feasible.

**A quantitative Agreement Theorem.** While Aumann’s Agreement Theorem provides obstructions to feasibility, a joint posterior distribution can be infeasible even when it does not imply that agents agree to disagree. A larger set of necessary conditions follows from our first main result, which is a quantitative version of the Agreement Theorem:

**Theorem 1.** *Let  $((S_i)_i, \mathbb{P})$  be an information structure, and let  $B_1 \subseteq S_1$  and  $B_2 \subseteq S_2$  be sets of possible signal realizations for agents 1 and 2, respectively. Then*

$$\mathbb{P}(s_1 \in B_1, s_2 \notin B_2) \geq \mathbb{E}(x_1 \cdot \mathbb{1}_{s_1 \in B_1}) - \mathbb{E}(x_2 \cdot \mathbb{1}_{s_2 \in B_2}) \geq -\mathbb{P}(s_2 \in B_2, s_1 \notin B_1).$$

A comparison to the Agreement Theorem (Theorem A) is illustrative. In Theorem A the common knowledge assumption (1) implies equality of average posteriors. Here (1) has been removed, and we instead bound the difference in the average posteriors by the extent to which (1) is violated. Thus, one can think of Theorem 1 as quantifying the extent to which approximate common knowledge implies approximate agreement.<sup>1</sup> The Agreement Theorem becomes the special case in which (1) holds. The proof of this theorem is simple and is provided in §A.<sup>2</sup>

In analogy to Corollary A, we use Theorem 1 to derive further necessary conditions for feasibility.

**Corollary 1.** *Let  $P \in \Delta([0, 1]^2)$  be  $p$ -feasible for some  $p$ , and let  $A_1$  and  $A_2$  be measurable subsets of  $[0, 1]$ . Then*

$$P(A_1 \times \overline{A_2}) \geq \int_{A_1} x dP_1(x) - \int_{A_2} x dP_2(x) \geq -P(\overline{A_1} \times A_2). \quad (4)$$

Corollary 1 admits a simple interpretation in terms of the No Trade Theorem (Milgrom and Stokey, 1982). Consider three risk-neutral agents: two traders and a mediator. Trader 2 owns a good

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<sup>1</sup>An alternative approach to quantitative extensions of the Agreement Theorem is by the concept of common  $p$ -beliefs (Monderer and Samet, 1989). Neeman (1996) shows that when posteriors are common  $p$ -belief then they cannot differ by more than  $1 - p$ . However, we are not aware of any formal connection between this extension and our Theorem 2.

<sup>2</sup>Aumann (1976) reflects on his Agreement Theorem, writing: “We publish this observation with some diffidence, since once one has the appropriate framework, it is mathematically trivial.”

with an unknown quality  $\omega \in \{0, 1\}$ . The mediator also owns a copy of the same good. The two traders receive private information regarding the quality of the good, with a joint belief distribution  $P \in \Delta([0, 1]^2)$ . The mediator knows  $P$  and the realized pair  $(x_1, x_2)$ .

Let  $A_1, A_2 \subseteq [0, 1]$  be any measurable sets and consider the following trading scheme: The mediator buys the good from trader 2 whenever  $x_2 \in A_2$  at a price of  $x_2$ . The mediator sells one copy of good to trader 1 whenever  $x_1 \in A_1$  at a price of  $x_1$ . Thus the mediator may need to use her copy of the good, in case she sells but does not buy.

We argue that the mediator's expected profit is at least

$$\int_{A_1} x \, dP_1(x) - \int_{A_2} x \, dP_2(x) - P(A_1 \times \overline{A_2}).$$

The first two addends correspond to the expected transfer between each trader and the mediator. The last addend corresponds to the event that the mediator has to sell his own good to trader 1 since trader 2's belief  $x_2$  is not in  $A_2$  and trader 1's belief is in  $A_1$ . In which case he loses at most 1.

Clearly, the mediator does not provide any additional information to the two players and so their expected profit is zero. Thus the mediator's expected profit is also zero, and so we have arrived at the left inequality of (4). The right inequality follows by symmetry.

**A characterization of feasible distributions for two agents.** Our second main result is a simple characterization of feasibility joint posterior beliefs for two agents. Perhaps surprisingly, the necessary condition of Corollary 1 is also sufficient.

**Theorem 2.** *A probability measure  $P \in \Delta([0, 1]^2)$  is  $p$ -feasible for some  $p$  if and only if*

$$P(A_1 \times \overline{A_2}) \geq \int_{A_1} x \, dP_1(x) - \int_{A_2} x \, dP_2(x) \geq -P(\overline{A_1} \times A_2)$$

for all measurable  $A_1, A_2 \subseteq [0, 1]$ . (5)

It follows from Theorem 2 and the single agent martingale condition that when (5) holds then  $P$  is  $p$ -feasible for

$$p = \int_0^1 x \, dP_1(x) = \int_0^1 x \, dP_2(x).$$

The necessity of (5) is Corollary 1. The sufficiency requires another argument, which uses a theorem of Kellerer (1961). We present the complete proof of Theorem 2 in §5.

It is natural to wonder if (5) can be relaxed to a simpler sufficient condition, and in particular if it suffices to check it on  $A_1, A_2$  that are intervals. As we show in §B, restricting (5) to intervals results in a condition that is not sufficient: we construct a measure that is not feasible, but satisfies

(4) for all intervals  $A_1, A_2$ . The constructed measure demonstrates that the condition derived by Ziegler (2020) for feasibility is necessary but insufficient.

**More than two agents.** Beyond the two agent case, Theorem 2 provides a necessary condition for feasibility, as the joint belief distribution of each pair of agents must clearly satisfy (5). However, for three agents or more, this condition is not sufficient: we construct below an example of three agents whose beliefs satisfy (5) for each pair of agents, and yet are not feasible. The violation of feasibility stems from a violation of the No Trade Theorem, in a manner similar to the one illustrated above for two agents. We use this idea to provide a necessary and sufficient condition for feasibility for an arbitrary number of agents.

A *trading scheme* consists of  $n$  measurable functions  $a_i: [0, 1] \rightarrow [-1, 1]$ ,  $i = 1, \dots, n$ . Given agent  $i$ 's posterior  $x_i$ , a mediator sells  $a_i(x_i)$  units of the good to agent  $i$  for the price of  $x_i$  per unit, so that the total transfer is  $a_i(x_i)x_i$ .

Clearly, each agent's expected profit is zero, since she is buying or selling the good at her expected value. Hence the mediator's expected profit is also zero. we argue that

$$\int_{[0,1]^n} \left( \sum_{i=1}^n a_i(x_i)x_i - \max \left\{ 0, \sum_{i=1}^n a_i(x_i) \right\} \right) dP(x_1, \dots, x_n) \quad (6)$$

is a lower bound on the mediator's profit. Indeed, the first addend in the integral is the total transfer to the mediator. The second is equal to the total number of units of the good that the mediator needs to contribute to the transaction, in case the total amount that she sells exceeds the total amount that she buys. Since each unit is worth at most 1, this is at most  $\sum a_i(x_i)$ , whenever this sum is positive. Thus, since the mediator's profit is zero, it follows that (6) cannot be positive if  $P$  is feasible.

Our characterization shows that this condition is also sufficient for feasibility. Its proof is given in §D.

**Theorem 3.** *A probability measure  $P \in \Delta([0, 1]^n)$  is  $p$ -feasible for some  $p$  if and only if for every trading scheme  $(a_1, \dots, a_n)$*

$$\int_{[0,1]^n} \left( \sum_{i=1}^n a_i(x_i)x_i - \max \left\{ 0, \sum_{i=1}^n a_i(x_i) \right\} \right) dP(x_1, \dots, x_n) \leq 0. \quad (7)$$

We end this section with an example that shows that Theorem 3 provides additional obstructions for feasibility when  $n \geq 3$ , beyond the pairwise condition implied by Theorem 2.

Consider three agents whose beliefs  $(x_1, x_2, x_3)$  are distributed uniformly and independently on  $[0, 1]$ . That is, let their joint belief distribution  $P \in \Delta([0, 1]^3)$  be the Lebesgue measure. By

Proposition 2 below, the agents *pairwise* satisfy the condition of Theorem 2. We argue that this is nevertheless not a feasible distribution.

To see this, consider the trading scheme given by

$$a_1(x) = \mathbb{1}_{x \geq 2/3} \quad a_2(x) = a_3(x) = -\frac{1}{2} \cdot \mathbb{1}_{x \leq 2/3}$$

Thus agent 1 buys a unit whenever her belief is above  $2/3$ , and agents 2 and 3 sell half a unit whenever their belief is below  $2/3$ . A simple calculation shows that condition (7) of Theorem 3 is violated.

## 4 Applications

**Identically Distributed Binary Signals.** As a first application of Theorem 2 we consider a setting with prior  $p = 1/2$ , and two agents who each receive a binary signal that equals the state with some probability  $r > 1/2$ . We ask: what joint distributions are feasible?

The canonical setting is the one in which signals are independent, conditioned on the state. Note that they are not unconditionally independent: while each agent has each posterior with probability  $1/2$ , conditioned on the first agent acquiring a high posterior, the second agent is more likely to also have a high posterior than a low one. In this case the induced belief distribution is

$$P = \frac{r^2 + (1-r)^2}{2} [\delta_{r,r} + \delta_{1-r,1-r}] + r(1-r) [\delta_{1-r,r} + \delta_{r,1-r}].$$

Another simple case is the one in which both agents observe the same signal, in which case the posteriors are of course perfectly correlated, and the distribution is

$$P = \frac{1}{2} \delta_{r,r} + \frac{1}{2} \delta_{1-r,1-r},$$

as in the signal agent case. In both of these cases  $P$  is feasible, since we derive it from a joint signal distribution.

The case in which agents' posteriors are perfectly anti-correlated, i.e.,

$$P = \frac{1}{2} \delta_{r,1-r} + \frac{1}{2} \delta_{1-r,r}$$

is precluded by the Agreement Theorem and its Corollary A, as agents here agree to disagree on their posteriors.

More generally, we can consider the case in which conditioned on an agent's posterior, the other agent has the same posterior with probability  $c$ . That is,

$$P = \frac{c}{2} \delta_{r,r} + \frac{c}{2} \delta_{1-r,1-r} + \frac{1-c}{2} \delta_{1-r,r} + \frac{1-c}{2} \delta_{r,1-r}. \quad (8)$$

In this case there is never common knowledge of beliefs, as long as  $c < 1$ . The natural question is: to what extent can the signals be anti-correlated? Can they for example be (unconditionally) independent, so that after observing a signal, the probability that an agent assigns to the other agent's posterior is still uniform over  $\{r, 1-r\}$ ? A common intuition suggests that this is impossible, since even if signals are independent given the state, the induced unconditional distribution of posteriors inherits the dependence on the state and thus the posteriors must be dependent; on the other hand, if the conditional distribution of signals is the same in both states, they convey no information and thus the posterior just equals the prior. Perhaps surprisingly, this intuition is wrong, and posteriors can be independent and even anti-correlated.

**Proposition 1.** *A joint belief distribution  $P$  as given in (8) is feasible if and only if  $c \geq 2r - 1$ .*

This proposition follows directly from Theorem 2: one only needs to check the sets  $A_i = \{r\}$  and  $A_i = \{1 - r\}$ . More generally, for finitely supported  $P$  only finitely many conditions need be checked.

