# Using Entanglement in Quantum Multi-Prover Interactive Proofs

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## Abstract

The central question in quantum multi-prover interactive proof systems is whether or not entanglement shared between provers affects the verification power of the proof system. We study for the first time *positive* aspects of prior entanglement and show that entanglement *is* useful even for *honest* provers. We show how to use shared entanglement to parallelize any multi-prover quantum interactive proof system to a *one-round* system with *perfect completeness*, with one extra prover. Alternatively, we can also parallelize to a three-turn system with the same number of provers, where the verifier only broadcasts the outcome of a coin flip. This "public-coin" property is somewhat surprising, since in the classical case public-coin multi-prover interactive proofs are equivalent to single prover ones.

# 1 Introduction

Multi-prover interactive proof systems are a central notion in theoretical computer science. An important generalization of interactive proof systems [GMR89, Bab85], they were originally introduced in [BOGKW88] in a cryptographic context. Later it was shown [BFL91, FRS94] that the class MIP of languages having a multi-prover interactive proof system is equal to NEXP, which led to the development of the theory of inapproximability and probabilistically checkable proofs [FGL⁺96, AS98, ALM⁺98].

In a multi-prover interactive proof system, a verifier communicates with several provers, which do not communicate with each other. One of the central challenges in this area is to understand the power of *quantum* multi-prover interactive proof systems (QMIP systems). In particular, the major open question is how *entanglement* shared among the provers affects these systems. This question is unique to the quantum world, since the related classical resource of shared randomness is known not to affect the power of such systems. It is not even clear whether entanglement *increases* or *decreases* the verification power of QMIP systems. On one hand, using entanglement, dishonest provers might cheat more easily, thereby breaking the soundness of the system. On the other hand, the increased power that entanglement gives to honest provers could be harnessed by the verifier, increasing the expressivity of the proof system.

To the best of our knowledge, all previous results in this area (see below) have focused on the former case, studying the *negative* effects of entanglement, i.e., whether or not *dishonest* entangled provers can break proof systems that are sound for any dishonest *unentangled* provers. Our work is the first to focus on the *positive* aspects of entanglement, where shared entanglement may be advantageous to *honest* provers.

## 1.1 Previous and related work

Kobayashi and Matsumoto [KM03] introduced QMIP systems with a quantum verifier, and proved that the class of languages having a quantum multi-prover interactive proof system is equal to NEXP when the provers do not share any prior entanglement, and is contained in NEXP when they share at most polynomially many entangled qubits. Cleve, Høyer, Toner, and Watrous [CHTW04] studied multi-prover interactive proof systems in which the verifier remains classical but provers may initially share entanglement, and presented several protocols for which shared EPR pairs can increase the power of dishonest provers. They also proved that the class of languages having some restricted version of multi-prover interactive proof system, denoted by $\oplus\mathrm{MIP}^*(2,1)$, is contained in EXP when provers are allowed to share prior entanglement (Wehner [Weh06] improved the upper bound to $\mathrm{QIP}(2)$, the class of languages having a two-message quantum interactive proof system), which is in stark contrast to the corresponding class $\oplus\mathrm{MIP}(2,1)$ without prior entanglement, which is equal to NEXP[1]. Very recently, Kempe, Kobayashi, Matsumoto, Toner, and Vidick [KKM⁺07] gave limits on the cheating power of dishonest entangled provers in some quantum and classical multi-prover interactive proof systems, by showing how such proof systems can be "immunized" against the use of entanglement by dishonest provers. Ito, Kobayashi, Preda, Sun, and Yao [IKP⁺07] and Cleve, Gavinsky, and Jain [CGJ07] also gave limits on the cheating power of entangled provers for some classical multi-prover interactive proof systems.

All these studies focus only on the *negative* aspects of prior entanglement, i.e., whether or not *dishonest* but prior-entangled provers can break the soundness of the proof system.

## 1.2 Our Results

This paper studies the *positive* aspects of prior entanglement and shows a number of general properties of QMIP systems, extensively using prior entanglement for *honest* provers. This gives the first evidence that prior entanglement *is* useful even for honest provers. Our main theorem states that any quantum $k$-prover

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[1]for some two-sided bounded error

interactive proof system that may involve polynomially many rounds can be parallelized to a *one-round* quantum $(k + 1)$-prover interactive proof system of *perfect* completeness and such that the gap between completeness and soundness accepting probabilities is still bounded by an inverse-polynomial.

To state our results more precisely, let $\mathrm{QMIP}(k, m, c, s)$ denote the class of languages having an $m$-turn quantum $k$-prover interactive proof system with completeness at least $c$ and soundness at most $s$, where provers are allowed to share an arbitrary amount of entanglement. We call the difference $c - s$ the "gap" in this paper. As commonly used in classical multi-prover interactive proofs we use the term "round" to describe an interaction consisting of questions from the verifier followed by answers from the provers. We use the term "turn" for messages sent in one direction. One round consists of two turns: a turn for the verifier and a turn for the provers. Let $\mathrm{poly}$ and $\mathrm{poly}^{-1}$ be the sets of all polynomially bounded functions and all inverse-polynomial functions, respectively. Throughout this paper we assume that the number $m$ of turns and the number $k$ of provers are functions in $\mathrm{poly}$ with respect to the input size, and that completeness $c$ and soundness $s$ are functions of the input size $c, s \colon \mathbb{Z}^+ \to [0, 1]$. Then we show the following main theorem.

**Theorem 1.** *For any $k, m \in \mathrm{poly}$ and $c, s$ satisfying $c - s \in \mathrm{poly}^{-1}$ there exists a function $p \in \mathrm{poly}$ such that $\mathrm{QMIP}(k, m, c, s) \subseteq \mathrm{QMIP}\left(k + 1, 2, 1, 1 - \frac{1}{p}\right)$.*

Since it is easy to amplify the success probability without increasing the number of rounds by running multiple instances of a proof system in parallel using a different set of provers for every instance, the above theorem shows that one-round (i.e., two-turn) QMIP systems are as powerful as general QMIP systems.

**Corollary 2.** *For any $k, m \in \mathrm{poly}$ and $c, s$ satisfying $c - s \in \mathrm{poly}^{-1}$, and $p \in \mathrm{poly}$, there exists $k' \in \mathrm{poly}$ such that $\mathrm{QMIP}(k, m, c, s) \subseteq \mathrm{QMIP}(k', 2, 1, 2^{-p})$.*

The proof of our main theorem comes in three parts, corresponding to Sections 3, 4, and 5. The first part shows how to convert any QMIP system with two-sided bounded error into one with one-sided bounded error of perfect completeness without changing the number of provers. The second part shows that any QMIP system with polynomially many turns can be parallelized to one with only three turns (messages from the provers followed by questions from the verifier followed by responses from the provers) in which the gap between completeness and soundness is still bounded by an inverse-polynomial. Again the number of provers remains the same in this transformation. Finally, the third part shows that any three-turn QMIP system with sufficiently large gap can be converted into a two-turn (i.e., one-round) QMIP system with inverse-polynomial gap, by adding an extra prover.

Similar statements to our first and second parts have already been shown by Kitaev and Watrous [KW00] for *single-prover* quantum interactive proofs. Their proofs, however, heavily rely on the fact that a single quantum prover can apply arbitrary operators over all the space except for the private space of the verifier. This is not the case any more for quantum multi-prover interactive proofs, since now a quantum prover cannot access the qubits in the private spaces of the other quantum provers, in addition to those in the private space of the verifier. Hence new methods are required for the multi-prover case.

To transform proof systems so that they have perfect completeness, our basic idea is to use the quantum rewinding technique developed for quantum zero-knowledge proofs by Watrous [Wat06], but in a different way. In our case we use it to "rewind" an unsuccessful computation that would result in rejection into a successful one. To apply the quantum rewinding technique, we first modify the proof system so that the honest provers can convince the verifier with probability exactly $\frac{1}{2}$ using some initial shared state and moreover no other initial shared state achieves a higher acceptance probability. This initial shared state corresponds to the auxiliary input in the case of quantum zero-knowledge proofs, and thus, as in that scenario, the sequence of forward, backward, and forward executions of the protocol achieves perfect completeness. The obvious problem of this construction lies in proving soundness, as the dishonest provers may not use the same strategies for all of the three executions of the proof system. To settle this, we design a simple protocol

that tests if the second backward execution is indeed a backward simulation of the first forward execution. The verifier performs with equal probability either the original rewinding protocol or this invertibility test without revealing which test the provers are undergoing. This forces the provers to use essentially the same strategies for the first two executions of the protocol, which is sufficient to bound the soundness. As a result we prove the following.

**Theorem 3.** *For any $k, m \in$ poly and $c, s$ satisfying $c - s \in$ poly$^{-1}$, and $p \in$ poly, there exists $m' \in$ poly such that* QMIP$(k, m, c, s) \subseteq$ QMIP$(k, m', 1, 2^{-p})$.

For the parallelization to three turns, our approach is to first show that any QMIP system with sufficiently large gap can be converted into another QMIP system with the same number of provers, in which the number of rounds (turns) becomes almost half of that in the original proof system. The proof idea is that the verifier in the first turn receives the snapshot state from the original system after (almost) half of turns have been executed, and then with equal probability executes either a forward-simulation or a backward-simulation of the original system from that turn on. Honest provers only have to simulate the original system to convince the verifier, while any strategy of dishonest provers with unallowable high success probability would lead to a strategy of dishonest provers in the original system that contradicts the soundness condition. By repeatedly applying this modification, together with Theorem 3 as preprocessing, we can convert any QMIP system into a three-turn QMIP system with the same number of provers that still has an inverse polynomial gap.

