A Toy Model of Boundary States with Spurious Topological Entanglement Entropy

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Topological entanglement entropy has been extensively used as an indicator of topologically ordered phases. However, it has been observed that there exist ground states in the topologically trivial phase that has nonzero "spurious" contribution to the topological entanglement entropy. In this work, we study conditions for two-dimensional topologically trivial states to exhibit such a spurious contribution. We introduce a tensor network model of the degrees of freedom along the boundary of a subregion. We then characterize the spurious contribution in the model using the theory of operator-algebra quantum error correction. We show that if the state at the boundary is a stabilizer state, then it has non-zero spurious contribution if, and only if, the state is in a non-trivial one-dimensional $G_1 \times G_2$ symmetry-protected topological (SPT) phase. However, we provide a candidate of a boundary state that has a non-zero spurious contribution but does not belong to any of such SPT phases.

Introduction.– Topologically ordered phases are gapped quantum phases which have characteristic properties such as topology-dependent ground state degeneracy, anyonic excitations and robustness against local perturbations. In contrast to the conventional symmetry-breaking phases, topologically ordered phases cannot be detected by local order parameter, and therefore require a new type of indicators.

Topological entanglement entropy (TEE) [1, 2] has been widely used as an indicator of topological order. For ground states in gapped two-dimensional (2D) models, the entanglement entropy $S(A) := -\mathrm{Tr}\rho_A \log_2 \rho_A$ of a region A is expected to behave as

$$S(A) = \alpha|\partial A| - \gamma + o(1)\,, \quad (1)$$

where α is a constant, ∂A is the boundary length and $o(1)$ comprises terms vanishing in the limit of $|\partial A| \to \infty$. TEE is defined as the universal constant term γ [1]. The term γ is shown to be the logarithm of the total quantum dimension of the abstract anyon model under various conditions [1, 3–5]. By this formula, TEE only depends on the type of the phase and one can use its value as a diagnostic tool to detect topological orders.

To extract TEE from a ground state, one can simply extrapolate Eq. (1) for various region sizes, or more unambiguously, calculate suitable linear combinations of entropies for certain subsystems (e.g., Fig. 1a) so that the first leading terms cancel out [1, 2]. The linear combination for the region in Fig. 1a is known as conditional mutual information (CMI) in quantum information theory. Another linear combination called tri-information is also useful for a different type of partitions [1].

The extraction methods rely on an assumption that Eq. (1) holds for 2D gapped ground states. However, it has been pointed out that Eq. (1) in general could contain an additional term, and thus the avobe argument does not always work. This additional contribution, called spurious TEE [6, 7], causes positive CMI (or tri-information) for states in the trivial phase, even though the total quantum dimension is 1.

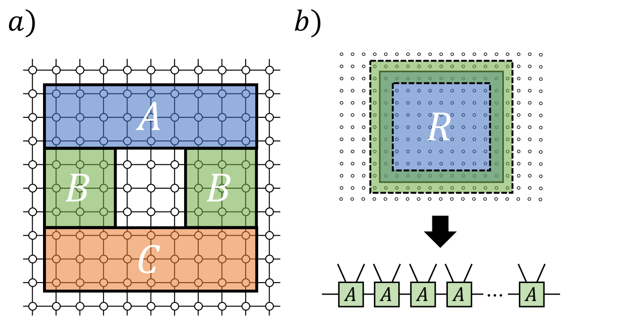

FIG. 1: *a*) A tripartition of a subsystem in a 2D spin lattice to calculate TEE. *b*) For ground states in the trivial phase, the entanglement entropy of region R is determined by a MPS located at the boundary of R (green region). Each tensor A has two physical legs associated to R and its complement respectively.

The existence of the spurious TEE was first pointed out by Bravyi [8]. It is the one-dimensional (1D) periodic cluster state embedded in 2D lattice, in a way that odd (even) sites are inside (outside) of the region we are considering. While the Bravyi's example is non-homogeneous, recently it has been shown that even translation-invariant system can obey spurious TEE [7].

In all known counter examples, the spurious TEE is connected to the existence of a 1D symmetry-protected topological (SPT) phase at the boundary of the region [6–11]. Spurious TEE seems to be fragile against general local perturbations or small deformation of the regions. However, the conditions in which the spurious TEE appears have not yet been fully understood.

A natural question is whether a SPT phase at the boundary is also a *necessary* condition for spurious TEE. In this paper, we study the underlying mechanism of spurious TEE in the trivial phase. We model the degrees of freedom at the boundaries of regions (Fig. 1b) by using Matrix Product States (MPS) [12]. Here we particulary focus on a renormalization fixed-point of the MPS in which the CMI is constant for all the length scales. We then characterize the fixed-points in terms of the operator-algebra quantum error correction [13–15], and derive a formula to calculate the value of the spurious TEE from algebras associated to the single tensor.

By using our characterization, we show that if the boundary MPS is a stabilizer state [16], a non-zero spurious TEE implies the MPS is in a non-trivial $G_1 \times G_2$ SPT phase. In contrast, we also provide a numerical evidence indicating that in general there are boundary states which have non-zero spurious TEE but does not belong to any such SPT phase. To the best of our knowledge, this is the first example of the mechanism of spurious TEE beyond $G_1 \times G_2$ SPT phase at the boundary.

MPS model of boundary states.– We consider a translation-invariant ground state $|\psi\rangle$ defined on a 2D spin lattice with size N. When a ground state is in the trivial phase, it can be (approximately) constructed from a product state only by a constant-depth local unitary circuit [17, 18]. More precisely, there exists a set of unitaries $\{V_i\}$ such that

$$|\psi\rangle = V_w V_{w-1} \dots V_1 |0\rangle^{\otimes N} ,$$

where the depth $w = \mathcal{O}(1)$ is a constant of N and each V_i is a product of local unitaries acting on disjoint sets of neighboring spins within radius $r = \mathcal{O}(1)$.

Let us divide the lattice into a connected region R and its complement R^c. Entanglement between R and R^c is invariant under local unitaries $U_R U_{R^c}$ and therefore we can undo some parts of the circuit. Hence, $S(R)_\rho$ is equivalent to that of a tensor product of an entangled state $|\phi\rangle_{RR^c}$ around the boundary ∂R and $|0\rangle$s at the rest part. We call $|\phi\rangle_{RR^c}$ as *the boundary state* of R (Fig. 1b).

A constant-depth circuit can increase the Schmidt-rank by at most a constant. Therefore $|\phi\rangle_{RR^c}$ is written as a Matrix Product State (MPS) (Fig. 1b):

$$|\phi\rangle_{RR^c} = \sum \mathrm{Tr}(A^{i_1 j_1} \dots A^{i_l j_l}) |i_1 \dots i_l\rangle_R |j_1 \dots j_l\rangle_{R^c} ,$$

where $A^{i_k j_k}$ is a $D \times D$ matrix with a constant bond dimension $D = \mathcal{O}(1)$ (here, we assume that all the tensors are the same due to the translation-invariance [30]). Each local basis $\{|i_k\rangle\}$ corresponds to a coarse-grained site which consists several neighboring spins so that the correlation length of the MPS is exactly zero. We denote by $\mathcal{H}$ and $\mathcal{K}$ the Hilbert spaces associated to $|i\rangle_R$ and $|j\rangle_{R^c}$. In this notation, there is an isometry $V : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathcal{H} \otimes \mathcal{K}$, $V^\dagger V = I$ such that the MPS has the form [19, 20]

$$|\phi\rangle_{RR^c} = V^{\otimes l} |\lambda_D\rangle^{\otimes l} , \tag{2}$$

where $|\lambda_D\rangle = \sum_{k=1}^{D} \lambda_k |kk\rangle$ is an entangled state with the Schmidt rank D. V acts on two separated sites of neighboring $|\lambda\rangle$s. In the following, we especially consider the case where $|\lambda_D\rangle$ is the maximally entangled state $|\omega_D\rangle := \sum_{i=1}^{D} \frac{1}{\sqrt{D}} |ii\rangle$ for the simplicity. We expect we do not lose much generality by this reduction, although we leave an extension for future works.

When R is an annulus like ABC in Fig. (1)a, we obtain two boundary states at the inner and outer boundaries. The ground state has a spurious TEE for R if one of these boundary states have a non-trivial CMI

$$I(A : C|B)_\rho := S(AB) + S(BC) - S(B) - S(ABC) > 0$$

for a tripartition $R = ABC$ such that B separates A from C. Importantly, the value of CMI matches to that of the tri-information [21] for the class of states we are considering.