Proposition 1 implies that indeed too much anti-correlation is infeasible, especially as the signals become more informative. Nevertheless, it is possible that the agents posteriors are independent of each other (i.e.,  $c = 1/2$ ) as long as  $r \leq 3/4$ . Moreover, for  $r < 3/4$ , the posteriors can be negatively correlated; for example,  $P$  is feasible for  $c = 1/3$  and  $r = 2/3$ . In this case, posteriors are either  $1/3$  or  $2/3$ , each obtained with probability  $1/2$ . When an agent has the high posterior, she assigns probability  $2/3$  to the event the the other agent has the low posterior.

**Unconditionally independent signals.** As another application of Theorem 2 we further explore joint signal distributions that are independent. To motivate this application consider a sender (e.g., a consulting agency) who wants to reveal some information to receivers (its clients). However, there is an additional concern: none of the receivers must be able to form any non-trivial guess about the information received by their counterpart. This can be motivated either by privacy concerns or by the desire to avoid complicated strategic reasoning on receivers' side.<sup>3</sup> We show in Lemma 1 that independent posteriors can be induced by independent signals, as a consequence of a revelation principle that allows the identification of signals and posteriors.

We already saw above that identically distributed binary signals can be independent for prior  $p = 1/2$  as long as  $r \leq 3/4$ . As a second example, let  $P$  be the uniform distribution on the unit square. We verify that it is  $\frac{1}{2}$ -feasible. It clearly satisfies the martingale condition so it remains to

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<sup>3</sup>For example, consider the case that the receivers are two firms competing on the same market and plan to use the received information to adjust their strategies. If their posteriors are not independent, they might engage in a complicated reasoning involving higher order beliefs, as in [Weinstein and Yildiz \(2007\)](#).

check that

$$P(A_1 \times \overline{A_2}) \geq \int_{A_1} x \, dP_1(x) - \int_{A_2} x \, dP_2(x).$$

The other inequality of (5) will follow by symmetry.

Let  $1-a$  be the Lebesgue measure of  $A_1$  and  $b$  of  $A_2$ . Then the left hand side equals  $(1-a)(1-b)$  and the right hand-side is maximized when  $A_1$  is pushed as much as possible towards high values of the integrand (i.e.,  $A_1 = [a, 1]$ ) and  $A_2$  is pushed towards low values ( $A_2 = [0, b]$ ). We get the following inequality

$$(1-a)(1-b) \geq \int_a^1 x \, dx - \int_0^b x \, dx = \frac{1}{2}(1-a^2-b^2).$$

Simplifying the expression, we get  $(a+b-1)^2 \geq 0$ , which holds for any  $a$  and  $b$ . Thus the uniform distribution is  $\frac{1}{2}$ -feasible.

We generalize this example to study the conditions for  $\frac{1}{2}$ -feasibility of more general product distributions. The next proposition provides a necessary and sufficient condition for feasibility of a large class of such distributions.

**Proposition 2.** *Let  $P = \nu \times \nu$ , where  $\nu \in \Delta([0, 1])$  has a density  $f$  that is symmetric with respect to  $\frac{1}{2}$  (i.e.,  $f(1-x) = f(x)$  for all  $x \in [0, 1]$ ). Denote  $S(a) = \nu([a, 1])$ . Then  $P$  is  $1/2$ -feasible if and only if*

$$\frac{1-a^2}{2} \geq \int_a^1 x f(x) \, dx, \tag{9}$$

for all  $a \in [1/2, 1]$  such that  $a = S(S(a))$ .

The proof is relegated to §A.

Since  $\frac{1-a^2}{2} = \int_a^1 x \, dx$ , Proposition 2 gives a simple sufficient condition for feasibility:  $P = \nu \times \nu$  is  $1/2$ -feasible if  $\nu$  is second-order stochastically dominated by the uniform distribution on  $[0, 1]$ . Hence, for example, the uniform distribution on  $[r, 1-r]^2$  is feasible for all  $0 \leq r < 1/2$ .

Interestingly, product distributions with a given marginal cease being feasible once the number of agents becomes large enough, as demonstrated in the following proposition.

**Proposition 3.** *For every probability measure  $\nu \in \Delta([0, 1])$  that differs from a Dirac measure, for sufficiently large  $n$  the product distribution  $\nu^n \in \Delta([0, 1]^n)$  is infeasible.*

The proof, that is relegated to §A, is based on an application of the No Trade Theorem. We show that with sufficiently large number of agents, a mediator can implement a strictly beneficial trading scheme, i.e. one that violates Theorem 3. In fact, we show that the product distribution  $\nu^n$  is infeasible whenever  $\lfloor \frac{n}{2} \rfloor > \frac{1}{8} \left( \int_m^1 x \, d\nu(x) - \int_0^m x \, d\nu(x) \right)^{-2}$ , where  $m$  is the median of the marginal distribution  $\nu$ .

## 5 Proof of Theorem 2

In this section we present the proof of Theorem 2. We begin with the following simple lemma, which gives a direct revelation principle for joint belief distributions. Similar statements recently appeared in the literature (e.g., Ziegler, 2020; Arieli et al., 2020). For the convenience of the reader, the proof is included in the Appendix.

**Lemma 1.** *Let  $P \in \Delta([0, 1]^n)$  be a  $p$ -feasible belief distribution. Then there exist  $P^\ell, P^h \in \Delta([0, 1]^n)$  such that  $P = (1 - p) \cdot P^\ell + p \cdot P^h$  and for every  $i \in \{1, \dots, n\}$  the marginal distributions  $P_i^h$  and  $P_i$  satisfy*

$$dP_i^h(x) = \frac{x}{p} dP_i(x). \quad (10)$$

*In particular, every feasible belief distribution can be induced by an information structure in which, for each  $i$ , the belief  $x_i$  is equal to the signal  $s_i$ .*

The next lemma gives a necessary and sufficient condition for feasibility in terms of the existence of a dominated measure with given marginals.

**Lemma 2.** *For  $n \geq 2$  agents, a distribution  $P \in \Delta([0, 1]^n)$  is  $p$ -feasible if and only if there exists a probability measure  $Q \in \Delta([0, 1]^n)$  such that*

1.  $Q$  is dominated by  $\frac{1}{p}P$ , i.e.  $Q(A) \leq \frac{1}{p}P(A)$  for any measurable  $A \subseteq [0, 1]^n$ , and
2. for every  $i \in \{1, \dots, n\}$  the marginal distribution  $Q_i$  is given by

$$dQ_i(x) = \frac{x}{p} dP_i(x). \quad (11)$$

*Proof.* If  $P$  is feasible, then by Lemma 1 there is a pair  $P^\ell, P^h \in \Delta([0, 1]^n)$  such that  $P = (1 - p)P^\ell + p \cdot P^h$  and such that (10) holds. Picking  $Q = P^h$ , it follows from (10) that  $Q \leq \frac{1}{p}P$  and that  $Q$  the desired marginals. In the opposite direction: if there is  $Q$  satisfying the theorem hypothesis, then let  $P^h = Q$  and  $P^\ell = \frac{P - p \cdot Q}{1 - p}$ . Consider the information structure  $I$  in which  $\omega = h$  with probability  $p$ , and  $(s_1, \dots, s_n)$  is chosen from  $P^h$  conditioned on  $\omega = h$ , and from  $P^\ell$  conditioned on  $\omega = \ell$ . It follows from (11) that  $P$  is the joint posterior distribution induced by  $I$ , and is hence  $p$ -feasible.  $\square$

Lemma 2 reduces the question of feasibility to the question of the existence of a dominated measure with given marginals. This is a well-studied question in the case  $n = 2$ . When the dominating measure is proportional to the Lebesgue measure, Lorentz (1949) provided the answer in terms of first-order stochastic dominance of marginals. The discrete analog of this result for

matrices is known as the Gale-Ryser Theorem (Gale et al., 1957; Ryser, 1957). The condition for general dominating measures was derived by Kellerer (1961, Satz 4.2); the formulation below is due to Strassen (1965, Theorem 6).

**Theorem B** (Kellerer). *Fix  $P \in \Delta([0, 1]^2)$ ,  $0 < p \leq 1$ , and  $M_1, M_2 \in \Delta([0, 1])$ . Then the following are equivalent:*

1. *There exists a probability measure  $Q \in \Delta([0, 1]^2)$  that is dominated by  $\frac{1}{p}P$ , and which has marginals  $Q_i = M_i$ .*
2. *For every measurable  $B_1, B_2 \subset [0, 1]$  it holds that*

$$\frac{1}{p}P(B_1 \times B_2) \geq M_1(B_1) + M_2(B_2) - 1. \quad (12)$$

Theorem 2 is a consequence of Kellerer's Theorem and Lemma 2.<sup>4</sup>

*Proof of Theorem 2.* When  $P$  is  $p$ -feasible, (5) follows immediately from Corollary 1. To prove the other direction, we assume that  $P$  satisfies (5), and prove that  $P$  is  $p$ -feasible for some  $p$ . To this end we will use Kellerer's Theorem to show that there exists a  $Q \in \Delta([0, 1]^2)$  satisfying the conditions of Lemma 2.

Assume then that  $P$  satisfies (5), i.e. for every measurable  $A_1, A_2 \subseteq [0, 1]$  it holds that

$$P(A_1 \times \overline{A_2}) \geq \int_{A_1} x \, dP_1(x) - \int_{A_2} x \, dP_2(x) \geq P(\overline{A_1} \times A_2). \quad (13)$$

Applying this to  $A_1 = A_2 = [0, 1]$  yields

$$\int_0^1 x \, dP_1(x) = \int_0^1 x \, dP_2(x).$$

We accordingly set  $p = \int_0^1 x \, dP_1(x)$ , and define the measures  $M_1, M_2 \in \Delta([0, 1])$  by  $dM_i(x) = \frac{x}{p} dP_i(x)$ . Given any two measurable subsets  $B_1, B_2 \subseteq [0, 1]$ , let  $A_1 = B_1$  and  $A_2 = \overline{B_2}$ . Applying the left hand inequality of (13) to  $A_1 = B_1$  and  $A_2 = \overline{B_2}$  yields

$$P(B_1 \times B_2) \geq \int_{B_1} x \, dP_1(x) - \int_{\overline{B_2}} x \, dP_2(x).$$

Since  $dM_i(x) = \frac{x}{p} dP_i(x)$  we can write this as

$$P(B_1 \times B_2) \geq p \cdot M_1(B_1) - p \cdot M_2(\overline{B_2}),$$

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<sup>4</sup>We are grateful to Kim Border and Luciano Pomatto for introducing us to Kellerer's theorem. Our original proof was more involved: it was based on interpreting the infinite-dimensional linear program for  $Q$  from Lemma 2 as a condition that there is a feasible flow of magnitude 1 in an auxiliary network (as in Gale et al. (1957), but with a continuum of edges) and then using max-flow/min-cut duality for infinite networks (see Neumann, 1985).

dividing by  $p$  and substituting  $M_2(\overline{B_2}) = 1 - M_2(B_2)$  yields.

$$\frac{1}{p}P(B_1 \times B_2) \geq M_1(B_1) + M_2(B_2) - 1.$$

Thus condition (12) holds, and we can directly apply Kellerer's theorem to conclude that there exists a measure  $Q$  that satisfies the conditions of Lemma 2, as condition (11) is simply  $Q_i = M_i$ . Hence  $P$  is  $p$ -feasible.  $\square$

## 6 Social Bayesian Persuasion and Extreme Feasible Joint Posterior Belief Distributions

**Social Bayesian persuasion.** In this section we consider a sender who sends information regarding an underlying state to a group of  $n$  receivers. The sender's utility depends on the action of the receivers. We assume that each receiver's utility depends only on the state and her own action, as is common in the social learning literature, and thus we call this setting *social Bayesian persuasion*.