**Theorem 4.** *For any $k, m \in$ poly and $c, s$ satisfying $c - s \in$ poly$^{-1}$, there exists $p \in$ poly such that* QMIP$(k, m, c, s) \subseteq$ QMIP $\left( k, 3, 1, 1 - \frac{1}{p} \right)$.

For $k = 1$, this gives an alternative proof of the parallelization theorem due to Kitaev and Watrous [KW00] for single-prover quantum interactive proofs. It is interesting to note that our parallelization method does not need the controlled-swap test at all, while it *is* the key test in the Kitaev-Watrous parallelization method. Another point worth mentioning in our method is that, at every time step of our parallelized protocol, the whole system has only one snapshot state of the original system. This is in contrast to the fact that the whole system has to simultaneously treat many snapshot states in the Kitaev-Watrous method. The merit of our method is, thus, that we do not need to treat the possible entanglement among different snapshot states when analyzing soundness, which may be a main reason why our method works well even for the multi-prover case. Moreover, our method is more space-efficient than the Kitaev-Watrous method, in particular when we parallelize a system with polynomially many rounds.

To prove the third part, we will take a detour by proving that

(i) any three-turn QMIP system with sufficiently large gap can be modified to a three-turn *public-coin* QMIP system with the same number of provers and a gap of roughly similar order of magnitude,

(ii) any three-turn public-coin QMIP system can be converted into a two-turn QMIP system without changing completeness and soundness, by adding one extra prover.

The notion of public-coin QMIP systems we use is a natural generalization of public-coin quantum interactive proofs in the single-prover case introduced by Marriott and Watrous [MW05]. The corresponding complexity class is denoted by QMIP$_{\text{pub}}(k, m, c, s)$ in this paper. Intuitively, at every round, a public-coin quantum verifier flips a fair classical coin at most polynomially many times, and then simply broadcasts the result of these coin-flips to all the provers. Property (i) is a generalization of the result by Marriott and Watrous [MW05] to the multi-prover case, whereas property (ii) is completely new. The idea to prove (ii), assuming that the number of provers in the original proof system is $k$, is to send questions only to the first $k$ provers in the new $(k + 1)$-prover system, requesting the original second messages from the $k$ provers in the original system. The verifier expects to receive from the $(k + 1)$-st prover the original first messages from

the $k$ provers in the original system without asking any question to that prover. The public-coin property of the original system implies the nonadaptiveness of the messages from the verifier, which is essential to prove (ii). In fact, there is a way to directly prove the third part, but our detour enables us to show another two important properties of QMIP systems. Specifically, property (i) essentially proves the equivalence of *public-coin* quantum $k$-prover interactive proofs and general quantum $k$-prover interactive proofs, for any $k$.

**Theorem 5.** *For any $k, m \in$ poly and $c, s$ satisfying $c - s \in$ poly$^{-1}$, and $p \in$ poly, there exists $m' \in$ poly such that* $\mathrm{QMIP}(k, m, c, s) \subseteq \mathrm{QMIP}_{\mathrm{pub}}(k, m', 1, 2^{-p})$.

Note that in the classical case, public-coin multi-prover interactive proofs are only as powerful as single-prover interactive proofs: because every prover receives the same question from the verifier it means that every prover knows how other provers will behave and the joint strategy of the provers can therefore be simulated by a single prover. Hence, these systems cannot be as powerful as general classical multi-prover interactive proofs unless NEXP = PSPACE. In contrast, our result shows that in the quantum case, public-coin QMIP systems *are* as powerful as general QMIP systems. The non-triviality of public-coin QMIP systems may be explained as follows: even if every quantum prover knows how other quantum provers will behave, still each quantum prover can apply only local transformations over a part of some state that may be entangled among the provers, which is not enough to simulate every possible strategy a single quantum prover could follow.

Property (ii) for the case $k = 1$ implies that any language in QIP (and thus in PSPACE) has a *two-prover one-round* quantum interactive proof system of perfect completeness with exponentially small error in soundness, since any language in QIP has a three-message public-coin quantum interactive proof system of perfect completeness with exponentially small error in soundness [MW05].

**Corollary 6.** *For any $p \in$ poly,* $\mathrm{QIP} \subseteq \mathrm{QMIP}(2, 2, 1, 2^{-p})$ *(and thus* $\mathrm{PSPACE} \subseteq \mathrm{QMIP}(2, 2, 1, 2^{-p})$*).*

In the classical case a similar statement to the last corollary was shown by Cai, Condon, and Lipton [CCL94] (and the stronger statement that two-prover one-round interactive proofs are as powerful as general multi-prover interactive proofs was shown later by Feige and Lovász [FL92]). All these results are, however, not known to hold under the existence of prior entanglement among the provers. Before our result, it has even been open if PSPACE has a two-prover one-round quantum interactive proof system. (Very recently, Kempe et al. [KKM+07] succeeded in proving that the classical two-prover one-round interactive proof system for PSPACE in Ref. [CCL94] is sound in a weak sense against any pair of dishonest prior-entangled provers: soundness is bounded away from one by an inverse-polynomial. Their result is incomparable to ours since on one hand we have a much stronger soundness condition, and on the other both the verifier and the honest provers must be quantum. In contrast, in Ref. [KKM+07] both of them just follow a classical protocol.)

Finally, we stress again that our constructions extensively use the prior shared entanglement of the provers in a positive sense. In particular, even if the honest provers in the original proof system do not need any prior entanglement at all, the honest provers in the constructed proof system do need prior entanglement in many cases. Most of the properties proved in this paper (Theorems 1 and 5 and Corollary 6 in particular) are not known to hold when considering only initially unentangled honest provers, and thus give first evidence that sharing prior entanglement may be advantageous even to honest provers.

## 2 Preliminaries

We assume that the reader is familiar with the quantum formalism, including the quantum circuit model and definitions of mixed quantum states (density operators) and fidelity (all of which are discussed in detail in

Refs. [NC00, KSV02], for instance). This section summarizes some of the notions and notations that are used in this paper, reviews the model of quantum multi-prover interactive proof systems and introduces the notion of *public-coin* quantum multi-prover interactive proof systems.

As in earlier work [Wat03, KW00, KM03], we define QMIP systems in terms of quantum circuits. It is assumed that our circuits consist of unitary gates, which is sufficient since non-unitary and unitary quantum circuits are equivalent in computational power [AKN98]. To avoid unnecessary complication, however, in the subsequent sections the descriptions of protocols often include non-unitary operations (measurements). Even in such cases, it is always possible to construct unitary quantum circuits that essentially achieve the same outcome. A notable exception is in the definition of the public-coin quantum verifier, where we want to define the public coin-flip to be a classical operation. This requires a non-unitary operation for the verifier, the (classical) public coin-flip.

When proving statements that involve the perfect-completeness property, we assume that our universal gate set satisfies some conditions, which may not hold with an arbitrary universal gate set. Specifically, we assume that the Hadamard transformation and any classical reversible transformations are exactly implementable in our gate set. Note that this condition is satisfied by most of the standard gate sets including the Shor basis [Sho96] consisting of the Hadamard gate, the controlled-$i$-phase-shift gate, and the Toffoli gate, and thus, we believe that this condition is not restrictive. We stress that most of our main statements do hold with an arbitrary choice of universal gate set (the completeness and soundness conditions may become worse by negligible amounts in some of the claims, which does not affect the final main statements).

All Hilbert spaces in this paper are of dimension a power of two, spanned by qubits. We will use the following property of fidelity.

**Lemma 7** ([SR02, NS03]). *For any density operators $\rho, \sigma, \xi$ over a Hilbert space $\mathcal{H}$,*
$$F(\rho, \sigma)^2 + F(\sigma, \xi)^2 \leq 1 + F(\rho, \xi).$$

**Quantum Multi-Prover Interactive Proof Systems (QMIP systems):**    Throughout this paper $k$ and $k'$ denote the number of provers and $m, m'$ denote the number of turns. All of these are from the set of polynomially bounded functions in the input size $|x|$, denoted by $\mathrm{poly}$. Further, $c$ and $s$ denote functions of the input size into $[0, 1]$ corresponding to completeness and soundness. For notational convenience in what follows we will omit the arguments of these functions.

A quantum $k$-prover interactive proof system consists of a verifier $V$ with private quantum register $\mathsf{V}$ and $k$ *provers* $P_1, \ldots, P_k$ with private quantum registers $\mathsf{P}_1, \ldots, \mathsf{P}_k$, as well as quantum message registers $\mathsf{M}_1, \ldots, \mathsf{M}_k$, which without loss of generality are assumed to have the same number of qubits, denoted by $q_\mathsf{M}$. One of the private qubits of the verifier is designated as the *output qubit*. At the beginning of the protocol, all the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ are initialized to $|0 \cdots 0\rangle$, and the qubits in $(\mathsf{P}_1, \ldots, \mathsf{P}_k)$ are in some *a priori shared state* $|\Phi\rangle$ prepared by the provers in advance (and hence possibly entangled), which w.l.o.g. can be assumed to be pure. No direct communication between the provers is allowed after that. The protocol consists of alternating turns of the provers and of the verifier, starting with the verifier, if $m$ is even, and with the provers otherwise. At a turn of the verifier, $V$ applies some polynomial-time circuit to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, and then sends each register $\mathsf{M}_i$ to prover $P_i$. At a turn of the provers each prover $P_i$ applies some transformation to the registers $(\mathsf{P}_i, \mathsf{M}_i)$ for $1 \leq i \leq k$ and sends $\mathsf{M}_i$ back to the verifier. The last turn is always a turn for the provers. After the last turn the verifier applies a polynomial-time circuit to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, and then measures the output qubit in the standard basis, accepting if the outcome is $|1\rangle$ and rejecting otherwise.