Due to the monotonicity of CMI, non-zero value of the spurious TEE implies that the CMI of an open boundary MPS must be positive as well. We formalize a family of such open boundary MPS $\{\phi^{(n)}\}_{n\ge 0}$ with different length n defined as

$$\phi^{(0)} := |\omega_D\rangle\langle\omega_D|_{A_1 A_2} ,$$
$$\phi^{(n)} := \mathcal{V}_{A_{2n} A_{2n+1} \to B_n E_n} \left(\phi^{(n-1)} \otimes |\omega_D\rangle\langle\omega_D|_{A_{2n+1} A_{2n+2}} \right) ,$$

where $\mathcal{V}_{A_{2n} A_{2n+1}}(X) = V X V^\dagger$ is the isometry map (Fig. 2). For the convenience we relabel A_{2n+2} by C_n so that each $\phi^{(n)}$ is a state on $A_1 \otimes (B_1 \otimes E_1) \otimes (B_2 \otimes E_2) \otimes \dots \otimes (B_n \otimes E_n) \otimes C_n$. A_1 and C_n represents the unfixed boundary condition.

After tracing out $R^c = E_1 ... E_n$, we have a family of mixed states $\{\rho^{(n)}\}_{n\ge 0}$ defined by

$$\rho^{(0)} := |\omega_D\rangle\langle\omega_D|_{A_1 A_2} ,$$
$$\rho^{(n)} := \mathcal{E}_{A_{2n} A_{2n+1} \to B_n} \left(\rho^{(n-1)} \otimes |\omega_D\rangle\langle\omega_D|_{A_{2n+1} A_{2n+2}} \right) ,$$

where $\mathcal{E} = \mathrm{Tr}_E \circ \mathcal{V}$ is a completely-positive and trace-preserving (CPTP) map. We denote CPTP-map $\mathcal{E}(\cdot \otimes |\omega_D\rangle\langle\omega_D|)$ by $\tilde{\mathcal{E}}$. We then have

$$\rho^{(n+1)} = \tilde{\mathcal{E}}_{C_n \to B_{n+1} C_{n+1}} (\rho^{(n)}) . \tag{3}$$

The whole family is obtained by iteratively applying $\tilde{\mathcal{E}}$:

$$\rho^{(n)} = \tilde{\mathcal{E}}_{C_n \to B_{n+1} C_{n+1}} \circ \dots \circ \tilde{\mathcal{E}}_{A_2 \to B_1 C_1} (\rho^{(0)}) \tag{4}$$

(recall that $C_0 = A_2$). We will simply denote the concatenated map in Eq. (4) by $\tilde{\mathcal{E}}^{(n)}$. When we trace out R instead of R^c, we obtain the complement chain which we will denote by $\{\sigma^{(n)}\}$. We also define $\mathcal{F} := \mathrm{Tr}_B \circ \mathcal{V}$ and $\tilde{\mathcal{F}}(\cdot) = \mathcal{E}^c(\cdot \otimes |\omega_D\rangle\langle\omega_D|)$.

$\{\phi^{(n)}\}$ has a spurious TEE if $I(A_1 : C_n | B_1 ... B_n)_{\rho^{(n)}}$ is bounded from below by a positive constant. Although $\rho^{(n)}$ has zero correlation length, we might still have a non-trivial length scale for the CMI [22]. We further remove

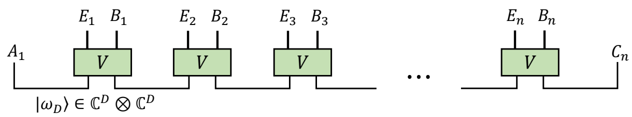

FIG. 2: A schematic picture of the family of states $\phi^{(n)}$. $|\omega_D\rangle = \sum_i \frac{1}{\sqrt{D}}|ii\rangle$ is the D-dimensional maximally entangled state and V is an isometry from $\mathbb{C}^D \otimes \mathbb{C}^D$ to $\mathcal{H} \otimes \mathcal{K}$.

such lengths scale by requiring the CMI saturates:

$$I(A_1 : C_1|B_1)_{\rho^{(1)}} = I(A_1 : C_n|B_1 \dots B_n)_{\rho^{(n)}} \,, \forall n. \quad (5)$$

Note that the LHS is always larger for any CPTP-map $\mathcal{E}$. In the rest of the paper we will simply denote $I(A_1 : C_n|B_1B_2 \dots B_n)_{\rho^{(n)}}$ by $I(A_1 : C_n|B_1B_2 \dots B_n)_{(n)}$.

While the definition (5) depends on n, it is equivalent to two independent conditions independent of n.

Proposition 1. *Eq.* (5) *is equivalent to*

$$I(A_1 : B_1C_1)_{(1)} = I(A_1 : B_1B_2C_2)_{(2)} \,, \quad (6)$$
$$I(A_1 : B_1)_{(1)} = I(A_1 : B_1B_2)_{(2)} \,. \quad (7)$$

Moreover, Eq. (7) *is equivalent to*

$$I(A_1 : E_1C_1)_{(1)} = I(A_1 : E_1E_2C_2)_{(2)} \,. \quad (8)$$

Therefore it is sufficient to consider up to $n = 2$.

Characterization by operator-algebra QEC.– We use the theory of operator-algebra quantum error correction (OAQEC) [13–15] to characterize $\{\phi^{(n)}\}$. OAQEC is a general framework of quantum error correction including standard quantum error correction codes [23] and subsystem codes [24]. It allows us to describe what kind of observables on a code space are correctable against a given error. For a given CPTP-map $\mathcal{E} : \mathcal{H} \to \mathcal{K}$ representing a "noise", one can always specify the *correctable algebra* $\mathcal{A}_{\mathcal{E}} \subset \mathcal{B}(\mathcal{H})$ that is a C^*-algebra containing all observables whose information is preserved under $\mathcal{E}$ (see Appendix A for more details).

In the following analysis the correctable algebras of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ play a crucial role. We first show that the saturation of the conditional mutual information (5) implies the saturation of these correctable algebras.

Proposition 2. *If Eq.* (5) *holds for* $\mathcal{E}$*, then*

$$\mathcal{A}_{\tilde{\mathcal{E}}} = \mathcal{A}_{\tilde{\mathcal{E}}^{(n)}}, \quad (9)$$
$$\mathcal{A}_{\tilde{\mathcal{F}}} = \mathcal{A}_{\tilde{\mathcal{F}}^{(n)}} \,, \quad (10)$$
$$\mathcal{A}_{\mathrm{Tr}_C \circ \tilde{\mathcal{E}}} = \mathcal{A}_{\mathrm{Tr}_C \circ \tilde{\mathcal{E}}^{(n)}}, \quad (11)$$
$$\mathcal{A}_{\mathrm{Tr}_C \circ \tilde{\mathcal{F}}} = \mathcal{A}_{\mathrm{Tr}_C \circ \tilde{\mathcal{F}}^{(n)}} \,, \forall n \,. \quad (12)$$

This proposition means that the algebra $\mathcal{A}_{\tilde{\mathcal{E}}}$ represents the information of the input which is faithfully encoded in the output on $B_1 ... B_n C_n$ for all n. In the same way, $\mathcal{A}_{\mathrm{Tr}_C \circ \tilde{\mathcal{E}}}$ represents the perfectly recoverable information encoded on $B_1 ... B_n$.

In general, there are operators carrying "unpreserved" information which are disturbed and cannot be recovered perfectly. Such operators also could contribute to CMI, but it may cause decrease of CMI with respect to n. Prop. 2 does not prevent to have such unpreserved operators and therefore the conditions (9)-(12) are not sufficient for Eq. (5). In fact, we can always assume these conditions by coarse-graining a finite number of channels [31].

Unpreserved information could be split into the local part and the non local part. The local part is outputted in B_1 and does not affect to CMI. The non-local part is outputted in B_1C_1, and therefore it is further disturbed by applying $\tilde{\mathcal{E}}$ on C_1, which induces the decay of CMI. We would like to disregard this unpreserved information, and thus we utilize the concept of complementary recovery property [25] to neglect such information.