Formally, a social Bayesian persuasion problem is given by  $B = (N, p, (A_i)_{i \in N}, (u_i)_{i \in N}, u_s)$ . As above,  $\omega \in \{\ell, h\}$  is a binary state regarding which  $n = |N|$  receivers have a common prior  $p \in (0, 1)$ . Each receiver  $i \in N$  has to choose an action  $a_i$  from a compact metric set of actions  $A_i$ . Her utility  $u_i(\omega, a_i)$  depends only on the state and her action. A single sender has utility  $u_s(a_1, \dots, a_n)$  which depends on the receivers' actions.

The sender chooses an information structure  $I = ((S_i)_{i \in N}, \mathbb{P})$  with prior  $p$ , and each receiver  $i$  observes a private signal  $s_i \in S_i$  and then chooses an action  $a_i$ .

In equilibrium, the action

$$\tilde{a}_i \in \operatorname{argmax}_{a_i \in A_i} \mathbb{E}(u_i(\omega, a_i) \mid s_i)$$

is chosen by  $i$  to maximize her expected utility conditioned on her information  $s_i$ . Note that since  $u_i$  depends only on  $\omega$  and  $a_i$ , it follows that given the information structure  $I$ , receiver  $i$ 's posterior  $x_i = \mathbb{P}(\omega = h \mid s_i)$  is a sufficient statistic for her utility, i.e.,

$$\max_{a_i \in A_i} \mathbb{E}(u_i(\omega, a_i) \mid s_i) = \max_{a_i \in A_i} \mathbb{E}(u_i(\omega, a_i) \mid x_i),$$

and so a receiver does not decrease her expected utility by discarding her private signal, retaining only the posterior. We accordingly consider the equilibrium refinement in which receivers choose their actions using only their posteriors:<sup>5</sup>

$$\tilde{a}_i \in \operatorname{argmax}_{a_i \in A_i} \mathbb{E}(u_i(\omega, a_i) \mid x_i).$$

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<sup>5</sup>No pun intended.

The information structure  $I$  is chosen to maximize the expectation of  $u_s$ . We assume that  $u_i$  and  $u_s$  are upper-semicontinuous, to ensure the existence of equilibria. The value  $V(B)$  is the sender's expected equilibrium utility in the social Bayesian persuasion problem  $B$ .

A crucial feature is the assumption that  $u_i$  does not depend on the other agents' actions. The case in which externalities are allowed is the general problem of Bayesian mechanism design, which is beyond the scope of this paper. In contrast, in social Bayesian persuasion the receivers have no strategic incentives. This implies that their higher order beliefs are irrelevant to the sender, who in turn is solely interested in their posterior beliefs. This is captured by the following proposition, which states that for every social Bayesian persuasion problem there is an indirect utility function that the sender maximizes by choosing the receivers' posteriors. Of course, the posterior distribution must be feasible.

Denote by  $\mathcal{P}_p^N$  the set of  $p$ -feasible  $P \in \Delta([0, 1]^N)$ .

**Proposition 4.** *For every social Bayesian persuasion problem  $B = (N, p, (A_i)_i, (u_i)_i, u_s)$  there is an indirect utility function  $v: [0, 1]^N \rightarrow \mathbb{R}$  such that the value  $V(B)$  is given by*

$$V(B) = \max_{P \in \mathcal{P}_p^N} \int v(x_1, \dots, x_n) dP(x_1, \dots, x_n).$$

**Example: Polarizing receivers.** In §4 we showed that joint belief distributions can be anti-correlated, and explored the extent to which identically distributed binary signals can be anti-correlated. For general signal distributions, this question can be formalized using a social Bayesian persuasion approach.

Consider a sender whose indirect utility for the posteriors of two receivers is

$$v(x_1, x_2) = |x_1 - x_2|^a,$$

for some parameter  $a > 0$ . Informally, the sender wishes to maximize the polarization between receivers, or the discrepancy between their posteriors.

For the case  $a = 2$  we solve the sender's problem completely.

**Proposition 5.** *Let  $V$  be the two receiver social Bayesian persuasion problem where the sender's indirect utility is*

$$v(x_1, x_2) = (x_1 - x_2)^2$$

*and the prior is  $p \in (0, 1)$ . Then  $V(B) = (1 - p)p$ .*

In this case of  $a = 2$ , the optimum can be achieved by completely informing one agent, and leaving the other completely uninformed. E.g., by letting  $s_1 = \omega$  and  $s_2 = 0$ . We show in §A that when  $p = 1/2$ , the same policy is also optimal for all  $a < 2$ .

For  $a \geq 3$ , it is no longer true that it is optimal to reveal the state to one receiver and leave the other uninformed, which yields utility  $2^{-a}$  to the sender. For example, the posterior distribution

$$P = \frac{1}{4}\delta_{0,2/3} + \frac{1}{4}\delta_{2/3,0} + \frac{1}{2}\delta_{2/3,2/3} \quad (14)$$

can be easily verified to be  $1/2$ -feasible using (5), and yields utility  $\frac{1}{2} \left(\frac{2}{3}\right)^a$ , which for  $a = 3$  (for example) is larger than  $2^{-a}$ .

Another approach to quantifying the extent to which beliefs can be anti-correlated is by directly minimizing their covariance. This question admits a simple, non-trivial solution: for the prior  $p = 1/2$ , the smallest possible covariance between posterior beliefs is  $-1/32$ , and is achieved on a distribution supported on four points.

**Proposition 6.** *Let  $V$  be the two receiver social Bayesian persuasion problem where the prior is  $p = 1/2$  and the sender's indirect utility is*

$$v(x_1, x_2) = -(x_1 - p) \cdot (x_2 - p).$$

*Then  $V(B) = 1/32$ , and is achieved by*

$$P = \frac{1}{8}\delta_{3/4,0} + \frac{3}{8}\delta_{3/4,1/2} + \frac{1}{8}\delta_{1/4,1} + \frac{3}{8}\delta_{1/4,1/2}.$$

The distribution  $P$  is depicted in Figure 1.

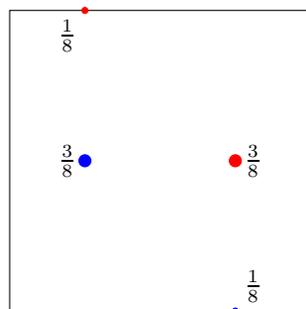


Figure 1: An optimal distribution of the social Bayesian persuasion problem given by the indirect utility  $v(x_1, x_2) = -(x_1 - p) \cdot (x_2 - p)$  and prior  $p = 1/2$ ; see Proposition 6. For  $p = 1/2$ , this distribution achieves the lowest possible covariance between posterior beliefs:  $-1/32$ . Blue points describe the distribution of posteriors conditional on  $\omega = 0$ . Red points describe the distribution of posteriors conditional on  $\omega = 1$ .

**Extreme feasible joint posterior belief distributions.** The set  $\mathcal{P}_p^N$  of feasible joint posterior belief distributions is a convex, compact subset of  $\Delta([0, 1]^N)$ , when the latter is naturally equipped with the weak\* topology. This is a consequence of Lemma 2, and is proved in Proposition 7 in the Appendix. Proposition 4 implies that  $V(B)$  is always achieved at an *extreme point* of the set of  $p$ -feasible distributions  $\mathcal{P}_p^N$ . It is thus natural to study this set.

In the single agent case the concavification argument of Aumann and Maschler (1995) and Kamenica and Gentzkow (2011) implies that every extreme point is a distribution with support size at most 2. This is not true for 2 or more agents. For example, the posterior distribution with support of size 3 defined in (14) is extreme in  $\mathcal{P}_{1/2}^2$ , since its restriction to any support of smaller cardinality is not feasible, as it cannot satisfy the martingale condition. The next theorem shows that there in fact exist extreme points with countably infinite support. It also states that the support cannot be too large, in the sense that every extreme point is supported on a set of zero Lebesgue measure.

**Theorem 4.** *Let  $|N| \geq 2$ . Then*

1. *For every  $p \in (0, 1)$  there exists an extreme point in  $\mathcal{P}_p^N$  whose support has an infinite countable number of points.*
2. *For every extreme  $P \in \mathcal{P}_p^N$  there exists a measurable  $A \subseteq [0, 1]^N$  such that  $P(A) = 1$ , and the Lebesgue measure of  $A$  is zero.*

To prove the first part of Theorem 4 we explicitly construct an extreme feasible belief distribution with countably infinite support; see §A. The construction is a variant of Rubinstein’s e-mail game (Rubinstein, 1989). Unlike Rubinstein’s email game, no agent in our construction is fully informed about the state. The resulting belief distribution for two agents is depicted in Figure 2. Interestingly, it is possible to modify our construction in a way that places the beliefs closer and closer to the diagonal. This results in a sequence of extreme points that converge to a distribution with a support size larger than two over the diagonal. Every extreme point that is supported on the diagonal is supported on at most two points (since this reduces to the single agent case). Therefore, for two agents or more, the set of extreme points is not closed within the set of feasible distributions. This demonstrates another distinction from the single agent case where the set of extreme points is closed.

The proof of the second part of Theorem 4 relies on a novel multidimensional extension of the classic result of Lindenstrauss (1965) regarding extreme points of the set of joint distributions with given marginals. This extension appears in §C.

Theorem 4 leaves a natural question open: Are there any non-atomic extreme points? Or conversely, does every extreme point have countable support?

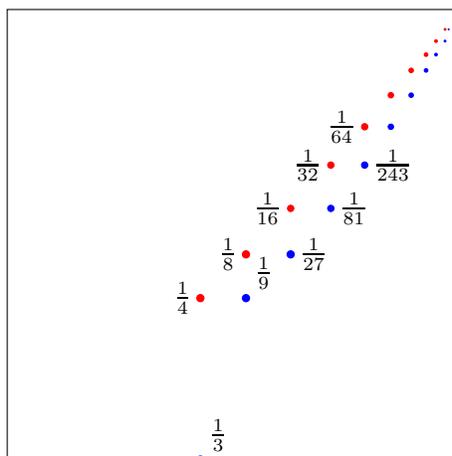


Figure 2: An extreme point of  $P_{1/2}^2$  with infinite countable support. The numbers near the points indicate their weights. The figure demonstrates the implementation of this distribution: Conditional on state  $\omega = \ell$  the pair of posteriors belongs to the set blue points. Conditional on state  $\omega = h$  the pair of posteriors belongs to the set of red points.