Formally, an *$m$-turn polynomial-time quantum verifier $V$* for $k$-prover QMIP systems is a polynomial-time computable mapping from input strings $x$ to a set of polynomial-time uniformly generated circuits $\{V^1, \ldots, V^{\lceil m+1/2 \rceil}\}$, and a partition of the space on which they act into registers $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, which consist of polynomially many qubits. Similarly an *$m$-turn quantum prover $P$* is a mapping from $x$ to a set of

circuits $\{P^1, \ldots, P^{\lceil m+1/2 \rceil}\}$ each acting on registers $(\mathsf{P}, \mathsf{M})$. No restrictions are placed on the complexity of this mapping or the size of $\mathsf{P}$. We will denote the $i$-th prover, his registers and transformations with a subscript $i$. We will always assume that each prover $P_i$ is *compatible* with the verifier, i.e. that the corresponding register $\mathsf{M}_i$ is the same for the verifier and the prover for $1 \leq i \leq k$.

The *protocol* $(V, P_1, \ldots, P_k, |\Phi\rangle)$ is the alternating application of prover's and verifier's circuits to the state $|0 \cdots 0\rangle \otimes |\Phi\rangle$ in registers $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{P}_1, \ldots, \mathsf{P}_k)$. For odd $m$, circuits $P_1^1 \otimes \cdots \otimes P_k^1$, $V^1$, $P_1^2 \otimes \cdots \otimes P_k^2$, $V^2$ and so on are applied in sequence terminating with $V^{m+1/2}$. If $m$ is even, the sequence begins with $V^1$ followed by $P_1^1 \otimes \cdots \otimes P_k^1$ and so on up to $V^{m+2/2}$. We say that $(V, P_1, \ldots, P_k, |\Phi\rangle)$ accepts $x$ if the designated output qubit in $\mathsf{V}$ is measured in $|1\rangle$ at the end of the protocol and call the probability with which this happens $p_{\mathrm{acc}}(x, V, P_1, \ldots, P_k, |\Phi\rangle)$.

**Definition 8.** A language $L$ is in $\mathrm{QMIP}(k, m, c, s)$ iff there exists an $m$-turn polynomial-time quantum verifier $V$ for quantum $k$-prover interactive proof systems such that, for every input $x$:

(Completeness) if $x \in L$, there exist $m$-turn quantum provers $P_1, \ldots, P_k$ and an a priori shared state $|\Phi\rangle$ such that $p_{\mathrm{acc}}(x, V, P_1, \ldots, P_k, |\Phi\rangle) \geq c$,

(Soundness) if $x \notin L$, for any $m$-turn quantum provers $P_1', \ldots, P_k'$ and any a priori shared state $|\Phi'\rangle$, $p_{\mathrm{acc}}(x, V, P_1', \ldots, P_k', |\Phi'\rangle) \leq s$.

Next, we introduce the notions of *public-coin* quantum verifier and *public-coin* QMIP systems. These are natural generalizations of the corresponding notions in the single-prover case introduced by Marriott and Watrous [MW05]. Intuitively, a quantum verifier for quantum multi-prover interactive proof systems is public-coin if, at each of his turns, after receiving the message registers from the provers, he first flips a fair classical coin at most a polynomial number of times, and then simply broadcasts the result of these coin-flips to all the provers. No other messages are sent from the verifier to the provers. At the end of the protocol, the verifier applies some quantum operation to the messages received so far, and decides acceptance or rejection.

Formally, an $m$-turn polynomial-time quantum verifier for $k$-prover interactive proof systems is *public-coin* if each of the circuits $V^1, V^2, \ldots, V^{\lceil m-1/2 \rceil}$ implements the following procedure: $V$ receives the message registers $\mathsf{M}_i$ from the provers, stores them in his private space, and then flips a classical fair coin at most $q_\mathsf{M}$ times to generate a public string $r_j$, records $r_j$ in his private space, and broadcasts $r_j$ to all the provers. The circuit $V^{\lceil (m+1)/2 \rceil}$ is some unitary transformation controlled by all the recorded random strings $r_j$ for $1 \leq j \leq \lceil (m-1)/2 \rceil$. A QMIP system is public-coin if the associated verifier is public-coin, and we define $\mathrm{QMIP}_{\mathrm{pub}}(k, m, c, s)$ to be the class of languages in $\mathrm{QMIP}(k, m, c, s)$ with a public-coin verifier.

# 3 QMIP with Perfect Completeness Equals General QMIP

In this section we prove Theorem 3, showing that any QMIP system with two-sided bounded error can be transformed into a one with one-sided bounded error of perfect completeness without changing the number of provers. For the case of a single prover, this was shown by Kitaev and Watrous [KW00], but their proof relies on the single prover performing a global unitary on the whole system, and therefore does not carry over to the multi-prover case (no prover has access to the other prover's private spaces and the private space of each prover might be arbitrarily large, so we cannot use the verifier to transfer those spaces from one prover to the other).

First, we introduce the notion of *perfectly rewindable* QMIP systems.

**Definition 9.** Let $s < \frac{1}{2}$. A language $L$ has a perfectly rewindable $m$-turn quantum $k$-prover interactive proof system with soundness at most $s$ iff there exists an $m$-turn polynomial-time quantum verifier $V$, such that, for every input $x$:

(Perfect Rewindability) if $x \in L$, there exists a set of $m$-turn quantum provers $P_1, \ldots, P_k$ such that $\max_{|\Phi\rangle} p_{\mathrm{acc}}(x, V, P_1, \ldots, P_k, |\Phi\rangle) = \frac{1}{2}$, where the maximum is taken over all a priori shared states $|\Phi\rangle$ prepared by $P_1, \ldots, P_k$.

(Soundness) if $x \notin L$, for any set of $m$-turn quantum provers $P_1', \ldots, P_k'$ and any a priori shared state $|\Phi'\rangle$, $p_{\mathrm{acc}}(x, V, P_1', \ldots, P_k', |\Phi'\rangle) \leq s$.[2]

We first show how to modify any general QMIP system (with some appropriate conditions on completeness and soundness) to a perfectly rewindable one with the same $k$ and $m$.

**Lemma 10.** *Let $c \geq \frac{1}{2} > s$. Then, any language $L$ in $\mathrm{QMIP}(k, m, c, s)$ has a perfectly rewindable $m$-turn quantum $k$-prover interactive proof system with soundness at most $s$.*

*Proof.* Let $L$ be a language in $\mathrm{QMIP}(k, m, c, s)$ and $V$ be the corresponding $m$-turn quantum verifier. We slightly modify $V$ to construct another $m$-turn quantum verifier $W$ for a perfectly rewindable proof system for $L$. The new verifier $W$, in addition to the registers of $V$, prepares another single-qubit register B, initialized to $|0\rangle$. For the first $m - 2$ turns, $W$ simply simulates $V$. In the $(m-1)$-st turn, a turn for the verifier, $W$ proceeds like $V$ would, but sends B to the first prover in addition to the qubits $V$ would send in the original proof system. In the $m$-th turn the first prover is requested to send B back to $W$, in addition to the qubits sent to $V$ in the original proof system. Then $W$ proceeds for the final decision procedure like $V$ would, but accepts iff $V$ would have accepted *and* B is in the state $|1\rangle$.[3] Notice that $W$ accepts only if $V$ would have accepted, so the soundness is obviously at most $s$ in the constructed proof system.

For perfect rewindability we slightly modify the protocol for honest provers in the case $x \in L$. Let $|\Phi^*\rangle$ be the a priori shared state in the original proof system that maximizes the acceptance probability for the original honest provers and let $p_{\max}$ be that maximal acceptance probability. The new provers use $|\Phi^*\rangle$ as the a priori shared state and simulate the original provers except for the last turn. The only difference is that in the last turn the first prover proceeds as $P_1$ would, *and* applies a one-qubit unitary $T$ to the qubit in B,

$$T : |0\rangle \to \sqrt{1 - \frac{1}{2p_{\max}}}|0\rangle + \sqrt{\frac{1}{2p_{\max}}}|1\rangle.$$

From the construction it is obvious that the maximum accepting probability is exactly equal to $\frac{1}{2}$ and that this maximum is achieved when the provers use the a priori shared state $|\Phi^*\rangle$. $\qquad\square$

Now, we are ready to show the following lemma.