Definition 3. *We say a CPTP-map* $\mathcal{E}$ *satisfy complementary recovery property if*

$$\mathcal{A}_{\mathcal{E}^c} = \mathcal{A}'_{\mathcal{E}} \,, \quad (13)$$

where $\mathcal{A}'_{\mathcal{E}}$ *is the commutant of* $\mathcal{A}_{\mathcal{E}}$*.*

Any CPTP-map satisfies $\mathcal{A}_{\mathcal{E}^c} \subset \mathcal{A}'_{\mathcal{E}}$, i.e. any operator recoverable from the output of the complementary channel $\mathcal{E}^c$ should commute with the correctable algebra of the original channel $\mathcal{E}$ (see also Appendix A). The complementary recovery property says the converse of this statement is also true. This property can be characterized by a projection map onto the correctable algebra.

Proposition 4. $\mathcal{E}$ *satisfies the complementary recovery property if, and only if,*

$$\mathcal{E}\left(\mathcal{P}_{\mathcal{A}_{\mathcal{E}}}(\rho)\right) = \mathcal{E}(\rho) \,, \forall \rho \quad (14)$$

or

$$\mathcal{P}_{\mathcal{A}_{\mathcal{E}}} \circ \mathcal{E}^{\dagger}(O) = \mathcal{E}^{\dagger}(O) \,, \forall O \,, \quad (15)$$

where $\mathcal{P}_{\mathcal{A}_{\mathcal{E}}} : \mathcal{B}(\mathcal{H}) \to \mathcal{A}_{\mathcal{E}}$ *is the idempotent projecting map onto* $\mathcal{A}_{\mathcal{E}}$*.*

Therefore the complementary recovery property restricts the support of the input to the correctable algebra.

Definition 5. *We say isometry* V *or CPTP-map* $\mathcal{E}$ *satisfies dual complementarity if* $\tilde{\mathcal{E}}$ *and* $\tilde{\mathcal{F}}$ *both satisfy the complementary recovery property.*

Dual complementarity reduces the four algebras in Prop. 2 to two algebras $\mathcal{A} := \mathcal{A}_{\tilde{\mathcal{E}}} = (\mathcal{A}_{Tr_C \circ \tilde{\mathcal{F}}})'$ and $\mathcal{B} := \mathcal{A}_{\tilde{\mathcal{F}}} = (\mathcal{A}_{Tr_C \circ \tilde{\mathcal{E}}})'$. In the following we only consider states satisfying this property. We show that the dual complementarity implies the saturation of the CMI. Furthermore, its value is determined by $\mathcal{A}$ and $\mathcal{B}$.

Theorem 6. *If V satisfies dual complementarity, then Eq.* (5) *holds. Let $\mathcal{A} = \bigoplus_k M_{n_k}(\mathbb{C}) \otimes I_{n'_k}$ and $\mathcal{B} = \bigoplus_l M_{m_l}(\mathbb{C}) \otimes I_{m'_l}$. Then, the value of the CMI is given by*

$$I(A_1 : C_1|B_1)_{(1)} = \sum_k p_k \log \frac{n_k}{n'_k} + \sum_l q_l \log \frac{m_l}{m'_l}\,, \quad (16)$$

where $p_k = \frac{n_k n'_k}{D}$ and $q_l = \frac{m_l m'_l}{D}$. Therefore, $I(A_1 : C_1|B_1)_{(1)} > 0$ if and only if

$$\mathcal{B}' \subsetneq \mathcal{A}\,. \quad (17)$$

Eq. (17) intuitively means there is an operator perfectly encoded in BC whose information cannot be read out by just looking B. Such a non-local information causes the spurious contribution to CMI.

Note that dual complementarity is not a necessary condition for Eq. 5. For example, one can consider $\mathcal{E} = \mathcal{E}'_{A_2} \otimes \mathrm{id}_{A_3}$ such that $\mathcal{E}'$ is not the completely depolarizing channel but $\mathcal{A}_{\mathcal{E}'} = \mathbb{C}I$. The corresponding family satisfies $I(A : C|B) = 0$ for any length, but the map does not satisfy dual complementarity. This is because all the unpreserved information is transfered to B and not C, and thus it cancels out in $I(A : C|B) = I(A : BC) - I(A : B)$.

Relation to SPT phases.– For any $O \in \mathcal{A}$, we always find a corresponding *logical operator* $\tilde{O}_{BC}$ such that

$$\mathcal{V}_{A_2 \to BEC}(O_{A_2}|\psi\rangle_{A_2}) = (\tilde{O}_{BC} \otimes I_E)\mathcal{V}_{A_2 \to BEC}|\psi\rangle_{A_2}\,,$$

for any $|\psi\rangle \in \mathbb{C}^D$, where

$$\mathcal{V}_{A_2 \to BEC}|\psi\rangle_{A_2} := V^{\otimes n}\left(|\psi\rangle_{A_2} \otimes \bigotimes_i |\omega_D\rangle_{A_{2i+1}A_{2i+2}}\right).$$

The logical operator is not unique in general. The set of all logical operators $\mathcal{L}_{\mathcal{A}}$ of operators in $\mathcal{A}$ is given as the pre-image of $\tilde{\mathcal{E}}^{(n)\dagger}$:

$$\mathcal{L}_{\mathcal{A}} := \{O_{BC}\,|\tilde{\mathcal{E}}^{(n)\dagger}(O_{BC}) \in \mathcal{A}\}\,.$$

$\tilde{\mathcal{E}}^{(n)\dagger}$ is a normal $*$-homomorphism from the pre-image to $\mathcal{A}$ [15]. By the first isomorphism theorem for algebra, the image of the homomorphism is isomorphic to the pre-image upto the kernel.

$$\mathcal{L}_{\mathcal{A}}/Ker\tilde{\mathcal{E}}^{(n)\dagger} \cong \mathcal{A}\,.$$

We denote the equivalence class of the logical operators of $O \in \mathcal{A}$ by $\mathcal{L}(O)$.

Suppose the boundary state is in a non-trivial SPT phase under a symmetry of group $G_1 \times G_2$ acting on each tensor as $U(g_1, g_2) = U(g_1)_B \otimes U'(g_2)_E$. The action induces a projective representation $V(g) \otimes V(g)^\dagger$ on the virtual degrees of freedom [26] (see also Appendix B). For instance, it holds that

$$U(g)_{B_1E_1}|\phi^{(1)}\rangle_{A_1B_1E_1C_1} = V(g)^T_{A_1} \otimes V(g)^\dagger_{C_1}|\phi^{(1)}\rangle_{A_1B_1E_1C_1}$$

for $n = 1$. This correspondence reads that $V(g_1) \in \mathcal{A}$ and $V(g_2) \in \mathcal{B}$. $V(g)$ has a logical unitary operator

$$U(g) \otimes U(g) \otimes \cdots \otimes U(g) \otimes V(g) \in \mathcal{L}(V(g))$$

whose support is BC (EC) if $g = (g_1, e)$ ($g = (e, g_2)$). Suppose the state is in a non-trivial SPT phase in the sense that $[V(g_1), V(g_2)] \neq 0$ for some g_1, g_2 [6]. This implies $\mathcal{B}' \subsetneq \mathcal{A}$. Therefore, we can reconfirm that non-trivial $G_1 \times G_2$ SPT phase implies non-zero CMI (under dual complementarity).

The converse direction is entirely non-trivial. The existence of tensor product logical unitaries $U_B \otimes U_C$ is necessary for $\phi^{(n)}$ to being a state in such a SPT phase, but not sufficient as we will see in later. A particular class of V in which the converse also holds is the isometry consists of Clifford gates, i.e. when the MPS is a stabilizer states [16].

Theorem 7. *Let V be an isometry composed of Clifford gates and ancillas $|0\rangle^{\otimes k}$. Then,*

$$I(A_1 : C_n|B_n)_{(n)} > 0\,,\ \forall n \quad (18)$$

if and only if there exists finite groups G_1 and G_2 such that the MPS generated by V is in a non-trivial $G_1 \times G_2$ SPT phase.

The proof is given in Appendix C. We first show the same result under more general conditions, and then show that all stabilizer states satisfy these conditions.

Theorem 7 can be applied for all 2D stabilizer states including the 2D cluster state [7]. However, the conclusion are not necessarily true outside of stabilizer states. In fact, one can find a family of boundary states such that all the non-identity logical operators cannot be written as $U_B \otimes U_C$.