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## A Missing Claims and Proofs

*Proof of Theorem 1.* By the law of total expectations, we have that

$$\mathbb{E}(x_i \cdot \mathbf{1}_{s_i \in B_i}) = \mathbb{E}(\mathbb{P}(\omega = h | s_i) \cdot \mathbf{1}_{s_i \in B_i}) = \mathbb{P}(\omega = h, s_i \in B_i).$$

We thus need to show that

$$\mathbb{P}(s_1 \in B_1, s_2 \notin B_2) \geq \mathbb{P}(\omega = h, s_1 \in B_1) - \mathbb{P}(\omega = h, s_2 \in B_2) \geq -\mathbb{P}(s_2 \in B_2, s_1 \notin B_1).$$

We show the first inequality; the second follows by an identical argument. Indeed, this inequality holds since, more generally, for any three events  $A, B, C$  it holds that

$$\mathbb{P}(A \cap \overline{B}) \geq \mathbb{P}(C \cap A) - \mathbb{P}(C \cap B).$$

This follows from the fact that

$$(A \cap \overline{B}) \cup (C \cap B) \supseteq (C \cap A \cap \overline{B}) \cup (C \cap B) \supseteq C \cap A,$$

and so

$$\mathbb{P}(A \cap \overline{B}) + \mathbb{P}(C \cap B) \geq \mathbb{P}(C \cap A).$$

□

*Proof of Proposition 2.* Because of the symmetry between agents, it suffices to check the first inequality in (5):

$$P(A_1 \times \overline{A_2}) \geq \int_{A_1} x dP_1(x) - \int_{A_2} x dP_2(x).$$

Keeping  $\nu(A_1) = \alpha$  and  $\nu(\overline{A_2}) = 1 - \beta$  fixed, the left-hand side of the inequality is equal to  $\alpha(1 - \beta)$ , and the worst-case right-hand side corresponds to  $A_1 = [a, 1]$  and  $A_2 = [0, b]$ , with  $a, b$  chosen to yield  $\nu([a, 1]) = \alpha$  and  $\nu([0, b]) = \beta$ . Note that the existence of such  $a$  and  $b$  for every  $\alpha$  and  $\beta$  relies on the continuity of  $\nu$ .

Inequality (5) thus boils down to

$$\nu([a, 1]) \cdot \nu([b, 1]) - \int_a^1 x f(x) dx + \int_0^b x f(x) dx \geq 0. \quad (15)$$

It is enough to check this inequality at the local extrema of the left-hand side. First order conditions give two equations:  $f(a)(a - \nu([b, 1])) = 0$  and  $f(b)(b - \nu([a, 1])) = 0$ . At zeros of the density  $f$ , partial derivatives do not change sign and hence extrema correspond to zeros of the expressions in the parentheses. Recalling that  $S(t) = \nu([t, 1])$ , we get the following system of equations:

$$\begin{cases} a = S(b) \\ b = S(a) \end{cases} \quad (16)$$

Clearly, every solution of (16) satisfied  $a = S(S(a))$ . We claim that furthermore, every solution also satisfies  $a + b = 1$ . To see this, assume that this identity is violated and  $a + b < 1$ . This is equivalent to  $a < 1 - S(a)$ . We apply  $S$  to both sides of the inequality and consider two cases. If  $S$  is not constant on the interval  $[a, 1 - S(a)]$ , we get  $S(a) > S(1 - S(a)) = 1 - S(S(a)) = 1 - a$  and hence  $a > 1 - S(a)$ , which contradicts the assumption. If  $S$  is constant on  $[a, 1 - S(a)]$ , then  $S(a) = S(1 - S(a)) = 1 - S(S(a)) = 1 - a$ , and we again get contradiction. The opposite inequality  $a + b > 1$  is ruled out by an identical argument, and thus we have that indeed  $a + b = 1$  for any solution of (16).

We have shown that it suffices to check (15) at  $a, b$  such that  $a = S(S(a))$  and  $b = 1 - a$ . We now simplify the inequality (15) using the identities (16) and  $a + b = 1$  and assuming  $a \geq \frac{1}{2}$  for definiteness. Using the facts that

$$\int_a^1 x d\nu - \int_0^{1-a} x d\nu = \int_a^1 x d\nu + \int_{1-a}^1 x d\nu - \frac{1}{2} = 2 \int_a^1 x d\nu + \frac{1}{2} \nu([1-a, a]) - \frac{1}{2}$$

and

$$\nu([1-a, a]) = 1 - 2\nu([a, 1]) = 1 - 2b = 2a - 1,$$

we get

$$\frac{1-a^2}{2} \geq \int_a^1 x d\nu(x),$$

and so we have proved that the symmetric distribution  $\nu \times \nu$  is  $\frac{1}{2}$ -feasible if and only if this inequality is satisfied for all  $a \geq \frac{1}{2}$  such that  $a = S(S(a))$ .  $\square$

*Proof of Proposition 3.* Let  $m \in (0, 1)$  be the median of  $\nu$ . We first prove the proposition for the case where  $\nu$  has no atom at  $m$ . We assume for simplicity that the number of agents is even; i.e.,  $n = 2k$  (otherwise, we simply ignore the last agent).

We consider the trading scheme  $(a_1, \dots, a_{2k})$  given by

$$a_i(x) = \mathbb{1}_{x < m}$$

for  $i \in \{1, \dots, k\}$ , and

$$a_i(x) = -\mathbb{1}_{x > m}$$

for  $i \in \{k+1, \dots, 2k\}$ . We argue that for large enough  $n$  (i.e., large enough  $k$ ) this trading scheme violates the condition of Theorem 3.

Let  $B_1 \sim \text{Bin}(k, \frac{1}{2})$  denote the random variable of the number of agents  $i = 1, \dots, k$  whose posterior is in  $[0, m)$  (i.e., those that sell the product). Let  $B_2 \sim \text{Bin}(k, \frac{1}{2})$  denote the random variable of the number of agents  $i = k+1, \dots, 2k$  whose posterior is in  $(m, 1]$  (i.e., those that buy the product). Since  $\nu^n$  is a product distribution we have that  $B_1$  is independent of  $B_2$ .

Note also that  $\sum_{i=1}^n a_i(x_i) = B_2 - B_1$ , and so we have that the lower bound to the mediator's profit given by (7) equals

$$k \int_m^1 x d\nu(x) - k \int_0^m x d\nu(x) - \mathbb{E}(\max\{0, B_2 - B_1\}).$$

We observe that  $\mathbb{E}(\max\{0, B_2 - B_1\}) \leq \sqrt{\frac{k}{8}}$ : We have  $\mathbb{E}(B_2 - B_1) = 0$  and  $\mathbb{E}((B_2 - B_1)^2) = \frac{k}{2}$ . Jensen's inequality implies that  $\mathbb{E}(|B_2 - B_1|) \leq \sqrt{\frac{k}{2}}$  and by symmetry we deduce that  $\mathbb{E}(\max\{B_2 - B_1, 0\}) \leq \frac{1}{2}\sqrt{\frac{k}{2}}$ .

It follows that for  $k > \frac{1}{8}(\int_m^1 x d\nu - \int_0^m x d\nu)^{-2}$  this lower bound on the mediator's profit is positive, and thus by Theorem 3 the belief distribution  $\nu^n$  is not feasible.

In case the distribution  $\nu$  has an atom at  $m$ , the only needed change is to choose  $a_i(m)$  so that the expected number of units that each agent buys or sells is  $1/2$ ; the same calculation applies in this case. □

*Proof of Lemma 1.* Let  $I_0 = ((S_i)_{i \in N}, \mathbb{P})$  be an information structure with prior  $p$  that induces  $P$ . By the law of total expectation,

$$x_i = \mathbb{P}(\omega = h \mid s_i) = \mathbb{P}(\omega = h \mid x_i)$$

almost everywhere. Hence, if we define a new information structure  $I'$  for which the signals are  $s'_i = x_i$  and the beliefs are, accordingly,  $x'_i = \mathbb{P}(\omega = h \mid s'_i)$ , we will have that

$$x'_i = s'_i,$$

and so we proved the second part of the claim.

Denote by  $P^\ell$  and  $P^h$  be the conditional distributions of  $(x'_1, \dots, x'_n)$ , conditioned on  $\omega = \ell$  and  $\omega = h$ , respectively. Then clearly  $P = (1-p) \cdot P^\ell + p \cdot P^h$ .

Finally, (10) holds, since by Bayes' law and the fact that  $x'_i = s'_i$ ,

$$x'_i = \mathbb{P}(\omega = h \mid x'_i) = p \cdot \frac{dP_i^h}{dP_i}(x'_i).$$

□

*Proof of Proposition 4.* Let  $B = (N, p, (A_i)_i, (u_i)_i, u_s)$  be a social Bayesian persuasion problem. By our equilibrium refinement assumption, there is for each receiver  $i$  a map  $\alpha_i: [0, 1] \rightarrow A_i$  such that in equilibrium receiver  $i$  chooses action  $\tilde{a}_i = \alpha_i(x_i)$  when their posterior is  $x_i$ . Hence, if we set

$$v(x_1, \dots, x_n) = u_s(\alpha_1(x_1), \dots, \alpha_n(x_n)),$$

Then, given an information structure chosen by the sender,

$$\mathbb{E}(u_s(\tilde{a}_1, \dots, \tilde{a}_n)) = \mathbb{E}(u_s(\alpha_1(x_1), \dots, \alpha_n(x_n))) = \mathbb{E}(v(x_1, \dots, x_n)).$$

Thus, if we denote by  $P$  the posterior belief distribution induced by the chosen information structure, we have shown that the sender's expected utility is

$$\int v(x_1, \dots, x_n) dP(x_1, \dots, x_n).$$

Since the sender, by choosing the information structure, can choose any  $p$ -feasible distribution, we arrive at the desired conclusion.  $\square$

*Proof of Proposition 5.* We first show that every information structure  $I = (S_1, S_2, \mathbb{P})$  with prior  $p$  chosen by the sender yields utility at most  $(1-p)p$ . This proof does not use our characterization of feasible beliefs, but rather an additional observation that follows from the fact that posteriors are conditional expectations, and the fact that conditional expectations are orthogonal projections in the Hilbert space of square-integrable random variables. We now review this fact.

For the probability space  $(\Omega \times S_1 \times S_2, \mathbb{P})$  denote by  $\mathcal{L}^2$  the Hilbert space of square-integrable random variables, equipped with the usual inner product  $(X, Y) = \mathbb{E}(X \cdot Y)$  and corresponding norm  $\|X\| = \sqrt{\mathbb{E}(X^2)}$ .

Given a sub-sigma-algebra  $\mathcal{G} \subseteq \mathcal{F}$ , denote by  $\mathcal{L}^2(\mathcal{G}) \subseteq \mathcal{L}^2$  the closed subspace of  $\mathcal{G}$ -measurable, square-integrable random variables. Recall that the map  $X \mapsto \mathbb{E}(X \mid \mathcal{G})$  is simply the orthogonal projection  $\mathcal{L}^2 \rightarrow \mathcal{L}^2(\mathcal{G})$ .

The following is an elementary lemma regarding Hilbert spaces. We will, of course, apply it to  $\mathcal{L}^2$ .