**Lemma 11.** *Let $c \geq \frac{1}{2}$ and $s < \frac{1}{25}$. Then, $\mathrm{QMIP}(k, m, c, s) \subseteq \mathrm{QMIP}\left(k, 3m, 1, \frac{1}{2} + 2\sqrt{s} + \frac{5s}{2}\right)$.*

*Proof.* The intuitive idea behind the proof of this lemma, using Watrous' "quantum rewinding technique", has already been explained in the introduction. We add some more intuition before proceeding to the technical proof. Using Lemma 10 we can assume that in the case of honest provers ($x \in L$) the acceptance probability with shared state $|\Phi^*\rangle$ is exactly $\frac{1}{2}$ and furthermore that no other a priori shared state achieves higher acceptance probability. The acceptance probability when the provers use any a priori shared state $|\Phi\rangle$ can be written as $p_{\mathrm{acc}} = \|\Pi_{\mathrm{acc}} Q |\Psi\rangle\|^2 = \|\Pi_{\mathrm{acc}} Q \Pi_{\mathrm{init}} |\Psi\rangle\|^2$, where $|\Psi\rangle = |0 \cdots 0\rangle_{(V, M_1, \ldots, M_k)} \otimes |\Phi\rangle$, $Q$ is the unitary transformation induced by the QMIP system just before the verifier's final measurement, $\Pi_{\mathrm{init}}$

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[2]Note that both for completeness and soundness we first fix the provers' transformations and then maximize over all a priori shared states, which hence have a fixed dimension.

[3]This protocol can be brought into the standard form where only one qubit is measured to decide acceptance. Call Y the register containing the designated output qubit for $V$. $W$ adds a new single-qubit output register X, initialized to $|0\rangle$. At the end of the protocol $W$ performs a Toffoli gate on the qubits in B, Y, X controlled by the qubits in (B, Y). Clearly X will contain $|1\rangle$ iff both B and Y contain $|1\rangle$.

is the projection on $|0\cdots0\rangle_{(\mathsf{V},\mathsf{M}_1,\ldots,\mathsf{M}_k)}$ and $\Pi_{\mathrm{acc}}$ is the projection on $|1\rangle$ of the designated output qubit. In other words the state $|\Psi^*\rangle = |0\cdots0\rangle_{(\mathsf{V},\mathsf{M}_1,\ldots,\mathsf{M}_k)} \otimes |\Phi^*\rangle$ maximizes the expression

$$\max_{|\Psi\rangle}\langle\Psi|\Pi_{\mathrm{init}}Q^\dagger\Pi_{\mathrm{acc}}Q\Pi_{\mathrm{init}}|\Psi\rangle,$$

meaning that the matrix $M = \Pi_{\mathrm{init}}Q^\dagger\Pi_{\mathrm{acc}}Q\Pi_{\mathrm{init}}$ has maximum eigenvalue $\frac{1}{2}$ with corresponding eigenvector $|\Psi^*\rangle$. Now we apply the quantum rewinding technique by performing forward, backward, and forward executions of the proof system in sequence. Perfect completeness follows from the fact that the initial state is an eigenvector of $M$ with the corresponding eigenvalue exactly $\frac{1}{2}$, exactly as in the zero-knowledge scenario of [Wat06].

The challenge of this construction lies in the proof of soundness. If the input is a no-instance, the maximum eigenvalue of any matrix $M$ corresponding to our proof system is small. This shows that if the dishonest provers are actually "not so dishonest", i.e., if they use the same strategies for all of the three (forward, backward, and forward) executions of the original proof system, the acceptance probability is still small. However, the problem arises when the dishonest provers change their strategies for some of the three executions. To settle this, we design a simple protocol that tests if the backward execution is indeed a backward simulation of the first forward execution. The verifier performs the original rewinding protocol or this invertibility test uniformly at random without revealing which test the provers are undergoing. Honest provers always pass this invertibility test, and thus perfect completeness is preserved. When the input is a no-instance, this forces the provers to use approximately the same strategies for the first two executions of the proof system, which is sufficient to bound the soundness.

We now proceed with the technical details. Let $L$ be a language in $\mathrm{QMIP}(k, m, c, s)$ and let $V$ be the verifier in the perfectly rewindable $m$-turn quantum $k$-prover interactive proof system for $L$ as per Lemma 10. We construct a $3m$-turn quantum verifier $W$ of a new quantum $k$-prover interactive proof system for $L$. $W$ has the same registers as $V$ in the original proof system, and performs one of two tests, which we call "REWINDING TEST" and "INVERTIBILITY TEST". The exact protocol is described in Figure 1, where for simplicity it is assumed that $m$ is even (the case in which $m$ is odd can be proved in a similar manner).

*Completeness:* Assume the input $x$ is in $L$. From the original provers $P_1,\ldots,P_k$ we design honest provers $R_1,\ldots,R_k$ for the constructed $3m$-turn system. Each new prover $R_i$ has the same quantum register $\mathsf{P}_i$ as $P_i$ has, and the new provers initially share $|\Phi^*\rangle$. For the first $m$ turns each $R_i$ simulates $P_i$. At the $(m + 2j)$-th turn for $1 \le j \le \frac{m}{2}$, $R_i$ applies $(P_i^{\frac{m}{2}-j+1})^\dagger$ (i.e. the inverse of the $(m - 2j + 2)$-nd turn of the original $P_i$). Finally, for the $(2m + 2j)$-th turn for $1 \le j \le \frac{m}{2}$, $R_i$ again applies $P_i^j$.

It is obvious from this construction that the provers $R_1,\ldots,R_k$ can convince $W$ with certainty when $W$ performs the INVERTIBILITY TEST. We show that $R_1,\ldots,R_k$ can convince $W$ with certainty even when $W$ performs the REWINDING TEST. In short, this holds for essentially the same reason that the quantum rewinding technique works well in the case of quantum zero-knowledge proofs, and we will closely follow that proof.

For notational convenience, let $\widetilde{P}^j = P_1^j \otimes \cdots \otimes P_k^j$ for $1 \le j \le \frac{m}{2}$, and let $Q = V^{\frac{m}{2}+1}\widetilde{P}^{\frac{m}{2}}V^{\frac{m}{2}}\cdots\widetilde{P}^1V^1$. Recall that $M|\Psi^*\rangle = \frac{1}{2}|\Psi^*\rangle$ where $M = \Pi_{\mathrm{init}}Q^\dagger\Pi_{\mathrm{acc}}Q\Pi_{\mathrm{init}}$. Define the unnormalized states $|\phi_0\rangle$, $|\phi_1\rangle$, $|\psi_0\rangle$, and $|\psi_1\rangle$ by

$$|\phi_0\rangle = \Pi_{\mathrm{acc}}Q|\Psi^*\rangle, \qquad |\phi_1\rangle = \Pi_{\mathrm{rej}}Q|\Psi^*\rangle, \qquad |\psi_0\rangle = \Pi_{\mathrm{init}}Q^\dagger|\phi_0\rangle, \qquad |\psi_1\rangle = \Pi_{\mathrm{illegal}}Q^\dagger|\phi_0\rangle,$$

where $\Pi_{\mathrm{illegal}} = I_{(\mathsf{V},\mathsf{M}_1,\ldots,\mathsf{M}_k)} - \Pi_{\mathrm{init}}$ is the projection onto states orthogonal to $|0\cdots0\rangle_{(\mathsf{V},\mathsf{M}_1,\ldots,\mathsf{M}_k)}$ and $\Pi_{\mathrm{rej}} = I_{(\mathsf{V},\mathsf{M}_1,\ldots,\mathsf{M}_k)} - \Pi_{\mathrm{acc}}$. Then, noticing that $|\Psi^*\rangle = \Pi_{\mathrm{init}}|\Psi^*\rangle$, we have

$$|\psi_0\rangle = \Pi_{\mathrm{init}}Q^\dagger\Pi_{\mathrm{acc}}Q|\Psi^*\rangle = \Pi_{\mathrm{init}}Q^\dagger\Pi_{\mathrm{acc}}Q\Pi_{\mathrm{init}}|\Psi^*\rangle = M|\Psi^*\rangle = \frac{1}{2}|\Psi^*\rangle,$$

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<div align="center">**Verifier's Protocol for Achieving Perfect Completeness**</div>

1. Simulate the original verifier for the first $m$ turns.

2. Choose $b \in \{0, 1\}$ uniformly at random. If $b = 0$, move to the REWINDING TEST described in Step 3, while if $b = 1$, move to the INVERTIBILITY TEST described in Step 4.

3. (REWINDING TEST)

   3.1 Apply $V^{\frac{m}{2}+1}$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Accept if the content of $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ corresponds to an accepting state in the original proof system. Otherwise apply $(V^{\frac{m}{2}+1})^{\dagger}$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, and send $\mathsf{M}_i$ to the $i$th prover, for $1 \le i \le k$.

   3.2 For $j = \frac{m}{2}$ down to 2, do the following:
   Receive $\mathsf{M}_i$ from the $i$th prover, for $1 \le i \le k$. Apply $(V^j)^{\dagger}$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, and send $\mathsf{M}_i$ to the $i$th prover, for $1 \le i \le k$.

   3.3 Receive $\mathsf{M}_i$ from the $i$th prover, for $1 \le i \le k$. Apply $(V^1)^{\dagger}$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Perform a controlled-phase-flip: multiply the phase by $-1$ if all the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ are in state $|0\rangle$. Apply $V_1$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, and send $\mathsf{M}_i$ to the $i$th prover, for $1 \le i \le k$.