A non-trivial example.– Let V_U be an isometry that is the Stinespring dilation of $\mathcal{E}_U(\sigma) = \frac{1}{4}\sum_{i=0}^3 (P_i \otimes P_i U)\sigma(P_i \otimes U^\dagger P_i)$, where P_i ($i = 0, 1, 2, 3$) are the Pauli Matrices. $\sigma \mapsto V_U(\sigma \otimes \Phi_+)V_U^\dagger$ is an encoding map of the 5-qubit code [27, 28] up to a local unitary. The correctable algebras are $\mathcal{A} = \mathcal{B} = M_2(\mathbb{C})$, therefore boundary states automatically satisfy dual complementarity. The CMI attains the maximum value $I(A : C|B)_{(n)} = 2$.

For $n = 1$, each Pauli operator P_i has an unique logical operator $(P_i P_i)_B \otimes U^T(P_i)_C U^*$ [32]. If U is not a Clifford unitary nor diagonal in X or Z-basis, both $U^T X_C U^*$ and $U^T Z_C U^*$ are non-Pauli matrices. This induces non-tensor product logical operators on B_2C_2, which are also unlikely to be a tensor product for $n > 2$ (Fig. 3). By coarse-graining $\tilde{\mathcal{E}} \equiv \tilde{\mathcal{E}}^{(2)}$, we obtain a model with no logical operator form like $U_B \otimes U_C$ but with

$I(A:C|B)_{(n)} = 2$.

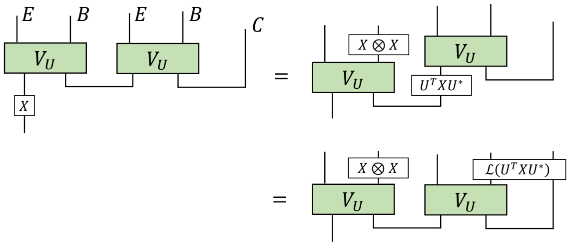

FIG. 3: A logical X operator in the example. This is a unique logical operator supported on BC. For general U, the logical operator is no longer a tensor product of unitaries on B and C for $n = 2$. We expect this to be hold for $n > 2$ as well.

To construct a 2D model, we can consider a 2D state which is a layer of tensor products of many copies of 1D MPS along the vertical and the horizontal direction (decoupled stacks) like a 2D weak subsystem SPT phase [29]. The resulting 2D translation-invariant state has a spurious TEE for an arbitrary large dumbbell-like region [7].

One may expect that for the periodic boundary condition the CMI could vanish in such a non-trivial example. Although we do not have any analytical result on that, we numerically sampled U from the Haar measure and then calculate the CMI for closed chains. Fig. 4 suggests that CMI remains to be a positive constant even for the closed boundary, while the value decreases from 2.

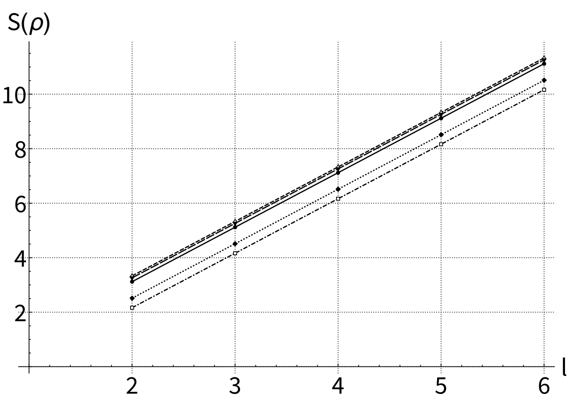

FIG. 4: The numerical result on the entropy $S(\rho_{ABC})$ of the example with closed boundary for 5 samples of U from the Haar measure. l is the length of the spin chain ($l = 6$ is 12-qubit). From the data, $S(\rho) = 2l - c_0$ with a constant $c_o > 0$ up to 10^{-7} error. Since any reduced state of the example is the completely mixed state, it shows that $I(A:C|B)_{(n)} = c_0 + O(10^{-7})$ for any tripartition ABC such that B separates A from C.

Future directions.– A crucial open question is how to classify the non-trivial example with spurious TEE discussed at the end. Although it should not be in a SPT phase under on-site $G_1 \times G_2$ symmetry, it could be in a SPT phase under other type of symmetry.

Generalization to more broad class of boundary MPS is desired. One possible extension is considering boundary states without dual complementarity. Dual complementarity neglect all information outside of $\mathcal{A}$, but in general one has some "noisy" information localized on B (or E). It may be possible to extend the correctable algebra by adding operators carrying such information. Another important direction is considering general injective MPS including (2). We expect that general injective MPS can be decomposed into protected and unprotected parts as in Ref. [11] such that the effect of the unprotected part vanishes exponentially as the conditioning system grows. We leave these problems for future works.

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[30] This assumption is slightly stronger than the translation-invariance, since generally tensors can depend on the direction of the edge and it can even contain some "corner" tensors. However, such corner contributions cancel out in the calculation of CMI.
[31] For any $\mathcal{E}$ we have $\mathcal{B}(\mathbb{C}^D) \supset \mathcal{A}_{\tilde{\mathcal{E}}} \supset \mathcal{A}_{\tilde{\mathcal{E}}^{(2)}} \supset \ldots \supset \mathbb{C}I$, and thus we cannot have an infinitely long sequence of strictly different C^*-algebras. Therefore there is $m \in \mathbb{N}$ such that the conditions hold by redefining $\tilde{\mathcal{E}} \equiv \tilde{\mathcal{E}}^{(m)}$ ($\mathcal{F} \equiv \tilde{\mathcal{F}}^{(m)}$).
[32] The uniqueness follows from the fact that there is no stabilizer supported on BC.

A. Operator Algebra Quantum Error Correction

In this appendix, we summarize the theory of operator-algebra quantum error correction (OAQEC) [1]. Formally, we say that a set $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ on a code subspace $\mathcal{H}_C = P_C\mathcal{H}$ ($P_C = I$ in the main text) is correctable for $\mathcal{E}$ if there exists a CPTP-map (also called a channel) $\mathcal{R}$ such that

$$P_C\left(\mathcal{R} \circ \mathcal{E}\right)^{\dagger}(X) P_C = P_C X P_C\,, \forall X \in \mathcal{S}\,. \tag{19}$$

We will call $\mathcal{R}$ *recovery map*. This is equivalent to say we require the information of not the whole code space but particular observables are preserved:

$$\mathrm{Tr}(X\rho) = \mathrm{Tr}\left(X(\mathcal{R} \circ \mathcal{E})(\rho)\right)\ \forall X \in S, \forall \rho \in \mathcal{S}(\mathcal{H}_C)\,. \tag{20}$$

This is in contrast to the standard (or subsystem) QEC in which we require that all information (= full density matrix, i.e. the expectation value for all observables) of the logical state is recoverable. A main result of Ref. [1] is the following:

Proposition 8. *[1] Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(P_C\mathcal{H})$. Then, $\mathcal{A}$ is correctable for $\mathcal{E}$ on $\mathcal{H}_C$ if, and only if,*

$$[P_C E_a^{\dagger} E_b P_C, X] = 0\ \forall X \in \mathcal{A}, \forall a, b \tag{21}$$

i.e. the algebra $Alg(\{P_C E_a^{\dagger} E_b P_C\}_{a,b})$ is a subalgebra of $\mathcal{A}'$, where $\mathcal{A}'$ is the commutant of $\mathcal{A}$.

In other words, any subalgebra of $Alg(\{P_C E_a^{\dagger} E_b P_C\}_{a,b})'$ is correctable. For a given code subspace $\mathcal{H}_C$ and an error $\mathcal{E}$, we define the *correctable algebra* of $\mathcal{E}$ by

$$\mathcal{A}_{\mathcal{E}} := Alg(\{P_C E_a^{\dagger} E_b P_C\}_{a,b})' \tag{22}$$

$$= \left\{X \in \mathcal{B}(\mathcal{H}_C) \mid [P_C E_a^{\dagger} E_b P_C, X] = 0\ \forall a, b\right\}\,. \tag{23}$$

This is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H}_C)$.