**Lemma 3.** *Let  $u$  be a vector in a Hilbert space  $U$ , and let  $w$  be an orthogonal projection of  $u$  to a closed subspace  $W \subseteq U$ . Then  $w$  lies on a sphere of radius  $\frac{1}{2}\|u\|$  around  $\frac{1}{2}u$ , i.e.  $\|w - \frac{1}{2}u\| = \frac{1}{2}\|u\|$ .*

*Proof.* Since  $w$  is an orthogonal projection of  $u$ , we can write  $u = w + w'$ , where  $w$  and  $w'$  are orthogonal. This orthogonality implies that  $\|w - w'\| = \|w + w'\| = \|u\|$ . It follows that

$$\left\|w - \frac{1}{2}u\right\| = \left\|w - \frac{1}{2}(w + w')\right\| = \left\|\frac{1}{2}w - \frac{1}{2}w'\right\| = \frac{1}{2}\|w - w'\| = \frac{1}{2}\|u\|.$$

$\square$

Let  $X = \mathbb{1}_{\omega=h}$  be the indicator of the event that the state is high. Hence  $x_i = \mathbb{E}(X | \mathcal{F}_i)$ , where  $\mathcal{F}_i = \sigma(s_i)$  is the sigma-algebra generated by the private signal  $s_i$ . Denote  $\hat{X} = X/p - 1$ . Hence  $\mathbb{P}(\hat{X} = 1/p - 1) = p$ ,  $\mathbb{P}(\hat{X} = -1) = 1 - p$ , and  $\|\hat{X}\|^2 = (1 - p)/p$ . Denote  $\hat{X}_i = x_i/p - 1$ , so that  $\hat{X}_i = \mathbb{E}(\hat{X} | \mathcal{F}_i)$ . Since  $\hat{X}_i$  is the projection of  $\hat{X}$  to the subspace  $\mathcal{L}^2(\mathcal{F}_i)$ , it follows from Lemma 3 that

$$\|\hat{X}_i - \hat{X}/2\| = \frac{1}{2} \sqrt{\frac{1-p}{p}}.$$

Therefore, by the triangle inequality,  $\|\hat{X}_1 - \hat{X}_2\| \leq \sqrt{(1-p)/p}$ , and since  $\hat{X}_i = x_i/p - 1$  we get that  $\|x_1/p - x_2/p\| \leq \sqrt{(1-p)/p}$ , or

$$\mathbb{E}((x_1/p - x_2/p)^2) \leq \frac{1-p}{p}.$$

Thus

$$\mathbb{E}((x_1 - x_2)^2) \leq (1-p)p.$$

To finish the proof of Proposition 5, we note that a simple calculation shows that the sender can achieve the expected utility  $(1-p)p$  by completely informing one agent, and giving no information to the other. That is, by inducing the joint posterior distribution

$$P = p \cdot \delta_{1,p} + (1-p)\delta_{0,p}.$$

This completes the proof of Proposition 5. □

**Remark on the polarizing social Bayesian persuasion problem with  $a < 2$  and  $p = 1/2$ .**

For the social Bayesian persuasion problem  $B_a$  given by  $v(x_1, x_2) = |x_1 - x_2|^a$  and symmetric prior  $p = \frac{1}{2}$ , we argue that  $P = \frac{1}{2}\delta_{1, \frac{1}{2}} + \frac{1}{2}\delta_{0, \frac{1}{2}}$  is the optimal posterior distribution for all  $a \in (0, 2)$ ; thus the value  $V(B_a)$  of the problem equals  $\frac{1}{2^a}$ . To prove this, it is enough to check the upper bound  $V(B_a) \leq \frac{1}{2^a}$ .

Consider an arbitrary  $\frac{1}{2}$ -feasible policy  $P'$ . Hölder's inequality implies

$$\int v(x_1, x_2) dP' = \int v(x_1, x_2) \cdot 1 dP' \leq \left( \int (v(x_1, x_2))^q dP' \right)^{\frac{1}{q}} \cdot \left( \int 1^{q'} dP' \right)^{\frac{1}{q'}}$$

where  $q, q' > 0$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ . Picking  $q = \frac{2}{a}$  and taking supremum over  $P'$  on both sides, we get  $V(B_a) \leq (V(B_2))^{\frac{a}{2}}$ . By Proposition 5,  $V(B_2) = \frac{1}{4}$  and we obtain the required upper bound.

*Proof of Proposition 6.* It is immediate to check that the expected utility for the distribution  $P$  given in the theorem statement is  $1/32$ . It thus remains to prove that no other distribution can achieve higher expected utility.

We retain the notation used in the proof of Proposition 4. Let  $(x_1, x_2)$  be the posterior beliefs of two agents induced by some information structure. Since the prior is  $p = 1/2$ ,  $\hat{X}_i = 2x_i - 1$ , and so

$$\mathbb{E}(v(x_1, x_2)) = -\mathbb{E}((x_1 - 1/2) \cdot (x_2 - 1/2)) = -\frac{1}{4}\mathbb{E}(\hat{X}_1 \cdot \hat{X}_2).$$

We recall that  $(\cdot, \cdot)$  denotes the inner product in the Hilbert space  $\mathcal{L}^2$ . Thus we can write

$$\mathbb{E}(v(x_1, x_2)) = -\frac{1}{4}(\hat{X}_1, \hat{X}_2). \tag{17}$$

By Lemma 3, the vectors  $\hat{X}_1, \hat{X}_2 \in \mathcal{L}^2$  lie on a sphere of radius  $\frac{1}{2}\|\hat{X}\|$  around  $\frac{1}{2}\hat{X} \in \mathcal{L}^2$ . Note that since  $p = 1/2$ ,  $\|\hat{X}\| = 1$ . Since  $\{\hat{X}, \hat{X}_1, \hat{X}_2\}$  span a subspace that is (at most) three dimensional, the question is reduced to the following elementary question in three dimensional geometry: given a vector  $w \in \mathbb{R}^3$ , what is the minimum of the inner product  $(u_1, u_2)$ , as  $u_1, u_2 \in \mathbb{R}^3$  range over the sphere of radius  $\|w\|$  around  $w$ ? Lemma 4 states that the minimum is  $-\frac{1}{2}\|w\|^2$ . Hence by (17)

$$\mathbb{E}(v(x_1, x_2)) \leq \frac{1}{32}.$$

□

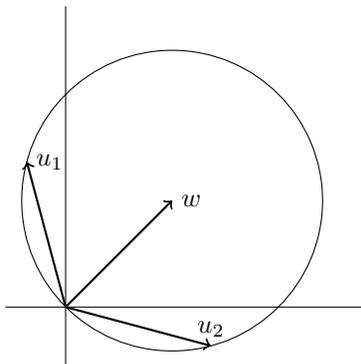


Figure 3: Lemma 4 states that  $-1/2$  is the smallest possible inner product between two vectors that lie on a unit sphere which intersects the origin. The depicted  $u_1$  and  $u_2$  achieve this minimum.

**Lemma 4.** *Let  $w \in \mathbb{R}^3$ . Then*

$$\min \left\{ (u_1, u_2) : \|w - u_1\| = \|w - u_2\| = \|w\| \right\} = -\frac{1}{2}\|w\|^2.$$

The proof is left as an exercise to the reader. Alternatively, this problem can be solved symbolically using the Mathematica command

Minimize[{x1\*y1 + x2\*y2 + x3\*y3,  
(x1 - 1)^2 + (x2)^2 + (x3)^2 == 1 && (y1 - 1)^2 + (y2)^2 + (y3)^2 == 1},  
{x1, x2, x3, y1, y2, y3}]

*Proof of Theorem 4.* We start with demonstrating an extreme point of  $\mathcal{P}_p^2$  with infinite support. Thereafter, we extend this construction for  $N \geq 2$  receivers. We define the pair of sequences

$$s_1(n) = \frac{2^{-n+1}p}{2^{-n+1}p + 2 \cdot 3^{-n}(1-p)} \text{ for } n \geq 2 \text{ and } s_1(1) = 0$$

$$s_2(n) = \frac{2^{-n}p}{2^{-n}p + 2 \cdot 3^{-n}(1-p)} \text{ for } n \geq 1.$$

We define two infinite-support probability measures over  $[0, 1]^2$  as follows.

$$P^\ell = \sum_{n=1}^{\infty} 2 \cdot 3^{-n} \delta_{s_1(n), s_2(n)} \text{ and}$$

$$P^h = \sum_{n=1}^{\infty} 2^{-n} \delta_{s_1(n+1), s_2(n)}.$$

Finally we set  $P = (1-p)P^\ell + pP^h$ . It is easy to verify that  $P \in \mathcal{P}_p^2$ . Indeed, if conditional on state  $\omega = l$  the pairs of signals is  $(s_1(n), s_2(n))$  where  $n$  is distributed Geometrically with parameter  $\frac{2}{3}$  and conditional on state  $\omega = h$  the pairs of signals is  $(s_1(n+1), s_2(n))$  where  $n$  is distributed Geometrically with parameter  $\frac{1}{2}$  then the induced joint posterior belief distribution is  $P$ . We argue that  $P$  is an extreme point of  $\mathcal{P}_p^2$ .

We first argue that  $(P^\ell, P^h)$  is the unique implementation of  $P$ . Namely, there is no other pair  $(\hat{P}^\ell, \hat{P}^h) \neq (P^\ell, P^h)$  such that we can write  $P$  as the average of the two conditional distributions  $P = (1-p)\hat{P}^\ell + p\hat{P}^h$ . All the mass of  $(1-p)2 \cdot 3^{-1}$  of the point  $(s_1(1), s_2(1)) = (0, s_2(1))$  must be assigned to  $\hat{P}^\ell$  because otherwise receiver 1 would not have a posterior of 0 after observing signal  $s_1(1)$ . All the mass of  $p \cdot 2^{-1}$  of the point  $(s_1(2), s_2(1))$  must be assigned to  $\hat{P}^h$  because otherwise receiver 2 would not have a posterior of  $\frac{2^{-n}p}{2^{-n}p + 2 \cdot 3^{-n}(1-p)}$  after observing signal  $s_2(1)$  but rather a lower posterior. All the mass of  $(1-p)2 \cdot 3^{-2}$  of the point  $(s_1(2), s_2(2))$  must be assigned to  $\hat{P}^\ell$  because otherwise receiver 1 would not have a posterior of  $\frac{2^{-n+1}p}{2^{-n+1}p + 2 \cdot 3^{-n}(1-p)}$  after observing signal  $s_1(2)$ , but rather a higher posterior. We proceed inductively and deduce that indeed  $P$  has a unique implementation.

Given a convex combination  $P = \alpha Q + (1-\alpha)\hat{Q}$  let  $(Q^\ell, Q^h)$  be some pair of conditional probability distributions that implements<sup>6</sup>  $Q$ . Similarly, let  $(\hat{Q}^\ell, \hat{Q}^h)$  be some pair of conditional probability distributions that implements  $\hat{Q}$ . The pair  $(\alpha Q^\ell + (1-\alpha)\hat{Q}^\ell, \alpha Q^h + (1-\alpha)\hat{Q}^h)$  implements  $P$ . From the uniqueness of  $P$ 's implementation we can deduce that  $\text{supp}(Q^\ell) \subset \text{supp}(P^\ell)$

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<sup>6</sup>There might be multiple pairs of conditional probabilities that implement  $Q \in \mathcal{P}_p^2$ .

and  $\text{supp}(Q^h) \subset \text{supp}(P^h)$ . Now we argue that  $\text{supp}(Q^\ell) = \text{supp}(P^\ell)$  and  $\text{supp}(Q^h) = \text{supp}(P^h)$ : If  $(s_1(1), s_2(1)) \notin \text{supp}(Q^\ell)$  then  $(s_1(2), s_2(1)) \notin \text{supp}(Q^h)$  because otherwise we would have posterior  $1 \neq s_2(1)$  for receiver 2 after observing  $s_2(1)$ . Therefore,  $(s_1(2), s_2(2)) \notin \text{supp}(Q^\ell)$  because otherwise we would have posterior  $0 \neq s_1(2)$  for receiver 1 after observing  $s_1(2)$ . Inductively, we get that all points are not in the support which is a contradiction. Therefore,  $(s_1(1), s_2(1)) \in \text{supp}(Q^\ell)$  and from that we can deduce  $(s_1(2), s_2(1)) \in \text{supp}(Q^h)$ , because otherwise we would have posterior  $1 \neq s_2(1)$  for receiver 2 after observing  $s_2(1)$ . We proceed inductively to deduce that  $\text{supp}(Q^\ell) = \text{supp}(P^\ell)$  and  $\text{supp}(Q^h) = \text{supp}(P^h)$ . Clearly,  $(P^\ell, P^h)$  is the unique Bayesian consistent pair with the given supports  $\text{supp}(P^\ell)$  and  $\text{supp}(P^h)$ . This proves that  $P = Q$  and hence  $P$  is an extreme point.