   3.4 For $j = 2$ to $\frac{m}{2}$, do the following:
   Receive $\mathsf{M}_i$ from the $i$th prover, for $1 \le i \le k$. Apply $V^j$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, and send $\mathsf{M}_i$ to the $i$th prover, for $1 \le i \le k$.

   3.5 Receive $\mathsf{M}_i$ from the $i$th prover, for $1 \le i \le k$. Apply $V^{\frac{m}{2}+1}$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Accept if the content of $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ corresponds to an accepting state in the original proof system, and reject otherwise.

4. (INVERTIBILITY TEST)

   4.1 Send $\mathsf{M}_i$ to the $i$th prover, for $1 \le i \le k$.

   4.2 For $j = \frac{m}{2}$ down to 2, do the following:
   Receive $\mathsf{M}_i$ from the $i$th prover, for $1 \le i \le k$. Apply $(V^j)^{\dagger}$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$, and send $\mathsf{M}_i$ to the $i$th prover, for $1 \le i \le k$.

   4.3 Receive $\mathsf{M}_i$ from the $i$th prover, for $1 \le i \le k$. Apply $(V^1)^{\dagger}$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Accept if all the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ are in state $|0\rangle$, and reject otherwise.

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<div align="center">Figure 1: Verifier's protocol for achieving perfect completeness</div>

and thus,

$$Q^{\dagger}|\phi_1\rangle = Q^{\dagger}\Pi_{\mathrm{rej}}Q|\Psi^*\rangle = |\Psi^*\rangle - Q^{\dagger}\Pi_{\mathrm{acc}}Q|\Psi^*\rangle = |\Psi^*\rangle - Q^{\dagger}|\phi_0\rangle = 2|\psi_0\rangle - (|\psi_0\rangle + |\psi_1\rangle) = |\psi_0\rangle - |\psi_1\rangle.$$

Hence, the state just before the controlled-phase-flip in Step 3.3 when entering the REWINDING TEST is exactly

$$\frac{1}{\||\phi_1\rangle\|}Q^{\dagger}|\phi_1\rangle = \frac{1}{\||\phi_1\rangle\|}(|\psi_0\rangle - |\psi_1\rangle).$$

Since $\Pi_{\text{init}}|\psi_0\rangle = |\psi_0\rangle$ and $\Pi_{\text{init}}|\psi_1\rangle = 0$, the controlled-phase-flip changes the state to

$$-\frac{1}{\||\phi_1\rangle\|}(|\psi_0\rangle + |\psi_1\rangle) = -\frac{1}{\||\phi_1\rangle\|}Q^\dagger|\phi_0\rangle.$$

Therefore, the state just after $V^{\frac{m}{2}+1}$ is applied in Step 3.5 is exactly

$$-\frac{1}{\||\phi_1\rangle\|}QQ^\dagger|\phi_0\rangle = -\frac{1}{\||\phi_1\rangle\|}|\phi_0\rangle,$$

and thus, the fact that $\Pi_{\text{acc}}|\phi_0\rangle = |\phi_0\rangle$ implies that the verifier $W$ always accepts in Step 3.5.

*Soundness:* Now suppose that the input $x$ is not in $L$. Let $R_1', \ldots, R_k'$ be any $k$ provers for the constructed $3m$-turn proof system, and let $|\psi\rangle$ be any a priori shared state. Let $R_i^j$ be the transformation that $R_i'$ applies at his $2j$-th turn, for $1 \le i \le k$ and $1 \le j \le \frac{3m}{2}$ and let $Z$ denote the controlled-phase-flip operator in Step 3.3. Call $\widetilde{R}^j = R_1^j \otimes \cdots \otimes R_k^j$ for $1 \le t \le \frac{3m}{2}$, and define

$$U_1 = \widetilde{R}^{\frac{m}{2}}V^{\frac{m}{2}}\cdots \widetilde{R}^2 V^2 \widetilde{R}^1 V^1,$$
$$U_2 = (V^1)^\dagger \widetilde{R}^m \cdots (V^{\frac{m}{2}-1})^\dagger \widetilde{R}^{\frac{m}{2}+2}(V^{\frac{m}{2}})^\dagger \widetilde{R}^{\frac{m}{2}+1},$$
$$U_3 = \widetilde{R}^{\frac{3m}{2}}V^{\frac{m}{2}}\cdots \widetilde{R}^{m+2}V^2 \widetilde{R}^{m+1}V^1.$$

There are three cases of acceptance in the constructed proof system. In the first case, the verifier $W$ performs the REWINDING TEST and accepts in Step 3.1. This happens with probability $\frac{p_1}{2}$, where

$$p_1 = \|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_1|\psi\rangle\|^2.$$

In the second case, the verifier $W$ performs the REWINDING TEST and accepts in Step 3.5. This happens with probability $\frac{p_2}{2}$, where

$$p_2 = \|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3ZU_2(V^{\frac{m}{2}+1})^\dagger \Pi_{\text{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle\|^2.$$

Finally, in the third case, the verifier $W$ performs the INVERTIBILITY TEST and accepts in Step 4.3. This happens with probability $\frac{p_3}{2}$, where
$$p_3 = \|\Pi_{\text{init}}U_2U_1|\psi\rangle\|^2.$$

Hence, the total probability $p_{\text{acc}}$ that $W$ accepts $x$ when communicating with $R_1', \ldots, R_k'$ is given by $p_{\text{acc}} = \frac{1}{2}(p_1 + p_2 + p_3)$. From the soundness condition of the original proof system, it is obvious that $p_1 \le s$. We shall show that $p_2 \le 1 + 4\sqrt{s} + 4s - p_3$. This implies that $p_{\text{acc}} \le \frac{1}{2} + 2\sqrt{s} + \frac{5s}{2}$, and the soundness condition follows.

Using the triangle inequality, we have that

$$\|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3ZU_2(V^{\frac{m}{2}+1})^\dagger \Pi_{\text{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle\|$$
$$\le \|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3ZU_2(V^{\frac{m}{2}+1})^\dagger \Pi_{\text{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle - \Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3ZU_2U_1|\psi\rangle\|$$
$$+ \|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3ZU_2U_1|\psi\rangle - \Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3Z\Pi_{\text{init}}U_2U_1|\psi\rangle\|$$
$$+ \|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3Z\Pi_{\text{init}}U_2U_1|\psi\rangle\|. \tag{1}$$

The first term of the right-hand side of inequality (1) can be bounded from above as follows:

$$\|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3ZU_2(V^{\frac{m}{2}+1})^\dagger \Pi_{\text{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle - \Pi_{\text{acc}}V^{\frac{m}{2}+1}U_3ZU_2U_1|\psi\rangle\|$$
$$\le \|V^{\frac{m}{2}+1}U_3ZU_2(V^{\frac{m}{2}+1})^\dagger \Pi_{\text{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle - V^{\frac{m}{2}+1}U_3ZU_2U_1|\psi\rangle\|$$
$$= \|(V^{\frac{m}{2}+1})^\dagger \Pi_{\text{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle - U_1|\psi\rangle\| = \|\Pi_{\text{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle - V^{\frac{m}{2}+1}U_1|\psi\rangle\|$$
$$= \|-\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_1|\psi\rangle\| = \|\Pi_{\text{acc}}V^{\frac{m}{2}+1}U_1|\psi\rangle\| = \sqrt{p_1} \le \sqrt{s}.$$

The second term of the right-hand side of inequality (1) can be bounded from above as follows:

$$\|\Pi_{\mathrm{acc}}V^{\frac{m}{2}+1}U_3ZU_2U_1|\psi\rangle - \Pi_{\mathrm{acc}}V^{\frac{m}{2}+1}U_3Z\Pi_{\mathrm{init}}U_2U_1|\psi\rangle\|$$

$$\leq \|V^{\frac{m}{2}+1}U_3ZU_2U_1|\psi\rangle - V^{\frac{m}{2}+1}U_3Z\Pi_{\mathrm{init}}U_2U_1|\psi\rangle\|$$

$$= \|U_2U_1|\psi\rangle - \Pi_{\mathrm{init}}U_2U_1|\psi\rangle\| = \|\Pi_{\mathrm{illegal}}U_2U_1|\psi\rangle\| = \sqrt{1-p_3}.$$

Here the last equality follows from the facts that $U_2U_1|\psi\rangle = \Pi_{\mathrm{init}}U_2U_1|\psi\rangle + \Pi_{\mathrm{illegal}}U_2U_1|\psi\rangle$ is a unit vector, that $\Pi_{\mathrm{init}}U_2U_1|\psi\rangle$ and $\Pi_{\mathrm{illegal}}U_2U_1|\psi\rangle$ are orthogonal, and that $\|\Pi_{\mathrm{init}}U_2U_1|\psi\rangle\|^2 = p_3$.