A complementary relation for OAQEC

In the standard QEC, we have a complementary relation implied by the no-cloning theorem: if one can exactly recover a logical state from a noise channel, then one cannot learn any information about the logical state from the output, i.e. the complimentary channel of the noise destroys all information about the logical state. In OAQEC, we can require to recover only partial information from noise. Hence, it might be possible to learn something from the outputs of the complementary channel.

Suppose $\{E_a\}$ are Kraus operators of $\mathcal{E}$. Then, we have a Steinespring isometry $V = \sum_a E_a \otimes |\phi_a\rangle$ for an orthonormal basis $\{|\phi_a\rangle\}$ on $\mathcal{H}_E$. The complementary channel $\mathcal{E}^c$ has a Kraus representation by $\{F_a\}$, defined as

$$F_a = \sum_b |\phi_b\rangle\langle a|E_b\,, \tag{24}$$

where $\{|a\rangle\}$ is an orthonormal basis of $\mathcal{H}_B$. The correctable algebra of $\mathcal{E}^c$ is spanned by operators commuting with all

$$P_C F_a^\dagger F_b P_C = P_C \mathcal{E}^\dagger(|a\rangle\langle b|) P_C \tag{25}$$

which span $P_C(Im\mathcal{E}^\dagger)P_C$. Hence, $\mathcal{A}_{\mathcal{E}^c} = Alg[(P_C(Im\mathcal{E}^\dagger)P_C)]'$.

Any $X \in \mathcal{A}_{\mathcal{E}}$ satisfies $X = P_C\mathcal{E}^\dagger \circ \mathcal{R}^\dagger(X)P_C$, and therefore $\mathcal{A}_{\mathcal{E}} \subset \mathcal{A}'_{\mathcal{E}^c}$: the correctable algebra of a channel is in the commutant of the correctable algebra of the complementary channel. Information of operators in $\mathcal{A}_{\mathcal{E}} \cap \mathcal{A}_{\mathcal{E}^c}$ can be extracted from the outputs of both channels and must be in the centers of $\mathcal{A}_{\mathcal{E}}$ and $\mathcal{A}_{\mathcal{E}^c}$. For example, if $\mathcal{A}_{\mathcal{E}} = \mathcal{B}(\mathcal{H}_C)$, then $\mathcal{A}_{\mathcal{E}^c} \subset \mathcal{B}(\mathcal{H}_C)' = \mathbb{C}I$ which corresponds to the complementary relation for the standard QEC.

B. MPS classification of $G_1 \times G_2$ SPT phases

Here we briefly summarize the MPS classification of 1D SPT phases under on-site $G = G_1 \times G_2$ symmetry [2, 3] to be self-contained. Consider a MPS defined as

$$|\psi_N\rangle = \sum_{\mathbf{i},\mathbf{j}} \mathrm{Tr}\left(A^{i_1 j_1} A^{i_2 j_2} \dots A^{i_N j_N}\right) |i_1 i_2 \dots i_N\rangle_R |j_1 j_2 \dots j_N\rangle_R\,, \tag{26}$$

where $i_k, j_k = 1, \dots, d$, $\forall k$ and A^{ij} is a $D \times D$ matrix. Consider a finite group $G_1 \times G_2$ with a unitary representation $(g_1, g_2) \mapsto U(g_1) \otimes U'(g_2)$ acting on $\mathbb{C}^d \otimes \mathbb{C}^d$. G_1 (G_2) then acts on R (R^c) as an on-site global symmetry $U(g_1)^{\otimes N}$ $\left(U'(g_2)^{\otimes N}\right)$. Suppose $|\psi_N\rangle$ is a symmetric state

$$\left(U(g_1) \otimes U'(g_2)\right)^{\otimes N} |\psi_N\rangle = |\psi_N\rangle\,, \quad \forall (g_1, g_2) \in G_1 \times G_2 \tag{27}$$

for all N (up to a global phase). This condition is shown to be equivalent to the following conditions for the single site:

$$\sum_k U(g_1)_{ik} A^{kj} = e^{i\theta(g_1)} V(g_1) A^{ij} V(g_1)^\dagger\,, \quad \sum_k U'(g_2)_{jk} A^{ik} = e^{i\theta(g_2)} V(g_2) A^{ij} V(g_2)^\dagger\,. \tag{28}$$

Here, $V(g_1)$ and $V(g_2)$ are unitary consisting a projective representation of $G_1 \times G_2$ in together.

1D SPT phase under symmetry G is classified by the second group cohomology $H^2(G; U(1))$ which is a group of the equivalence class $[\Theta]$ of the phase factor $\Theta : G \times G \to U(1)$. Each factor Θ is defined from the projective representation via

$$V(gh) = e^{i\Theta(g,h)} V(g) V(h) \tag{29}$$

up to the equivalence relation

$$\Theta(g,h) \sim \Theta(g,h) + \theta(gh) - \theta(g) - \theta(h) \mod 2\pi\,. \tag{30}$$

The symmetric state $|\psi_N\rangle$ is said to be in a non-trivial G SPT phase if the projective representation $V(g)$ is associated to a non-trivial element in the second cohomology class. In particular, in this paper, we employ the following definition used in Ref. [3]:

Definition 9. *We say symmetric MPS* $|\psi_N\rangle$ *is in a non-trivial* $G_1 \times G_2$ *SPT phase if there is* $(g_1, g_2) \in G_1 \times G_2$ *such that*

$$[V(g_1), V(g_2)] \neq 0\,. \tag{31}$$

This only happens for non-trivial projective representation since $[U(g_1), U'(g_2)] = 0$.

C. Proofs of the theorems

Proof of Proposition 1

proof. For any CPTP-map $\mathcal{E}$, the corresponding family $\{\rho^{(n)}\}$ satisfies

$$\begin{aligned} I(A_1 : C_1|B_1)_{(1)} &= I(A_1 : B_1C_1)_{(1)} - I(A_1 : B_1)_{(1)} \\ &\geq I(A_1 : B_1B_2C_2)_{(2)} - I(A_1 : B_1)_{(1)} \\ &= I(A_1 : B_1B_2C_2)_{(2)} - I(A_1 : B_1)_{(2)} \\ &= I(A_1 : C_2|B_1B_2)_{(2)} + I(A_1 : B_2|B_1)_{(2)} \\ &\geq I(A_1 : C_2|B_1B_2)_{(2)}\,. \end{aligned}$$

The first inequality follows from the data-processing inequality [4] and Eq. (3), and the second inequality follows from the strong subadditivity [5]. Equality holds when the both inequalities are saturated. The first inequality is saturated if, and only if, [4]:

$$I(A_1 : B_1)_{(1)} = I(A_1 : B_1B_2C_2)_{(2)}\,. \tag{32}$$

The second inequality saturated if and only if $I(A_1 : B_2|B_1)_{(2)} = 0$, i.e.

$$S^{(1)}_{A_1} + S^{(1)}_{B_1} - S^{(1)}_{A_1B_1} = S^{(2)}_{A_1} + S^{(2)}_{B_1B_2} - S^{(2)}_{A_1B_1B_2}\,,$$

which is equivalent to Eq. (7) by $\rho^{(1)}_{A_1B_1} = \rho^{(2)}_{A_1B_1}$. Since $\rho^{(1)}_{A_1B_1E_1C_1}$ and $\rho^{(2)}_{A_1B_1B_2E_1E_2C_2}$ are pure states, the above equality is equivalent to

$$-S^{(1)}_{A_1} + S^{(1)}_{A_1E_1C_1} - S^{(1)}_{E_1C_1} = -S^{(2)}_{A_1} + S^{(2)}_{A_1E_1E_2C_2} - S^{(2)}_{E_1E_2C_2}$$

after subtracting $2S^{(1)}_{A_1} = 2S^{(2)}_{A_1}$, which is Eq. (8). Therefore, $I(A_1 : C_1|B_1)_{(1)} = I(A_1 : C_2|B_1B_2)_{(2)}$ if and only if Eqs. (6) (8) holds. The same argument applies to $I(A_1 : C_1|E_1)_{(1)} = I(A_1 : C_2|E_1E_2)_{(2)}$.