This construction extends to  $N \geq 2$ , by sending no information to the remaining  $N - 2$  agents. The resulting distribution over posteriors is an extreme point due to the same arguments.

Now we check the singularity of the extreme points of  $\mathcal{P}_p^N$  with respect to the Lebesgue measure  $\lambda$  on the unit cube  $[0, 1]^N$ . The proof relies on the extension of the classical theorem of [Lindenstrauss \(1965\)](#), which states that all extreme points of the set of all probability measures with given marginals are singular. [Lindenstrauss \(1965\)](#) proved this result for  $n = 2$  and uniform marginals; in §C (Theorem 5), we check that the conclusion holds in general case. Here we show how to apply this result to our problem.

By Lemma 1, any  $p$ -feasible distribution  $P$  can be represented as  $(1 - p)P^\ell + p \cdot P^h$ , where the marginals of the pair  $(P^\ell, P^h)$  satisfy the identity

$$\frac{x}{p} dP_i^\ell(x) = \left(1 - \frac{x}{p}\right) dP_i^h(x). \quad (18)$$

Denote by  $\mathcal{H}_p^N$  the set of all pairs satisfying (18). The set of  $p$ -feasible distributions  $\mathcal{P}_p^N$  is, therefore, the image of the convex set  $\mathcal{H}_p^N$  under the linear map  $(P^\ell, P^h) \rightarrow (1 - p)P^\ell + p \cdot P^h$ . The extreme points of  $\mathcal{P}_p^N$  are hence all contained in the image of the extreme points of  $\mathcal{H}_p^N$ .

So it is enough to check that in any extreme pair  $(P^\ell, P^h)$ , both measures are necessary singular with respect to  $\lambda$ . We check singularity of  $P^\ell$ ; the argument for  $P^h$  is analogous. We assume towards a contradiction that  $P^\ell$  is not singular. Then by Theorem 5,  $P^\ell$  is not an extreme point among measures with the same marginals and thus can be represented as  $P^\ell = \frac{1}{2}Q + \frac{1}{2}Q'$ , where  $Q$  and  $Q'$  have the same marginals as  $P^\ell$  and  $Q \neq Q'$ . This induces the representation of  $(P^\ell, P^h)$  as the average of  $(Q, P^h)$  and  $(Q', P^h)$ , where both pairs satisfy (18). Contradiction.  $\square$

**Proposition 7.** *The set of  $p$ -feasible posterior distributions is a convex, weak\* compact subset of  $\Delta([0, 1]^n)$ .*

*Proof.* Pick a pair of feasible distributions  $P, P' \in \mathcal{P}_p^N$  and consider the corresponding  $Q, Q' \in \Delta([0, 1]^N)$  that satisfy the conditions of Lemma 2 for  $P$  and  $P'$ , respectively. For a convex combi-

nation  $P'' = \alpha P + (1 - \alpha)P'$ , the conditions of the lemma are satisfied by  $Q'' = \alpha Q + (1 - \alpha)Q'$ . Thus  $P''$  also belongs to  $\mathcal{P}_p^N$ , and so we have shown that the set of feasible distributions is convex.

To verify the weak\* compactness of  $\mathcal{P}_p^N$ , consider a sequence of  $p$ -feasible distributions  $P^{(k)}$  weakly converging to some  $P^{(\infty)} \in \Delta([0, 1]^N)$ , as  $k \rightarrow \infty$ . Weak\* compactness means that this limit distribution  $P^{(\infty)}$  is also feasible. For each  $P^{(k)}$ , select some  $Q^{(k)}$  from Lemma 2. The set of all probability distributions on  $[0, 1]^N$  is weak\* compact and, therefore, there is a subsequence  $Q^{(k_m)}$  weakly converging to some  $Q^{(\infty)} \in \Delta([0, 1]^N)$ .

The conditions of Lemma 2 can be rewritten in an equivalent integrated form. Condition (1) becomes

$$\int f(x_1, \dots, x_n) \left( \frac{1}{p} dP(x_1, \dots, x_n) - dQ(x_1, \dots, x_n) \right) \geq 0 \quad (19)$$

for any non-negative continuous function  $f$  on the unit cube. Condition (2) is equivalent to

$$\int g(x_i) \left( \frac{x_i}{p} dP(x_1, \dots, x_n) - dQ(x_1, \dots, x_n) \right) = 0 \quad (20)$$

for any agent  $i \in N$  and any continuous  $g$  of arbitrary sign on the unit interval.

With this reformulation it is immediate that both conditions are closed, and hence withstand the weak\* limit. Therefore, since  $(P^{(k_m)}, Q^{(k_m)})$  satisfy the conditions (19) and (20), the limiting pair  $(P^{(\infty)}, Q^{(\infty)})$  satisfies them as well. We deduce that  $P^{(\infty)}$  is feasible. Thus  $\mathcal{P}_p^N$  contains all its weak\* limit points.  $\square$

## B Checking on Intervals is not Sufficient

In this section we show that restricting the condition (5) to intervals  $A_1, A_2$  provides a condition that—while clearly necessary—is not sufficient for feasibility.

To see this, consider the following distribution  $P$ , depicted in Figure 4. It is parameterized by small  $\varepsilon > 0$  and is supported on four points: two “heavy” points,  $(\frac{1}{4} - \frac{\varepsilon}{4}, \frac{1}{2})$  and  $(\frac{3}{4}, \frac{1}{2})$  with probabilities  $\frac{1-\varepsilon}{2}$  and  $\frac{1}{2}$ , respectively, and two “light” points,  $(\frac{1}{2} - \frac{\varepsilon}{4}, 0)$  and  $(\frac{1}{2} - \frac{\varepsilon}{4}, 1)$ , each with probability  $\frac{\varepsilon}{4}$ . Thus

$$P = \frac{1-\varepsilon}{2} \delta_{\frac{1-\varepsilon}{4}, \frac{1}{2}} + \frac{1}{2} \delta_{\frac{3}{4}, \frac{1}{2}} + \frac{\varepsilon}{4} (\delta_{1/2-\varepsilon/4, 0} + \delta_{1/2-\varepsilon/4, 1}) \quad (21)$$

This distribution satisfies the martingale condition with prior  $\frac{1}{2}$ ; however, it is not  $\frac{1}{2}$ -feasible. To see that, pick  $A_1 = \{\frac{1-\varepsilon}{4}, \frac{3}{4}\}$  and  $A_2 = \{\frac{1}{2}\}$ , then  $A_1 \times \overline{A_2}$  has zero  $P$ -measure and the condition (5) is violated:

$$0 \geq \frac{1-\varepsilon}{4} \cdot \frac{1-\varepsilon}{2} + \frac{3}{4} \cdot \frac{1}{2} - \frac{1}{2} \left( \frac{1-\varepsilon}{2} + \frac{1}{2} \right) = \frac{\varepsilon^2}{8}.$$

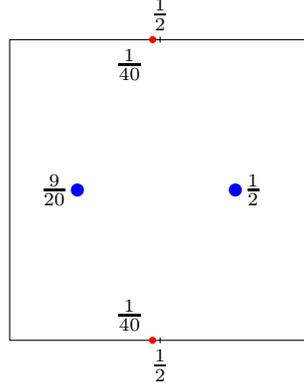


Figure 4: The distribution defined in (21), for  $\varepsilon = 1/10$ . The heavy points are large and blue, and the light points are small and red.

We now check that none of the inequalities (5) is violated for intervals  $A_1, A_2$ . Since  $P$  satisfies the martingale condition, it suffices to check the left inequality<sup>7</sup>

$$P(A_1 \times \overline{A_2}) \geq \int_{A_1} x dP_1(x) - \int_{A_2} x dP_2(x). \quad (22)$$

Since  $P$  has finite support, different choices of  $A_1, A_2$  yield the same inequality if each set contains the same points of the support of  $P_1$  and  $P_2$ , respectively. Thus, we need to check that (22) is satisfied if  $A_1$  is a subset of  $\{\frac{1-\varepsilon}{4}, \frac{1}{2} - \frac{\varepsilon}{4}, \frac{3}{4}\}$  and  $A_2$ , a subset of  $\{0, \frac{1}{2}, 1\}$ , *except* the cases of  $A_1 = \{\frac{1-\varepsilon}{4}, \frac{3}{4}\}$  and  $A_2 = \{0, 1\}$ , which exclude the middle points and do not correspond to any interval.

We consider the following cases:

- Inequality (22) holds if one of the sets  $A_i$  is empty or contains the whole support of  $P_i$ , as it then boils down to the martingale condition, which we already verified.
- Consider the case when  $A_1 \times \overline{A_2}$  contains exactly one heavy point; by the interval constraint, if it contains two, then  $A_1$  contains the support of  $P_1$ , which is the case we already considered. In this case,  $P(A_1 \times \overline{A_2}) \geq \frac{1-\varepsilon}{2}$ . On the other hand, the integral  $\int_{A_1} x dP_1(x)$  does not exceed  $\frac{1}{2} - \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{8}$  if  $A_1$  excludes the rightmost heavy point and it does not exceed  $\frac{1}{2} - \frac{1-\varepsilon}{4} \cdot \frac{1-\varepsilon}{2}$ . We see that for  $\varepsilon$  small enough (e.g.,  $\varepsilon = \frac{1}{10}$ ), condition (22) is satisfied regardless of the choice of  $A_2$ , since  $\frac{1-\varepsilon}{2} \geq \max\{\frac{1}{8}, \frac{1}{2} - \frac{1-\varepsilon}{4} \cdot \frac{1-\varepsilon}{2}\}$ .

<sup>7</sup>Using the martingale condition  $\int_{A_1} x dP_1(x) + \int_{\overline{A_1}} x dP_1(x) = p$  the right inequality in (5) follows from the left by a simple calculation.