Finally, since $\Pi_{\mathrm{init}}U_2U_1|\psi\rangle$ is an unnormalized state parallel to some legal initial state and $Z\Pi_{\mathrm{init}} = -\Pi_{\mathrm{init}}$ from the definitions of $Z$ and $\Pi_{\mathrm{init}}$, the third term of the right-hand side of inequality (1) can be bounded as follows by using the soundness condition of the original proof system:

$$\|\Pi_{\mathrm{acc}}V^{\frac{m}{2}+1}U_3Z\Pi_{\mathrm{init}}U_2U_1|\psi\rangle\| = \|-\Pi_{\mathrm{acc}}V^{\frac{m}{2}+1}U_3\Pi_{\mathrm{init}}U_2U_1|\psi\rangle\| = \|\Pi_{\mathrm{acc}}V^{\frac{m}{2}+1}U_3\Pi_{\mathrm{init}}U_2U_1|\psi\rangle\| \leq \sqrt{s}.$$

Putting everything together, we have

$$p_2 = \|\Pi_{\mathrm{acc}}V^{\frac{m}{2}+1}U_3ZU_2(V^{\frac{m}{2}+1})^\dagger \Pi_{\mathrm{rej}}V^{\frac{m}{2}+1}U_1|\psi\rangle\|^2$$

$$\leq (2\sqrt{s} + \sqrt{1-p_3})^2 = 1 + 4\sqrt{s(1-p_3)} + 4s - p_3 \leq 1 + 4\sqrt{s} + 4s - p_3,$$

as desired.  $\square$

Now Theorem 3 follows immediately from Lemma 11 by appropriately applying sequential repetition.

# 4    Parallelizing to Three Turns

In this section we prove Theorem 4, which reduces the number of turns to three without changing the number of provers. This is done by repeatedly converting any $(2^l + 1)$-turn QMIP system into a $(2^{l-1} + 1)$-turn QMIP system where the gap decreases, but is still bounded by an inverse-polynomial. We first show the following lemma.

**Lemma 12.** *Let* $c^2 > s$. *Then,* $\mathrm{QMIP}(k, 4m + 1, c, s) \subseteq \mathrm{QMIP}\left(k, 2m + 1, \frac{1+c}{2}, \frac{1+\sqrt{s}}{2}\right)$.

*Proof.* Let $L$ be a language in $\mathrm{QMIP}(k, 4m + 1, c, s)$ and let $V$ be the corresponding $(4m + 1)$-turn quantum verifier. We construct a $(2m + 1)$-turn quantum verifier $W$ for the new quantum $k$-prover interactive proof system for $L$. The idea is that $W$ first receives the snapshot state that $V$ would have in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ just after the $(2m + 1)$-st turn of the original system. $W$ then executes with equal probability either a forward-simulation of the original system from the $(2m + 1)$-st turn or a backward-simulation of the original system from the $(2m + 1)$-st turn. In the former case, $W$ accepts if and only if the simulation results in acceptance in the original proof system, while in the latter case $W$ accepts if and only if the qubits in $\mathsf{V}$ are in state $|0 \cdots 0\rangle$.[4] The details are given in Figure 2.

*Completeness:* Assume the input $x$ is in $L$. Let $P_1, \ldots, P_k$ be the honest quantum provers in the original proof system with a priori shared state $|\Phi\rangle$. Let $|\psi_{2m+1}\rangle$ be the quantum state in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{P}_1, \ldots, \mathsf{P}_k)$ just after the $(2m + 1)$-st turn in the original proof system. We construct honest provers $R_1, \ldots, R_k$ for the new $(2m + 1)$-turn system. In addition to $\mathsf{V}$ and $\mathsf{M}_1$, $R_1$ prepares $\mathsf{P}_1$ in his private space. Similarly, in addition to $\mathsf{M}_i$, $R_i$ prepares $\mathsf{P}_i$ in his private space for $2 \leq i \leq k$. $R_1, \ldots, R_k$ initially share $|\psi_{2m+1}\rangle$ in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{P}_1, \ldots, \mathsf{P}_k)$. At the first turn of the constructed proof system,

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[4]Recall that in the original proof system the first turn was done by the provers, hence we do not measure the qubits in each $\mathsf{M}_i$ here.

**Verifier's Protocol to Reduce the Number of Turns by Half**

1. Receive $V$ and $M_1$ from the first prover and $M_i$ from the $i$th prover for $2 \le i \le k$.

2. Choose $b \in \{0, 1\}$ uniformly at random.

3. If $b = 0$, execute a forward-simulation of the original proof system as follows:

   3.1 Apply $V^{m+1}$ to the qubits in $(V, M_1, \ldots, M_k)$. Send $b$ and $M_i$ to the $i$th prover, for $1 \le i \le k$.

   3.2 For $j = m + 2$ to $2m$, do the following:
   Receive $M_i$ from the $i$th prover, for $1 \le i \le k$. Apply $V^j$ to the qubits in $(V, M_1, \ldots, M_k)$. Send $M_i$ to the $i$th prover, for $1 \le i \le k$.

   3.3 Receive $M_i$ from the $i$th prover, for $1 \le i \le k$. Apply $V^{2m+1}$ to the qubits in $(V, M_1, \ldots, M_k)$. Accept if the content of $(V, M_1, \ldots, M_k)$ is an accepting state of the original proof system, and reject otherwise.

4. If $b = 1$, execute a backward-simulation of the original proof system as follows:

   4.1 Send $b$ and $M_i$ to the $i$th prover, for $1 \le i \le k$.

   4.2 For $j = m$ down to 2, do the following:
   Receive $M_i$ from the $i$th prover, for $1 \le i \le k$. Apply $(V^j)^\dagger$ to the qubits in $(V, M_1, \ldots, M_k)$. Send $M_i$ to the $i$th prover, for $1 \le i \le k$.

   4.3 Receive $M_i$ from the $i$th prover, for $1 \le i \le k$. Apply $(V^1)^\dagger$ to the qubits in $(V, M_1, \ldots, M_k)$. Accept if the qubits in $V$ are in state $|0 \ldots 0\rangle$, and reject otherwise.

Figure 2: Verifier's protocol to reduce the number of turns by half.

$R_1$ sends $V$ and $M_1$ to $W$, while each $R_i$, for $2 \le i \le k$, sends $M_i$ to $W$. At the $(2j-1)$-st turn for $2 \le j \le m+1$, if $b = 0$, each $R_i$ applies $P_i^{m+j}$ (i.e. $P_i$'s transformation at the $(2m + 2j - 1)$-st turn in the original system) while if $b = 1$, each $R_i$ applies $(P_i^{m-j+3})^\dagger$ (i.e. the inverse of $P_i$'s transformation at the $(2m - 2j + 5)$-th turn in the original system) to the qubits in $(P_i, M_i)$, for $1 \le i \le k$. The provers $R_1, \ldots, R_k$ can then clearly convince $W$ with probability at least $c$ if $b = 0$, and with certainty if $b = 1$. Hence, $W$ accepts every input $x \in L$ with probability at least $\frac{1+c}{2}$.

*Soundness:* Now suppose that $x$ is not in $L$. Let $R'_1, \ldots, R'_k$ be arbitrary provers for the constructed proof system, and let $|\psi\rangle$ be an arbitrary quantum state that represents the state just after the first turn in the constructed system. Suppose that, at the $(2j-1)$-st turn for $2 \le j \le m+1$, each $R'_i$ applies $X_i^j$ if $b = 0$ and $Y_i^j$ if $b = 1$, for $1 \le i \le k$ and write $\widetilde{X}^j = X_1^j \otimes \cdots \otimes X_k^j$ and $\widetilde{Y}^j = Y_1^j \otimes \cdots \otimes Y_k^j$. Define unitary transformations $U_0$ and $U_1$ by $U_0 = V^{2m+1} \widetilde{X}^{m+1} V^{2m} \cdots \widetilde{X}^2 V^{m+1}$ and $U_1 = (V^1)^\dagger \widetilde{Y}^{m+1} \cdots (V^m)^\dagger \widetilde{Y}^2$, and let $|\alpha\rangle = \frac{1}{\|\Pi_{\mathrm{acc}} U_0 |\psi\rangle\|} \Pi_{\mathrm{acc}} U_0 |\psi\rangle$ and $|\beta\rangle = \frac{1}{\|\Pi_{\mathrm{init}} U_1 |\psi\rangle\|} \Pi_{\mathrm{init}} U_1 |\psi\rangle$, where $\Pi_{\mathrm{acc}}$ is the projection onto accepting states in the original proof system and $\Pi_{\mathrm{init}}$ is the projection on $|0 \cdots 0\rangle_V$ in $V$. Then

$$\|\Pi_{\mathrm{acc}} U_0 |\psi\rangle\| = \frac{1}{\|\Pi_{\mathrm{acc}} U_0 |\psi\rangle\|} |\langle\psi| U_0^\dagger \Pi_{\mathrm{acc}} U_0 |\psi\rangle| = F(|\alpha\rangle\langle\alpha|, U_0 |\psi\rangle\langle\psi| U_0^\dagger) = F(U_0^\dagger |\alpha\rangle\langle\alpha| U_0, |\psi\rangle\langle\psi|),$$

and thus, the probability $p_0$ of acceptance when $b = 0$ is given by $p_0 = F(U_0^\dagger |\alpha\rangle\langle\alpha| U_0, |\psi\rangle\langle\psi|)^2$. Similarly, the probability $p_1$ of acceptance when $b = 1$ is given by $p_1 = F(U_1^\dagger |\beta\rangle\langle\beta| U_1, |\psi\rangle\langle\psi|)^2$. Hence the

probability $p_{\mathrm{acc}}$ that $W$ accepts $x$ when communicating with $R'_1, \ldots, R'_k$ is given by

$$p_{\mathrm{acc}} = \frac{1}{2}(p_0 + p_1) = \frac{1}{2}\left(F\big(U_0^\dagger|\alpha\rangle\langle\alpha|U_0, |\psi\rangle\langle\psi|\big)^2 + F\big(U_1^\dagger|\beta\rangle\langle\beta|U_1, |\psi\rangle\langle\psi|\big)^2\right).$$