We now show Eqs. (6) (8) imply

$$\begin{aligned} I(A_1 : B_1C_1)_{(1)} &= I(A_1 : B_1 \ldots B_nC_n)_{(n)}\,, \forall n\,, \\ I(A_1 : E_1C_1)_{(1)} &= I(A_1 : E_1 \ldots E_nC_n)_{(n)}\,, \forall n\,. \end{aligned}$$

The converse is clear. Since $\rho^{(2)} = \tilde{\mathcal{E}}_{C_1\to B_2C_2}(\rho^{(1)})$, Eq. (6) implies there exists a *recovery map* $\hat{\mathcal{R}}_{B_1B_2C_2\to C_1}$ such that

$$\begin{aligned} \hat{\mathcal{R}}_{B_1B_2C_2\to C_1}(\rho^{(2)}) &= \hat{\mathcal{R}}_{B_1B_2C_2\to C_1} \circ \tilde{\mathcal{E}}_{C_1\to B_2C_2}(\rho^{(1)}) &\quad (33) \\ &= \rho^{(1)}\,. &\quad (34) \end{aligned}$$

By definition, we can write $\rho^{(3)}$ as $\tilde{\mathcal{E}}^L_{A_3\to A_1B_1}(\rho^{(2)}_{A_3B_2B_3C_3})$, where $\tilde{\mathcal{E}}^L_{A_3\to A_1B_1}(\cdot) := \mathcal{E}_{A_2A_3\to B_1}(\omega_{A_1A_2} \otimes \cdot)$ is a CPTP-map (here we used the notation $\rho^{(2)}_{A_3B_2B_3C_3}$ for the state after shifting $\rho^{(2)}_{A_1B_1B_2C_2}$). From this expression, Eq. (34) implies

$$\begin{aligned}
\hat{\mathcal{R}}_{B_2B_3C_3\to C_2}(\rho^{(3)}) &= \hat{\mathcal{R}}_{B_2B_3C_3\to C_2}\circ\tilde{\mathcal{E}}_{C_2\to B_3C_3}(\rho^{(2)}) && (35)\\
&= \tilde{\mathcal{E}}^L\circ\hat{\mathcal{R}}_{B_2B_3C_3\to C_2}\circ\tilde{\mathcal{E}}_{C_2\to B_3C_3}(\rho^{(1)}) && (36)\\
&= \tilde{\mathcal{E}}^L(\rho^{(1)}) && (37)\\
&= \rho^{(2)}\,. && (38)
\end{aligned}$$

This is equivalent to

$$I(A_1:B_1B_2C_2)_{(2)} = I(A_1:B_1B_2B_3C_3)_{(3)}\,. \tag{39}$$

The same argument works for $n>3$ and under exchanging of $B\leftrightarrow E$. Therefore Eqs. (6) (8) are equivalent to Eq. (5), which completes the proof. □

Proof of Proposition 2

Proof. By definition, it is clear that $\mathcal{A}_{\tilde{\mathcal{E}}}\supset\mathcal{A}_{\tilde{\mathcal{E}}^{(n)}}$ (information which is recoverable after an additional noise is recoverable without the noise). Suppose that Eq. (5) holds. Then, as shown in the proof of Prop. 1, there exists a recovery map associated with the data-processing inequality such that

$$\hat{\mathcal{R}}_{B_1B_2C_2\to B_1C_1}\circ\tilde{\mathcal{E}}^{(2)}_{C_0\to B_1B_2C_2} = \tilde{\mathcal{E}}^{(1)}_{C_0\to B_1C_1}\,. \tag{40}$$

There also exists another recovery map $\mathcal{R}$ associated with OAQEC, such that

$$(\tilde{\mathcal{E}}^{(1)}_{C_0\to B_1C_1})^\dagger\circ\mathcal{R}^\dagger_{B_1C_1\to C_0}(X_{C_0}) = X_{C_0} \tag{41}$$

for any $X_{C_0}\in\mathcal{A}_{\tilde{\mathcal{E}}}$. By combining these two recovery maps together, we obtain that

$$\begin{aligned}
(\tilde{\mathcal{E}}^{(1)}_{C_0\to B_1C_1})^\dagger\circ\mathcal{R}^\dagger_{B_1C_1\to C_0}(X_{C_0}) &= (\tilde{\mathcal{E}}^{(2)}_{C_0\to B_1B_2C_2})^\dagger\circ\hat{\mathcal{R}}^\dagger_{B_1B_2C_2\to B_1C_1}\circ\mathcal{R}^\dagger_{B_1C_1\to C_0}(X_{C_0})\\
&= X_{C_0}\,.
\end{aligned} \tag{42}$$

Hence $\mathcal{R}\circ\hat{\mathcal{R}}$ can be regarded as the recovery map (in the sense of OAQEC) against the noise $\tilde{\mathcal{E}}^{(2)}$. Therefore, any $X_{C_0}\in\mathcal{A}_{\tilde{\mathcal{E}}}$ is recoverable after applying $\tilde{\mathcal{E}}^{(2)}$ and $X_{C_0}\in\mathcal{A}_{\tilde{\mathcal{E}}^{(2)}}$. By repeating the same argument for all n and the complementary channel, we complete the proof. □

Proof of Theorem 6

Proof. Recall that we have shown $I(A_1:B_1C_1)_{(1)} = I(A_1:B_1B_2C_2)_{(2)}$ (and that of E) is sufficient for Eq. (5). $\mathcal{A}_{\tilde{\mathcal{E}}}=\mathcal{A}_{\tilde{\mathcal{E}}^{(2)}}$ (we assume this w.l.o.g.) implies that there exists $\mathcal{R}$ such that $\mathcal{R}\circ\tilde{\mathcal{E}}^{(2)}$ act as the identity map on $\mathcal{A}_{\tilde{\mathcal{E}}}$. It holds that $\tilde{\mathcal{E}}\circ\mathcal{R}\circ\tilde{\mathcal{E}}^{(2)} = \tilde{\mathcal{E}}\circ\mathcal{P}_{\mathcal{A}_{\tilde{\mathcal{E}}}}\circ\mathcal{R}\circ\tilde{\mathcal{E}}^{(2)} = \tilde{\mathcal{E}}$, so we can set $\hat{\mathcal{R}} := \tilde{\mathcal{E}}\circ\mathcal{R}$. This completes the proof by the data-processing inequality.

To calculate CMI, let us show $I(A_1:B_1C_1)_{(1)} = I(A_1:A_2)_{\omega_D^{\mathcal{A}}}$. Since $\tilde{\mathcal{E}}\circ\mathcal{P}_{\mathcal{A}}=\tilde{\mathcal{E}}$ by dual complementarity, we have $\tilde{\mathcal{E}}_{A_2\to B_1C_1}(\omega^{\mathcal{A}}_{DA_1A_2}) = \rho^{(1)}_{A_1B_1C_1}$. By the monotonicity of the mutual information under local CPTP-maps, this implies $I(A_1:B_1C_1)_{(1)}\le I(A_1:A_2)_{\omega_D^{\mathcal{A}}}$. By applying the recovery map of OAQEC $\mathcal{R}_{B_1C_1\to A_2}$, we also have $\mathcal{P}_{\mathcal{A}}\circ\mathcal{R}_{B_1C_1\to A_2}(\rho^{(1)}_{A_1B_1C_1}) = \omega_D^{\mathcal{A}}$, therefore $I(A_1:B_1C_1)_{(1)}\ge I(A_1:A_2)_{\omega_D^{\mathcal{A}}}$. The same arguments hold for the complementary channel and $\mathcal{B}$, and thus we have

$$\begin{aligned}
I(A_1:C_1|B_1)_{(1)} &= I(A_1:B_1C_1)_{(1)} - I(A_1:B_1)_{(1)} && (43)\\
&= I(A_1:B_1C_1)_{(1)} + I(A_1:E_1C_1)_{(1)} - 2S(A_1) && (44)\\
&= I(A_1:A_2)_{\omega_D^{\mathcal{A}}} + I(A_1:A_2)_{\omega_D^{\mathcal{B}}} - 2S(A_1) && (45)\\
&= I_c(A_1\rangle A_2)_{\omega_D^{\mathcal{A}}} + I_c(A_1\rangle A_2)_{\omega_D^{\mathcal{B}}}\,, && (46)
\end{aligned}$$

where $\omega_D^{\mathcal{A}} := (\mathrm{id} \otimes \mathcal{P}_{\mathcal{A}})(|\omega_D\rangle\langle\omega_D|)$ and $\omega_D^{\mathcal{B}} := (\mathrm{id} \otimes \mathcal{P}_{\mathcal{B}})(|\omega_D\rangle\langle\omega_D|)$. Here $I_c(A\rangle B) := S(B) - S(AB)$ is the coherent information [6].