- Consider the remaining case, in which there are no heavy points in  $A_1 \times \overline{A_2}$ , and both  $A_1$  and  $A_2$  are nonempty strict subsets of the supports. This can only be possible if  $A_2$  contains  $\frac{1}{2}$  or  $A_1 = \{\frac{1}{2} - \frac{\varepsilon}{4}\}$ . In the former case,  $\int_{A_2} x dP_2(x) \geq \frac{1}{2} - \frac{\varepsilon}{4}$ , which for small  $\varepsilon$  ( $\frac{1}{10}$  suffices) exceeds  $\int_{A_1} x dP_1(x)$  for any  $A_1$  excluding at least one of the heavy points (see the bounds above). Hence, the right-hand side of (22) is negative and the inequality is satisfied. Consider the remaining case of  $A_1 = \{\frac{1}{2} - \frac{\varepsilon}{4}\}$  and  $A_2$  either  $\{0\}$  or  $\{1\}$ . The left-hand side of (22) equals  $\frac{\varepsilon}{4}$  and  $\int_{A_1} x dP_1(x) = (\frac{1}{2} - \frac{\varepsilon}{4}) \cdot \frac{\varepsilon}{2} \leq \frac{\varepsilon}{4}$ . Thus (22) is satisfied on all intervals.

This example also demonstrates that the condition of Ziegler (2020) (Theorem 1) is not sufficient for feasibility. Using our notation his condition can be written as follows:

$$\begin{aligned} & \max \left\{ \int_0^a x dP_1(x) + \int_0^b x dP_2(x) - p, \quad \int_0^a (1-x) dP_1(x) + \int_0^b (1-x) dP_2(x) - (1-p) \right\} \leq \\ & \leq P([0, a] \times [0, b]) \leq \\ & \leq \min \left\{ \int_0^a (1-x) dP_1(x) + \int_0^b x dP_2(x), \quad \int_0^a x dP_1(x) + \int_0^b (1-x) dP_2(x) \right\}. \end{aligned}$$

for every  $a, b \in [0, 1]$

Simple computations show that for every given  $a, b \in [0, 1]$  this condition is *equivalent* to four of our conditions (see Equation 5) for all possible combinations of  $A_1 = [0, a]$  or  $A_1 = [a, 1]$  and  $A_2 = [0, b]$  or  $A_2 = [b, 1]$ . Therefore, the condition in Ziegler (2020) is essentially the same condition as ours (Equation 5) but restricted to interval subsets  $A_1, A_2 \subset [0, 1]$ , where each interval has a boundary point either at 0 or at 1. If the marginals  $P_1$  and  $P_2$  are supported on at most two points, such intervals exhaust all possible non-trivial sets in Equation 5; in this case, Ziegler's condition is necessary and sufficient for feasibility. However, the example above demonstrates that this condition becomes insufficient if the support contains at least 3 points.

## C Singularity of extreme measures with given marginals

In this section we prove an extension of a celebrated result of Lindenstrauss (1965) regarding extreme points of the set of measures with given marginals. The extension is two-fold: the classic result assumes  $n = 2$  and uniform marginals and we get rid of both assumptions. The proof is inspired by the original ideas of Lindenstrauss (1965).

**Theorem 5** (Extension of Theorem 2 from (Lindenstrauss, 1965)). *Any extreme point  $P$  of the set of probability measures on  $[0, 1]^N$  with given one-dimensional marginals  $\nu_i \in \Delta([0, 1])$ ,  $i \in N$ , is singular with respect to the Lebesgue measure  $\lambda$  on the unit cube. In other words, for each extreme  $P$  there exists a measurable set  $B$  such that  $P(B) = 1$  and  $\lambda(B) = 0$ .*

*Proof.* Assume the converse:  $P$  is not singular. By the Lebesgue decomposition theorem the continuous part of  $P$  can be singled out:  $P = \mu + \lambda^\perp$ , where  $\mu \neq 0$  is absolutely continuous with respect to  $\lambda$  ( $d\mu = f d\lambda$  with a non-negative integrable density  $f$ ),  $\lambda^\perp$  is singular with respect to  $\lambda$ .

Consider the Lebesgue space  $\mathcal{L}^1(\mu)$  of integrable functions with respect to  $\mu$  (defined  $\mu$ -everywhere) and its closed subspace  $\mathcal{S}$  generated by “separable”<sup>8</sup> functions  $g(x_1, \dots, x_n) = \sum_{i \in N} g_i(x_i)$ , where  $g_i \in \mathcal{L}^1(\mu)$ ,  $i \in N$ , depends on the variable  $x_i$  only.

Let’s assume that  $\mathcal{S} \neq \mathcal{L}^1(\mu)$ , i.e., separable integrable functions are not dense in all integrable (we check this condition at the end of the proof). Now we show that under this assumption,  $\mu$  can be represented as a convex combination of  $\frac{1}{2}\mu' + \frac{1}{2}\mu''$ , where  $\mu'$  and  $\mu''$  are distinct but have the same marginals. By the Hahn-Banach theorem there exists a continuous functional  $\theta$  of norm 1 such that  $\theta$  is identically zero on  $\mathcal{S}$ . Since the dual space to  $\mathcal{L}^1(\mu)$  is the space  $\mathcal{L}^\infty(\mu)$  of essentially-bounded functions, the functional  $\theta$  can be identified with a non-zero function  $\theta(x_1, \dots, x_n)$  bounded by 1 in absolute value. The condition of vanishing on  $\mathcal{S}$  reads as

$$\int \sum_{i \in N} g_i(x_i) \cdot \theta(x_1, \dots, x_n) d\mu = 0, \quad \forall g_i = g_i(x_i) \in \mathcal{L}^1(\mu), \quad i \in N. \quad (23)$$

We see that the measure  $\theta d\mu$  is non-zero but has zero marginals! Define  $\mu'$  and  $\mu''$  as  $d\mu' = (1 - \theta) d\mu$  and  $d\mu'' = (1 + \theta) d\mu$ . By the construction,  $\mu$  is the average of  $\mu'$  and  $\mu''$ ,  $\mu'$  and  $\mu''$  are distinct, and all measures  $\mu$ ,  $\mu'$ , and  $\mu''$  have the same marginals. Thus  $P$  is also represented as the average of  $\mu' + \lambda^\perp$  and  $\mu'' + \lambda^\perp$ , i.e.,  $P$  is not an extreme point. Contradiction. This contradiction completes the proof (under the assumption that  $\mathcal{S} \neq \mathcal{L}^1(\mu)$ ).

Now we check that  $\mathcal{S} \neq \mathcal{L}^1(\mu)$ . Our goal is to construct a function  $h \in \mathcal{L}^1(\mu)$  such that for any  $g(x_1, \dots, x_n) = \sum_{i \in N} g_i(x_i) \in \mathcal{L}^1(\mu)$  the  $\mathcal{L}^1$ -distance  $\int |h - g| d\mu$  is bounded below by some positive constant independent of  $g$ .

Recall that  $d\mu = f d\lambda$ . Fix  $\delta > 0$  such that the set  $A_\delta = \{x \in [0, 1]^N : f(x) \geq \delta\}$  has non-zero Lebesgue measure. Fix another small constant  $\varepsilon > 0$ ; by the Lebesgue density theorem applied to  $A_\delta$ , there exist a point  $x^0 \in (0, 1)^N$  and a number  $a > 0$  such that for the cube  $C = \prod_{i \in N} [x_i^0 - a, x_i^0 + a] \subset [0, 1]^N$  of size  $2a$  centered at  $x^0$  the following inequality holds  $\frac{\lambda(C \cap A_\delta)}{\lambda(C)} \leq \varepsilon$ .

Define  $h(x) = 1$  if  $x_i \geq x_i^0$  for all  $i \in N$ , and  $h(x) = 0$ , otherwise.

Cut the cube  $C$  into  $2^n$  small cubes indexed by subsets of  $N$ : for  $M \subset N$  the cube  $C_M$  is given by  $\prod_{i \in N \setminus M} [x_i^0 - a, x_i^0] \times \prod_{i \in M} [x_i^0, x_i^0 + a)$ . No function  $g(x) = \sum_{i \in N} g_i(x_i)$  can approximate  $h$  well on all the small cubes at the same time. Intuition is the following: since  $h$  is zero in all the cubes except  $C_N$ , then  $g$  must be close to zero on these cubes; however, values on these cubes

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<sup>8</sup>We refer here to separability in economic sense.

determine values of  $g$  on  $C_N$  by

$$g(x) = \frac{1}{n-1} \left( \left( \sum_{i \in N} g(x_1, \dots, x_{i-1}, x_i - a, x_{i+1}, \dots, x_n) \right) - g(x_1 - a, x_2 - a, \dots, x_n - a) \right), \quad (24)$$

therefore  $g$  is close to zero on  $C_N$  and cannot approximate  $h$  well.

To formalize this intuition, we assume that  $\int_{C_M} |h - g| d\mu$  is less than some  $\alpha \cdot \lambda(C_M)$  for any  $M \subset N$  and show that this constant  $\alpha$  cannot be too small. For  $M \neq N$  we get  $\int_{C_M} |g| \cdot f d\lambda \leq \alpha \lambda(C_M)$ . Applying the Markov inequality on the set  $C_M \cap A_\delta$  and taking into account that this set is big enough (by the construction of the original cube,  $\frac{\lambda(C_M \setminus A_\delta)}{\lambda(C_M)} \leq 2^n \varepsilon$ ), we obtain existence of a set  $B_M \subset C_M$  such that  $|g(x)| \leq \frac{\sqrt{\alpha}}{\delta}$  on  $B_M$  and  $\frac{\lambda(C_M \setminus B_M)}{\lambda(C_M)} \leq 2^n \varepsilon + \sqrt{\alpha}$ .

Consider a subset  $B^*$  of  $C_N$  such that, whenever  $x \in B^*$ , all the arguments of the right-hand side in (24) belong to respective subsets  $B_M$ , i.e.,  $B^* = \bigcap_{i \in N} (B_N \setminus \{i\} + a \cdot e_i) \cap (B_\emptyset + a \cdot \sum_{i \in N} e_i)$ , where sets  $B_M$  are translated by the elements of the standard basis  $(e_i)_{i \in N}$ . The union bound implies that the set  $B^*$  is dense enough in  $C_N$ :  $\frac{\lambda(C_N \setminus B^*)}{\lambda(C_N)} \leq (n+1)(2^n \varepsilon + \sqrt{\alpha})$ . By formula (24), the absolute value of  $g$  is bounded by  $\frac{n}{n-1} \frac{\sqrt{\alpha}}{\delta}$  on  $B^*$ . We get the following chain of inequalities:

$$\begin{aligned} \alpha &\geq \frac{1}{\lambda(C_N)} \int_{C_N} |h - g| d\mu \\ &\geq \frac{1}{\lambda(C_N)} \int_{B^* \cap A_\delta} |h - g| \cdot f d\lambda \\ &\geq \left( 1 - \frac{n}{n-1} \frac{\sqrt{\alpha}}{\delta} \right) \cdot \delta \cdot \frac{\lambda(B^* \cap A_\delta)}{\lambda(C_N)} \\ &\geq \left( \delta - \frac{n}{n-1} \sqrt{\alpha} \right) (1 - (n+1)(2^n \varepsilon + \sqrt{\alpha}) - 2^n \varepsilon). \end{aligned} \quad (25)$$

Denote by  $\alpha^* = \alpha^*(\delta, n, \varepsilon)$  the minimal value of  $\alpha \geq 0$  satisfying the inequality created by the head and the tail of (25). The constant  $\varepsilon$  is a free parameter in the construction. Selecting it to be small enough (namely  $(n+2)2^n \varepsilon < 1$ ), we ensure that  $\alpha^* > 0$ .