Therefore, from Lemma 7, we have

$$p_{\mathrm{acc}} \leq \frac{1}{2}\left(1 + F\big(U_0^\dagger|\alpha\rangle\langle\alpha|U_0, U_1^\dagger|\beta\rangle\langle\beta|U_1\big)\right) = \frac{1}{2}\left(1 + F\big(|\alpha\rangle\langle\alpha|, U_0 U_1^\dagger|\beta\rangle\langle\beta|U_1 U_0^\dagger\big)\right).$$

Note that $\Pi_{\mathrm{init}}|\beta\rangle = |\beta\rangle$ and that $|\beta\rangle$ is a legal quantum state which could appear in the original proof system just after the first turn. Hence, from the soundness property of the original proof system,

$$\left\|\Pi_{\mathrm{acc}} U_0 U_1^\dagger|\beta\rangle\right\|^2 = \left\|\Pi_{\mathrm{acc}} V^{2m+1} \widetilde{X}^{m+1} V^{2m} \cdots \widetilde{X}^2 V^{m+1} (\widetilde{Y}^2)^\dagger V^m \cdots (\widetilde{Y}^{m+1})^\dagger V^1|\beta\rangle\right\|^2 \leq s,$$

since $V^1, (\widetilde{Y}^{m+1})^\dagger, \cdots, V^m, (\widetilde{Y}^2)^\dagger, V^{m+1}, \widetilde{X}^2, \cdots, V^{2m}, \widetilde{X}^{m+1}, V^{2m+1}$ form a legal sequence of transformations in the original proof system.

Now, from the fact that $\Pi_{\mathrm{acc}}|\alpha\rangle = |\alpha\rangle$, we have

$$F\big(|\alpha\rangle\langle\alpha|, U_0 U_1^\dagger|\beta\rangle\langle\beta|U_1 U_0^\dagger\big) = \big|\langle\alpha|U_0 U_1^\dagger|\beta\rangle\big| = \big|\langle\alpha|\Pi_{\mathrm{acc}} U_0 U_1^\dagger|\beta\rangle\big| \leq \left\|\Pi_{\mathrm{acc}} U_0 U_1^\dagger|\beta\rangle\right\| \leq \sqrt{s}.$$

Hence the probability $p_{\mathrm{acc}}$ that $W$ accepts $x$ is bounded by $p_{\mathrm{acc}} \leq \frac{1}{2} + \frac{\sqrt{s}}{2}$, which completes the proof. $\square$

Now, by repeatedly applying the construction in the proof of Lemma 12, we can reduce the number of turns to three. The proof is straightforward, but we need to carefully keep track of the efficiency of the constructed verifiers in each application, since the construction is sequentially applied a logarithmic number of times.

**Lemma 13.** *For any $m \geq 4$ and any $c, s$ such that $\varepsilon = 1 - c$ and $\delta = 1 - s$ satisfy $\delta > 2(m-1)\varepsilon$, $\mathrm{QMIP}(k, m, 1 - \varepsilon, 1 - \delta) \subseteq \mathrm{QMIP}\left(k, 3, 1 - \frac{2\varepsilon}{m-1}, 1 - \frac{\delta}{(m-1)^2}\right).$*

*Proof.* Let $l$ be such that $2^l + 1 \leq m \leq 2^{l+1} + 1$. Trivially, $\mathrm{QMIP}(k, m, c, s) \subseteq \mathrm{QMIP}(k, 2^{l+1} + 1, c, s)$. We show $\mathrm{QMIP}(k, 2^{l+1} + 1, 1 - \varepsilon, 1 - \delta) \subseteq \mathrm{QMIP}(k, 3, 1 - \frac{2\varepsilon}{m-1}, 1 - \frac{\delta}{(m-1)^2})$.

Let $L$ be a language in $\mathrm{QMIP}(k, 2^{l+1} + 1, 1 - \varepsilon, 1 - \delta)$ and let $V^{(0)}$ be the corresponding $(2^{l+1} + 1)$-turn quantum verifier. Given a description of $V^{(0)}$ one can compute in polynomial time a description of a $(2^l + 1)$-turn quantum verifier $V^{(1)}$ following the proof of Lemma 12. The resulting proof system has completeness at least $1 - \frac{\varepsilon}{2}$ and soundness at most $\frac{1}{2} + \frac{\sqrt{1-\delta}}{2} \leq 1 - \frac{\delta}{4}$. Crucially, the description of $V^{(1)}$ is at most some constant times the size of the description of $V^{(0)}$ plus an amount bounded by a polynomial in $|x|$. Hence it is obvious that, given a description of $V^{(0)}$, one can compute in polynomial time a description of a three-turn quantum verifier $V^{(l)}$ by repeatedly applying the construction in the proof of Lemma 12 $l$ times. The resulting proof system has completeness at least $1 - \frac{\varepsilon}{2^l} \geq 1 - \frac{2\varepsilon}{m-1}$ and soundness at most $1 - \frac{\delta}{4^l} \leq 1 - \frac{\delta}{(m-1)^2}$, as desired. $\square$

Theorem 4 now follows immediately from Theorem 3 and Lemma 13: For every $p \in \mathrm{poly}$ there is an $m' \in \mathrm{poly}$ such that $\mathrm{QMIP}(k, m, c, s) \subseteq \mathrm{QMIP}(k, m', 1, 2^{-p}) \subseteq \mathrm{QMIP}\left(k, 3, 1, 1 - \frac{1-2^{-p}}{(m'-1)^2}\right)$. Now it suffices to observe that $\frac{1-2^{-p}}{(m'-1)^2} \in \mathrm{poly}^{-1}$.

# 5 Public-Coin Systems

In this section we present the last part to complete the proof of Theorem 1. We show how any three-turn QMIP system with sufficiently large gap can be converted into a two-turn QMIP system with one extra prover, in which the gap is bounded by an inverse-polynomial. Although we also have a direct proof for this, given in Appendix B, we will take a detour by showing how (i) any three-turn QMIP system with sufficiently large gap can be modified to a three-turn *public-coin* QMIP system with inverse-polynomial gap without changing the number of provers, and (ii) any three-turn public-coin QMIP system can be converted into a two-turn QMIP system without changing completeness and soundness, by adding an extra prover. The added benefits of our detour are a proof of the equivalence of public-coin QMIP systems and general QMIP systems (Theorem 5) and a proof that QIP and hence PSPACE has a two-prover one-round quantum interactive proof system of perfect completeness and exponentially small soundness (Corollary 6).[5]

## 5.1 Converting to Public-Coin Systems

In this subsection we prove Theorem 5 showing that any language that has a quantum $k$-prover interactive proof system with two-sided bounded error also has a *public-coin* quantum $k$-prover interactive proof system of perfect completeness and exponentially small soundness.

We first show that any three-turn QMIP system with sufficiently large gap can be modified to a three-turn public-coin QMIP system with the same number of provers and inverse-polynomial gap. In the single-prover case, Marriott and Watrous [MW05] proved a similar statement. Our proof is a generalization of their proof (Theorem 5.4 in Ref. [MW05]) to the multi-prover case.

**Lemma 14.** *For any $c, s$ satisfying $c^2 > s$, $\mathrm{QMIP}(k, 3, c, s) \subseteq \mathrm{QMIP}_{\mathrm{pub}}(k, 3, \frac{1+c}{2}, \frac{1+\sqrt{s}}{2})$. Moreover, the message from the verifier to each prover in the public-coin system consists of only one classical bit.*

*Proof.* Let $L$ be a language in $\mathrm{QMIP}(k, 3, c, s)$ and let $V$ be the corresponding three-turn quantum verifier. We construct a new verifier $W$ for the public-coin system. The idea is that in the first turn $W$ receives the reduced state in the original register $\mathsf{V}$ of the snapshot state just after the second turn (i.e., just after the first transformation of $V$) in the original proof system. $W$ then flips a fair classical coin $b \in \{0, 1\}$ and broadcasts $b$ to the provers. At the third turn the $i$th prover is requested to send the register $\mathsf{M}_i$ of the original proof system, for $1 \leq i \leq k$. If $b = 0$ the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ should form the quantum state the original verifier $V$ would possess just after the third turn of the original proof system. Now $W$ applies $V^2$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ and accepts if and only if the content of $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ is an accepting state of the original proof system. On the other hand, if $b = 1$, the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ should form the quantum state the original verifier $V$ would possess just after the second turn of the original proof system. Now $W$ applies $(V^1)^\dagger$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ and accepts if and only if all the qubits in $\mathsf{V}$ are in state $|0\rangle$. The detailed description of the protocol of $W$ is given in Figure 3. The analysis of completeness and soundness of the constructed proof system is nearly identical to the one in Lemma 12, and is relegated to Appendix A. $\qquad\square$

Theorem 5 now follows directly from Theorem 4 and Lemma 14 together with sequential repetition: Theorem 4 and Lemma 14 imply that there is a $p' \in$ poly such that $\mathrm{QMIP}(k, m, c, s) \subseteq \mathrm{QMIP}\left(k, 3, 1, 1 - \frac{1}{p'}\right) \subseteq \mathrm{QMIP}_{\mathrm{pub}}\left(k, 3, 1, 1 - \frac{1}{4p'}\right)$, since

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[5]The direct proof in Appendix B would only give the weaker corollary that QIP has a two-prover one-round quantum interactive proof system of perfect completeness, but with soundness only exponentially close to $\frac{1}{2}$. This is indeed weaker than what we can show with the detour, since it is not known how to amplify the success probability of QMIP systems without increasing either the number of provers or the number of turns.