The correctable algebras have decompositions

$$\mathcal{A} \cong \bigoplus_k M_{n_k}(\mathbb{C}) \otimes I_{n'_k}\,, \quad \mathcal{B} \cong \bigoplus_l M_{m_l}(\mathbb{C}) \otimes I_{m'_l}, \tag{47}$$

where $\sum_k n_k n'_k = \sum_l m_l m'_l = D$. From these representations, it turns out that

$$\omega_D^{\mathcal{A}} = \bigoplus_k p_k (\omega_{n_k})_{A_1 A_2} \otimes \left(\frac{1}{n_k} I_{n_k}\right)_{A_2} \tag{48}$$

(up to local unitary) with $p_k = \frac{n_k n'_k}{D}$ and

$$I_c(A_1\rangle A_2)_{\omega_D^{\mathcal{A}}} = S(A_2)_{\omega_D^{\mathcal{A}}} - S(A_1 A_2)_{\omega_D^{\mathcal{A}}} \tag{49}$$

$$= \log D - \left(H(\{p_k\}) + 2\sum_k p_k \log n'_k\right) \tag{50}$$

$$= \sum_k p_k \log \frac{n_k}{n'_k}\,. \tag{51}$$

Therefore, by applying the same argument for $\mathcal{B}$,

$$I(A_1 : C_1|B_1)_{(1)} = \sum_k p_k \log \frac{n_k}{n'_k} + \sum_l q_l \log \frac{m_l}{m'_l} \tag{52}$$

holds with $q_l = \frac{m_l m'_l}{D}$. One may obtain more detailed formula by employing $\mathcal{B}' \subset \mathcal{A}$. The algebras satisfy $\mathcal{A} = \mathcal{B}'$ if and only if $|\{k\}| = |\{l\}|$, $p_k = q_k$, $m_k = n'_k$ and $m'_k = n_k$. This leads $I(A_1 : C_1|B_1)_{(1)} = 0$.

Proof of Theorem 7

In this section, we prove Theorem 7. We start from deriving the group structure of tensor product logical unitaries. We then reveal a sufficient condition to imply the existence of SPT phase from the non-zero value of CMI. We finally show that all stabilizer states satisfy the sufficient condition.

Recall that it is necessary to have a tensor product logical operator for $\phi^{(n)}$ to be in a SPT phase (under on-site symmetry). Let $\mathcal{G}_{\mathcal{A}} := \mathcal{U}(\mathcal{L}_{\mathcal{A}}) \cap (\mathcal{B}(\mathcal{H}_B) \otimes \mathcal{B}(\mathcal{H}_C))$ be the set of all tensor product logical operators. Since $\mathcal{L}_{\mathcal{A}}$ is the pre-image of $*$-homomorphism, it is a finite-dimensional C^*-algebra and therefore $\mathcal{U}(\mathcal{L}_{\mathcal{A}})$ is a compact and connected Lie group. $\mathcal{G}_{\mathcal{A}}$ is then a Lie subgroup of $\mathcal{U}(\mathcal{L}_{\mathcal{A}})$.

Proposition 10. *Define $\mathcal{C}_{\mathcal{A}}$ as the subgroup of logical operators*

$$\mathcal{C}_{\mathcal{A}} = \{U_B \otimes I_C, U_B \in \mathcal{L}_{\mathcal{A}}\}\,. \tag{53}$$

$\mathcal{C}_{\mathcal{A}}$ is the identity component of $\mathcal{G}_{\mathcal{A}}$.

Proof. Suppose $U = e^{itO} \in \mathcal{U}(\mathcal{L}_{\mathcal{A}})$. Then $\tilde{\mathcal{E}}^\dagger(e^{itO}) = e^{it\tilde{\mathcal{E}}^\dagger(O)} \in \mathcal{U}(\mathcal{A})$ ($\tilde{\mathcal{E}}^\dagger$ is a $*-$homomorphism) and therefore $\tilde{\mathcal{E}}^\dagger(O) \in \mathcal{A}$, i.e. $O \in \mathcal{L}_{\mathcal{A}}$. The Lie algebra of $\mathcal{U}(\mathcal{L}_{\mathcal{A}})$, which we denote by $\mathfrak{u}(\mathcal{L}_{\mathcal{A}})$, is therefore Hermitian logical operator on BC. Furthermore, $\mathfrak{u}(\mathcal{L}_{\mathcal{A}})$ should not contain operators like $I_B \otimes O_C$ for $O_C \neq I_C$, since the input of $\tilde{\mathcal{E}}$ and C is uncorrelated. Therefore,

$$\mathfrak{u}(\mathcal{L}_{\mathcal{A}}) = \left\{L_B \otimes I_C\,,\ L_{BC} \in \mathcal{L}_{\mathcal{A}} \,\middle|\, L_B = L_B^\dagger\,,\ L_{BC} = L_{BC}^\dagger\right\}, \tag{54}$$

where $\mathrm{Tr}_B L_{BC} = \mathrm{Tr}_C L_{BC} = 0$. The Lie algebra of $\mathcal{G}_{\mathcal{A}}$ is a subalgebra of $\mathfrak{u}(\mathcal{L}_{\mathcal{A}})$. For any non-local term L_{BC}, $e^{itL_{BC}}$ cannot be a tensor product for all $t \in \mathbb{R}$. Therefore, the identity component is given by $\{e^{itL_B \otimes I_C}\}$, which is Eq. (53). □

A well-known result on Lie group implies that the identity component is a closed normal subgroup and the quotient group $\mathcal{G}_\mathcal{A}/\mathcal{C}_\mathcal{A}$ is a discrete group (see e.g. Ref. [7] and references therein for more details). Since $\mathcal{G}_\mathcal{A}$ is compact, we obtain the following.

Corollary 11. *$\mathcal{G}_\mathcal{A}/\mathcal{C}_\mathcal{A}$ is a finite group.*

The exactly same arguments holds for $\mathcal{G}_\mathcal{B}$ and $\mathcal{C}_\mathcal{B}$. We denote an abstract group isomorphic to $\mathcal{G}_\mathcal{A}/\mathcal{C}_\mathcal{A}$ by G_1 and similarly use G_2 for $\mathcal{G}_\mathcal{B}/\mathcal{C}_\mathcal{B}$. Under dual complementarity, we have $\mathcal{C}_\mathcal{A} = \mathcal{U}(\mathcal{L}_{\mathcal{B}'})$. Let $\hat{\mathcal{G}}_\mathcal{A} \leqslant \mathcal{U}(\mathcal{A})$ be a subgroup of unitaries that have logical operators in $\mathcal{G}_\mathcal{A}$. It is easy to check $\mathcal{U}(\mathcal{B}') \lhd \hat{\mathcal{G}}_\mathcal{A}$ and we obtain $\hat{\mathcal{G}}_\mathcal{A}/\mathcal{U}(\mathcal{B}') \cong G_1$ as well. For any $g \in G_1$ there exists $V(g) \neq I \in \hat{\mathcal{G}}_\mathcal{A}$ such that the corresponding logical operator is in the form $U_B \otimes U_C(g)$ with unitaries $U_B, U_C(g) \neq I_B, I_C$. $U_C(g)$ is independent of particular choice of element from the equivalence class $[V(g)]$, which has one-to-one correspondence with g by definition.

If G_1 represents the physical symmetry of the state, then $U_C(g) \in [V(g)]$ for any $g \in G_1$ by $C \cong A$. This guarantees that we obtain tensor product logical operators for arbitrary length n. However, in general there is no guarantee that $U_C(g)$ again has a tensor product logical operator. Actually, the non-trivial example shown in the main text violates this condition. We denote by H_1 the subgroup of G_1 such that $U_C(h) \in [V(h)]$ for any $h \in H_1$.