For any  $g = \sum_{i \in N} g_i(x_i)$

$$\int_{[0,1]^N} |h - g| d\mu \geq \max_{M \subset N} \int_{C_M} |h - g| d\mu \geq \alpha^*(\delta, n, \varepsilon) \cdot 2^{-n} \lambda(C) > 0.$$

Since the constant on the right-hand side is independent of  $g$  and positive, we see that  $h$  cannot be approximated by separable functions. Thus  $h$  does not belong to  $\mathcal{S}$  and  $\mathcal{S} \neq \mathcal{L}^1$ .  $\square$

## D A characterization of feasible distributions for finitely many agents

In this section we prove Theorem 3. The proof of necessity is straightforward and explained before the theorem statement.

The proof of sufficiency uses Théorème 2.6 of Hansel and Trollic (1986), which is a generalization of Kellerer's Theorem. Recall that a *paving* of a set  $X$  is a set of subsets of  $X$  that includes the empty set, and an *algebra* is a paving that is closed under unions and complements. By a *collection* we mean a multiset, i.e., the same element can enter the collection several times. Denote  $\bar{\mathbb{R}}_+ = [0, \infty]$ . A finitely additive measure is a map from an algebra to  $\bar{\mathbb{R}}_+$  that is additive for disjoint sets.

**Theorem C** (Hansel and Trollic). *Let  $X$  be a set, and  $\mathcal{F}$  an algebra of subsets of  $X$ . Let  $\mathcal{F}_1, \dots, \mathcal{F}_n, \mathcal{G}_1, \dots, \mathcal{G}_m$  be subpavings of  $\mathcal{F}$ . For  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, m\}$  let  $\alpha_i: \mathcal{F}_i \rightarrow \bar{\mathbb{R}}_+$ , and  $\beta_j: \mathcal{G}_j \rightarrow \bar{\mathbb{R}}_+$  be maps that vanish on the empty set. Then the following are equivalent:*

1. *There is a finitely additive measure  $Q: \mathcal{F} \rightarrow \bar{\mathbb{R}}_+$  such that, for every  $i \in \{1, \dots, n\}$  and every  $A \in \mathcal{F}_i$  it holds that  $\alpha_i(A) \leq Q(A)$ , and for every  $j \in \{1, \dots, m\}$  and every  $B \in \mathcal{G}_j$  it holds that  $Q(B) \leq \beta_j(B)$ .*
2. *For every finite collections of sets  $\mathcal{A}_1 \subseteq \mathcal{F}_1, \dots, \mathcal{A}_n \subseteq \mathcal{F}_n, \mathcal{B}_1 \subseteq \mathcal{G}_1, \dots, \mathcal{B}_m \subseteq \mathcal{G}_m$ , if*

$$\sum_{i=1}^n \sum_{A \in \mathcal{A}_i} \mathbb{1}_A \leq \sum_{j=1}^m \sum_{B \in \mathcal{B}_j} \mathbb{1}_B$$

then

$$\sum_{i=1}^n \sum_{A \in \mathcal{A}_i} \alpha_i(A) \leq \sum_{j=1}^m \sum_{B \in \mathcal{B}_j} \beta_j(B).$$

We will need the following corollary of this result.

**Corollary C.** *Fix  $P \in \Delta([0, 1]^n)$ ,  $0 < p \leq 1$ , and  $M_1, \dots, M_n \in \Delta([0, 1])$ . Then the following are equivalent:*

1. *There exists a probability measure  $Q \in \Delta([0, 1]^n)$  that is dominated by  $\frac{1}{p}P$ , and which has marginals  $Q_i = M_i$ .*
2. *For every  $\mathcal{A}_1, \dots, \mathcal{A}_n$ , finite collections of Borel subsets of  $[0, 1]$ , every  $\mathcal{C}$  a finite collection of Borel subsets of  $[0, 1]^n$ , and for every non-negative integer  $K$ , if*

$$\sum_{i=1}^n \sum_{A \in \mathcal{A}_i} \mathbb{1}_A(x_i) \leq K + \sum_{C \in \mathcal{C}} \mathbb{1}_C(x_1, \dots, x_n) \tag{26}$$

for all  $(x_1, \dots, x_n)$ , then

$$\sum_{i=1}^n \sum_{A \in \mathcal{A}_i} M_i(A) \leq K + \sum_{C \in \mathcal{C}} \frac{1}{P} P(C). \quad (27)$$

*Proof.* The proof that 1. implies 2. is simple and omitted.<sup>9</sup>

Let  $\mathcal{F}$  be the Borel sigma-algebra of  $[0, 1]^n$ . For each  $i \in \{1, \dots, n\}$  let  $\mathcal{F}_i$  be the sub-sigma-algebra of sets that are measurable in the  $i^{\text{th}}$  coordinate. That is,  $\mathcal{F}_i$  consists of sets of the form  $[0, 1]^{i-1} \times A \times [0, 1]^{n-i}$ , where  $A$  is any Borel subset of  $[0, 1]$ . Denote by  $\pi_i: [0, 1]^n \rightarrow [0, 1]$  the projection on the  $i^{\text{th}}$  coordinate.

Let  $m = 2$ . Define  $\mathcal{G}_1$  as the trivial algebra  $\{\emptyset, [0, 1]^n\}$  and  $\mathcal{G}_2 = \mathcal{F}$ . Let  $\alpha_i = M_i$ ,  $\beta_1([0, 1]^n) = 1$ ,  $\beta_1(\emptyset) = 0$ , and  $\beta_2 = \frac{1}{P}P$ . Let  $\mathcal{A}_1, \dots, \mathcal{A}_n$  and  $\mathcal{C}$  satisfy (26). By abuse of notation we identify each  $A \in \mathcal{A}_i$  with its preimage  $\pi_i^{-1}(A) = [0, 1]^{i-1} \times A \times [0, 1]^{n-i}$ . Thus  $\mathcal{A}_i \subseteq \mathcal{F}_i$ . We define  $\mathcal{B}_1$  and  $\mathcal{B}_2$  as follows. The collection  $\mathcal{B}_1 \subseteq \mathcal{G}_1$  contains  $K$  copies of  $[0, 1]^n$  and  $\mathcal{B}_m = \mathcal{C} \subseteq \mathcal{G}_2$ .

We can apply Theorem C directly to conclude that there is a finitely additive measure  $Q$  that is dominated by  $\frac{1}{P}P$ , has  $Q([0, 1]^n) \leq 1$ , and whose marginals  $Q_i$  dominate  $M_i$  for  $i \in N$ . Each  $M_i$  is a probability measure and the total mass of  $Q$  is at most 1; hence,  $Q_i$  can dominate  $M_i$  only if  $Q_i = M_i$ . We conclude that  $Q$  is a probability measure with marginals  $M_i$  for each  $i$ . Since  $Q$  is dominated by a sigma-additive measure, it is also itself sigma-additive (see Lemma 5 below).  $\square$

**Lemma 5.** *Let  $\mathcal{F}$  be a sigma-algebra of subsets of  $X$ . Let  $\mu: \mathcal{F} \rightarrow \bar{\mathbb{R}}_+$  be a finitely additive measure, and let  $\nu: \mathcal{F} \rightarrow \bar{\mathbb{R}}_+$  be a sigma-additive measure. If  $\mu \leq \nu$  and  $\nu(X) < \infty$  then  $\mu$  is sigma-additive.*

*Proof.* Let  $A_1, A_2, \dots \in \mathcal{F}$  be pairwise disjoint. Denote  $A = \cup_i A_i$ . Then, by additivity, we have that

$$\mu(A) = \mu(\cup_{i=1}^n A_i) + \mu(\cup_{i=n+1}^{\infty} A_i) \geq \mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$$

Hence  $\mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)$ . For the other direction,

$$\mu(A) = \mu(\cup_{i=1}^n A_i) + \mu(\cup_{i=n+1}^{\infty} A_i) \leq \sum_{i=1}^n \mu(A_i) + \nu(\cup_{i=n+1}^{\infty} A_i),$$

since  $\mu \leq \nu$ . By the sigma-additivity and finiteness of  $\nu$ , the last addend vanishes as  $n$  tends to infinity. Hence  $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ .  $\square$

We are now ready to finish the proof of Theorem 3. We will show that (7) suffices for feasibility by checking that it implies the condition of Lemma 2. Assume then that  $P$  satisfies (7), i.e. for

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<sup>9</sup>We will only make use of the other, non-trivial direction.

every trading scheme  $(a_1, \dots, a_i)$  it holds that

$$\int_{[0,1]^n} \left( \sum_{i=1}^n a_i(x_i)x_i - \max \left\{ 0, \sum_{i=1}^n a_i(x_i) \right\} \right) dP(x_1, \dots, x_n) \leq 0. \quad (28)$$

Choose  $i, j \in \{1, \dots, n\}$ , and consider the trading scheme in which  $a_i = 1$ ,  $a_j = -1$  and  $a_k = 0$  for all  $k \notin \{i, j\}$ . Then (28) implies that  $\int_0^1 x dP_i(x) = \int_0^1 x dP_j(x)$ . We accordingly set  $p = \int_0^1 x dP_i(x)$ , and note that this definition is independent of the choice of  $i$ . Define the measures  $M_i \in \Delta([0, 1]^n)$  by  $dM_i(x) = \frac{x}{p} dP_i(x)$ .

Let  $\mathcal{A}_1, \dots, \mathcal{A}_n$ ,  $K$ , and  $\mathcal{C}$  satisfy (26). We will show that (27) must hold. This will conclude the proof, since then it follows from Corollary C that there exists a measure  $Q$  that satisfies the conditions of Lemma 2, as condition (11) is simply  $Q_i = M_i$ .

To show that (27) holds, define the trading scheme  $(a_1, \dots, a_n)$  by

$$a_i(x) = c \cdot \left( \sum_{A \in \mathcal{A}_i} \mathbb{1}_A(x) - \frac{K}{n} \right),$$

where  $c$  is a normalization constant chosen small enough so that the image of all the  $a_i$ 's is in  $[-1, 1]$ . Then by (28)

$$\int_{[0,1]^n} \sum_{i=1}^n a_i(x_i)x_i dP(x_1, \dots, x_n) \leq \int_{[0,1]^n} \max \left\{ 0, \sum_{i=1}^n a_i(x_i) \right\} dP(x_1, \dots, x_n).$$

We substitute the definition of the trading scheme and the measures  $M_i$ . This yields

$$p \cdot \sum_{i=1}^n \sum_{A \in \mathcal{A}_i} M_i(A) - p \cdot K \leq \int_{[0,1]^n} \max \left\{ 0, \sum_{i=1}^n \sum_{A \in \mathcal{A}_i} \mathbb{1}_A(x_i) - K \right\} dP(x_1, \dots, x_n).$$

By (26), the integrand on the right-hand side is at most  $\sum_{C \in \mathcal{C}} \mathbb{1}_C(x_1, \dots, x_n)$ . Hence

$$p \cdot \sum_{i=1}^n \sum_{A \in \mathcal{A}_i} M_i(A) - p \cdot K \leq \sum_{C \in \mathcal{C}} P(C).$$

Dividing by  $p$  and rearranging yields

$$\sum_{i=1}^n \sum_{A \in \mathcal{A}_i} M_i(A) \leq K + \sum_{C \in \mathcal{C}} \frac{1}{p} P(C).$$

Thus condition (27) holds, and by Lemma 2 the distribution  $P$  is  $p$ -feasible. This completes the proof of Theorem 3.