A simple situation is that $H_1 = G_1$. However, the group G_1 can still be small compare to $\mathcal{A}$. To avoid this problem, we further assume $\hat{\mathcal{G}}_\mathcal{A}$ spans the whole correctable algebra. We show these two conditions are strong enough to obtain a non-trivial SPT phase from the value of CMI:

Proposition 12. *Let V be an isometry satisfying all of the following properties:*

- *dual complementarity*
- *$H_1 = G_1$, $H_2 = G_2$*
- *$Alg(\hat{\mathcal{G}}_\mathcal{A}) = \mathcal{A}$, $Alg(\hat{\mathcal{G}}_\mathcal{B}) = \mathcal{B}$*

Then the MPS generated by V is in a non-trivial $G_1 \times G_2$ SPT phase after corse-graining a constant number of sites if, and only if,

$$I(A_1 : C_n|B_n)_{(n)} > 0, \ \forall n.$$

Proof. Since $H_1 = G_1$, any $V(g)$ has a logical operator in the form $U_B \otimes U_C(g)$ with $U_C(g) \in \hat{\mathcal{G}}_\mathcal{A}$. The equivalence class of unitaries $U_C(g)$ again consists a group isomorphic to G_1. It is clear that $f : g \mapsto [U_C(g)]$ is surjective from its definition. To show it is also injective, suppose that $U_C(g) = U_C(g')$ for $g, g' \in G_1$. Then we can find logical operators of $V(g)V(g')^\dagger$ in the form $U_B \otimes I_C$, and thus $V(g)V(g')^\dagger \in \mathcal{U}(\mathcal{B}')$. This implies $[V(g)] = [V(g')] \Leftrightarrow g = g'$. Moreover, if $U_C(g) = VU_C(g')$ for $V \in \mathcal{U}(\mathcal{B}')$, then $g = g'$. This is because we can apply $\mathcal{V}_{C\to E'B'C'}$ and then logical operators of $U_C(g)$ and $VU_C(g')$ has the same unitary on C', so the injectivity discussed before applies to this case as well. Therefore $f : g \mapsto [U_C(g)]$ is a well-defined group isomomorphism.

Both $[V(g)]$ and $[U_C(g)]$ form $\hat{\mathcal{G}}_\mathcal{A}/\mathcal{U}(\mathcal{B}')$, but the labeling could be different, i.e. it might be true that $[V(g)] = [U_C(g')]$. The correspondence between $[V(g)]$ and $[U(g)]$ is a permutation (or automorphism) $Per : g' \mapsto g$ on G_1. Therefore, there exists a constant $m \in \mathbb{N}$ such that $(Per)^m = id_{G_1}$. By coarse-graining m channels, we obtain the desired relation

$$\mathcal{V}_{A_2\to B_m E_m C}(V(g)|\psi\rangle_{A_2}) = (U_{B_m}(g) \otimes V(g))\mathcal{V}_{A_2\to B_m E_m C}(|\psi\rangle_{A_2}) \ \forall g \in G_1 \tag{55}$$

with a unitary $U_{B_m}(g)$, where B_m and E_m are the coarse-grained systems. These arguments are also applicable for G_2, with possibly different coarse-graining scale m'. If $m \neq m'$, we coarse-grain $\tilde{m} := \mathrm{lcm}(m, m')$ sites. The unitaries $U_{B_{\tilde{m}}}(g_1) \otimes U_{E_{\tilde{m}}}(g_2)$ for $(g_1, g_2) \in G_1 \times G_2$, constracted in this way, form a unitary representation of $G_1 \times G_2$.

The states generated by V satisfy $I(A_1 : C_n|B_n)_{(n)} > 0$ if and only if $\mathcal{B} \subsetneq \mathcal{A}'$ (Theorem 6) due to dual complementarity. Moreover, $V(g_1)$ and $V(g_2)$ form a non-trivial projective representation. By assumption, $\hat{\mathcal{G}}_\mathcal{A}$ and $\hat{\mathcal{G}}_\mathcal{A}$ contain the basis of $\mathcal{A}$ and $\mathcal{B}$. Moreover, for every g_1, g_2 $V(g_2) \in \hat{\mathcal{G}}_\mathcal{B} \subset \mathcal{B}'$ and $V(g_1) \in \hat{\mathcal{G}}_\mathcal{A}\backslash\mathcal{U}(\mathcal{B}')$ and therefore there exists a pair (g_1, g_2) such that $[V(g_1), V(g_2)] \neq 0$ (note that the corresponding symmetery actions $U_{B_{\tilde{m}}}(g_1) \otimes I_{E_{\tilde{m}}}$ and $I_{B_{\tilde{m}}} \otimes U_{E_{\tilde{m}}}(g_2)$ commute each other). This completes the proof. ∎

Proof of Theorem 7

Proof. We show the theorem by proving that all the conditions in Proposition 12 are satisfied when V is Clifford. Since V is Clifford, $\mathcal{V}_{A_2\to BEC}$ is an encoding map of a stabilizer code. In stabilizer codes, all the logical operators are spanned by logical Pauli operators [9]. The pre-image of $\tilde{\mathcal{E}}^\dagger$ is thus spanned by Pauli operators and it follows that $\mathcal{A} = Alg\{P_i | \exists \tilde{P}_B \otimes \tilde{P}_C \in \mathcal{L}(P_i)\}$, where $\tilde{P}_B$ and $\tilde{P}_C$ are Pauli operators. Let $D = 2^K$ without loss of generality. A is regarded as a K-qubit system and Pauli operators on A are generated by Z_i, X_i operators acting on ith qubit. Then, the generators of $\mathcal{A}$ (up to a local Clifford) are summarized as a table:

$$\mathcal{A} \cong Alg \begin{pmatrix} 1 & \cdots & l & l+1 & \cdots & m & m+1 & \cdots & K \\ Z & \cdots & Z & Z & \cdots & Z & I & \cdots & I \\ I & \cdots & I & X & \cdots & X & I & \cdots & I \end{pmatrix}. \tag{56}$$

Here, the first column means $\mathcal{A}$ contains Z_1, but not contains X_1. In the same way, $m+1$th column means $\mathcal{A}$ does not contain both Z_{m+1} and X_{m+1}. The commutant of $\mathcal{A}$ is immediately given as

$$\mathcal{A}' \cong Alg \begin{pmatrix} 1 & \cdots & l & l+1 & \cdots & m & m+1 & \cdots & K \\ Z & \cdots & Z & I & \cdots & I & Z & \cdots & Z \\ I & \cdots & I & I & \cdots & I & X & \cdots & X \end{pmatrix}. \tag{57}$$

From these expressions it is clear that $\mathcal{A} = Alg(\hat{\mathcal{G}}_{\mathcal{A}})$. Let $g_{BC}(g_E)$ is the number of independent logical Pauli operators supported on BC (E). For stabilizer codes, it is known that they satisfy the formula $g_{BC} + g_E = 2K$ [8]. It is then clear that the number of logical operators found on E is $l + 2(K-m)$, which is the number of independent generators of $\mathcal{A}'$. Since the correctable algebra corresponding to output on E should be a subalgebra of $\mathcal{A}'$, they must be equivalent. Therefore dual complementarity is satisfied.

$H_1 = G_1$ follows from $\mathcal{A} = \mathcal{A}_{\tilde{\mathcal{E}}} = \mathcal{A}_{\tilde{\mathcal{E}}^{(2)}}$. Let us consider $P_i \in \mathcal{A}\backslash\mathcal{B}'$. For $n = 1$, P_i have a logical operator $\tilde{P}_{B_1} \otimes \tilde{P}_{C_1} \in \mathcal{L}(P_i)$ such that $\tilde{P}_C \neq I_C$. Since $\tilde{P}_C$ is also a Pauli operator, it has a logical Pauli operator on $B_2E_2C_2$. Suppose every such $\tilde{P}_C$ has no logical operator on B_2C_2. Then $\tilde{P}_C \in \mathcal{A}_{\tilde{\mathcal{E}}}\backslash\mathcal{A}'_{\tilde{\mathcal{F}}}$ but $\tilde{P}_C \notin \mathcal{A}_{\tilde{\mathcal{E}}^{(2)}}\backslash\mathcal{A}'_{\tilde{\mathcal{F}}^{(2)}}$, which conflicts to $\mathcal{A} = \mathcal{A}_{\tilde{\mathcal{E}}} = \mathcal{A}_{\tilde{\mathcal{E}}^{(2)}}$ and $\mathcal{B} = \mathcal{A}_{\tilde{\mathcal{F}}} = \mathcal{A}_{\tilde{\mathcal{F}}^{(2)}}$. Therefore $\mathcal{L}(\tilde{P}_C)$ contains operator on B_2C_2, which is also a Pauli operator and thus has a tensor product form. This proves $H_1 = G_1$. The same arguments hold for H_2. □

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