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**Verifier's Protocol in Three-Turn Public-Coin System**

1. Receive V from the first prover and receive nothing from the $i$th prover, for $2 \leq i \leq k$.

2. Choose $b \in \{0, 1\}$ uniformly at random. Send $b$ to each prover.

3. Receive $\mathsf{M}_i$ from the $i$th prover for $1 \leq i \leq k$.

   3.1 If $b = 0$, apply $V^2$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Accept if the content of $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ is an accepting state of the original proof system, and reject otherwise.

   3.2 If $b = 1$, apply $(V^1)^\dagger$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Accept if all the qubits in V are in state $|0\rangle$, and reject otherwise.

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Figure 3: Verifier's protocol in three-turn public-coin system.

$\frac{1}{2}\left(1 + \sqrt{1 - \frac{1}{p'}}\right) \leq 1 - \frac{1}{4p'}$. Finally, sequential repetition gives that for all $p \in \mathrm{poly}$ there exists an $m' \in \mathrm{poly}$ such that $\mathrm{QMIP}_{\mathrm{pub}}\left(k, 3, 1, 1 - \frac{1}{4p'}\right) \subseteq \mathrm{QMIP}_{\mathrm{pub}}(k, m', 1, 2^{-p})$.

## 5.2 Parallelizing to Two Turns

Finally, we prove the last piece of Theorem 1 by showing that any three-turn public-coin quantum $k$-prover interactive proof system can be converted into a two-turn (i.e., one-round) $(k+1)$-prover system without changing completeness and soundness. The idea of the proof is to send questions only to the first $k$ provers to request the original second messages from the $k$ provers in the original system and to receive from the $(k+1)$-st prover the original first messages of the $k$ provers in the original system without asking him any question.

**Lemma 15.** $\mathrm{QMIP}_{\mathrm{pub}}(k, 3, c, s) \subseteq \mathrm{QMIP}(k+1, 2, c, s)$.

*Proof.* Let $L$ be a language in $\mathrm{QMIP}_{\mathrm{pub}}(k, m, c, s)$ and let $V$ be the corresponding verifier.

The protocol can be viewed as follows: At the first turn, $V$ first receives a quantum register $\mathsf{M}_i$ from the $i$th prover, for each $1 \leq i \leq k$. $V$ flips a fair classical coin $q_\mathsf{M}$ times to generate a random string $r$ of length $q_\mathsf{M}$, and broadcasts $r$ to all the provers. $V$ also stores $r$ in a quantum register $\mathsf{Q}$ in his private space. Finally, at the third turn, $V$ receives a quantum register $\mathsf{N}_i$ from the $i$th prover, for each $1 \leq i \leq k$. $V$ then prepares a quantum register V for his work space, where all the qubits in V are initialized to state $|0\rangle$, applies the transformation $V^{\mathrm{final}}$ to the qubits in $(\mathsf{Q}, \mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{N}_1, \ldots, \mathsf{N}_k)$, and performs the measurement $\Pi = \{\Pi_{\mathrm{acc}}, \Pi_{\mathrm{rej}}\}$ to decide acceptance or rejection. We construct a two-turn quantum verifier $W$ for the new quantum $(k+1)$-prover interactive proof system for $L$.

The constructed prover $W$ starts with generating a random string $r$ of length $q_\mathsf{M}$ in the first turn, and sends $r$ to the first $k$ provers. $W$ does not send any question to the last prover. In the second turn $W$ receives $\mathsf{N}_i$ from the $i$th prover expecting the original second message from the original $i$th prover, for $1 \leq i \leq k$. From the $(k+1)$-st prover $W$ receives $k$ quantum registers $\mathsf{M}_1, \ldots, \mathsf{M}_k$, expecting the original first messages of the original $k$ provers. $W$ then proceeds like $V$ would. A detailed description of the protocol of $W$ is given in Figure 4.

*Completeness:* Assume the input $x$ is in $L$. Let $P_1, \ldots, P_k$ be the honest provers in the original proof system. Let $|\psi_1\rangle$ be the quantum state in $(\mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{P}_1, \ldots, \mathsf{P}_k)$ in the original proof system just after the first turn. We construct honest provers $R_1, \ldots, R_{k+1}$ for the two-turn system. For $1 \leq i \leq k$, $R_i$ prepares

**Verifier's Protocol in One-Round System**

1. Prepare a quantum register $\mathsf{V}$, and initialize all the qubits in $\mathsf{V}$ to state $|0\rangle$. Flip a fair classical coin $q_\mathsf{M}$ times to generate a random string $r$ of length $q_\mathsf{M}$. Store $r$ in a quantum register $\mathsf{Q}$, and send $r$ to the $i$th prover for $1 \leq i \leq k$. Send nothing to the $(k+1)$-st prover.

2. Receive a quantum register $\mathsf{N}_i$ from the $i$th prover, for $1 \leq i \leq k$, and $k$ quantum registers $\mathsf{M}_1, \ldots, \mathsf{M}_k$ from the $(k+1)$-st prover. Apply $V^{\text{final}}$ to the qubits in $(\mathsf{Q}, \mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{N}_1, \ldots, \mathsf{N}_k)$ and accept if and only if the content of $(\mathsf{Q}, \mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{N}_1, \ldots, \mathsf{N}_k)$ is an accepting state of the original proof system.

Figure 4: Verifier's protocol to reduce the number of turns to two.

quantum register $\mathsf{P}_i$ in his private space, where some of the qubits in $\mathsf{P}_i$ form the quantum register $\mathsf{N}_i$, while $R_{k+1}$ prepares the quantum registers $\mathsf{M}_1, \ldots, \mathsf{M}_k$ in his private space. $R_1, \ldots, R_{k+1}$ initially share $|\psi_1\rangle$ in $(\mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{P}_1, \ldots, \mathsf{P}_k)$. At the second turn, $R_{k+1}$ just sends the qubits in $(\mathsf{M}_1, \ldots, \mathsf{M}_k)$ to $W$, while each $R_i$, after receiving $r$, just behaves like $P_i$ would at the third turn of the original system, and then sends $\mathsf{N}_i$, which is a part of $\mathsf{P}_i$, to $W$, for $1 \leq i \leq k$. It is obvious from the construction that the provers $R_1, \ldots, R_{k+1}$ can convince $W$ with the same probability with which $P_1, \ldots, P_k$ could convince $V$, which is at least $c$.

*Soundness:* Now assume the input $x$ is not in $L$. Let $R'_1, \ldots, R'_{k+1}$ be any provers for the constructed proof system and let $\mathsf{R}'_i$ be the quantum register consisting of all the qubits in the private space of $R'_i$, for $1 \leq i \leq k+1$. For $R'_{k+1}$, some of the qubits in $\mathsf{R}'_{k+1}$ form the register $\mathsf{M} = (\mathsf{M}_1, \ldots, \mathsf{M}_k)$. Let $|\psi\rangle$ be an arbitrary quantum state in $(\mathsf{R}'_1, \ldots, \mathsf{R}'_{k+1})$ that is initially shared by $R'_1, \ldots, R'_{k+1}$. Suppose that, at the second turn, each $R'_i$ applies $X_i^{(r)}$, for $1 \leq i \leq k$, if the message from $W$ is $r$. Without loss of generality, we assume that $R'_{k+1}$ does nothing, and just sends the qubits in $(\mathsf{M}_1, \ldots, \mathsf{M}_k)$ at the second turn, since $R'_{k+1}$ receives nothing from $W$ (that $R'_{k+1}$ applies some transformation $Z$ is equivalent to sharing $Z|\psi\rangle$ at the beginning).

Consider three-turn quantum provers $P'_1, \ldots, P'_k$ for the original proof system with the following properties: (1) each $P'_i$ prepares the quantum register $\mathsf{R}'_i$ in his private space, for $1 \leq i \leq k$, (2) $P'_1, \ldots, P'_k$ initially share $|\psi\rangle$ in $(\mathsf{R}'_1, \ldots, \mathsf{R}'_{k+1})$, where all the qubits in $\mathsf{R}'_{k+1}$ except for those in $(\mathsf{M}_1, \ldots, \mathsf{M}_k)$ are shared arbitrarily, (3) at the first turn, each $P'_i$ sends $\mathsf{M}_i$ to $V$, for $1 \leq i \leq k$, and (4) if the message from $V$ is $r$, at the third turn, each $P'_i$ applies $X_i^{(r)}$ to the qubits in $\mathsf{R}'_i$, for $1 \leq i \leq k$. It is obvious that these provers $P'_1, \ldots, P'_k$ can convince the original verifier $V$ with the same probability that $R'_1, \ldots, R'_{k+1}$ can convince $W$. Hence, the probability $W$ accepts $x$ is at most $s$, as desired. $\square$

Now Theorem 1 follows from Theorem 4 and Lemmas 14 and 15. Corollary 6, claiming $\text{QIP} \subseteq \text{QMIP}(2, 2, 1, 2^{-p})$ for any $p \in \text{poly}$ follows directly from Lemma 15 and the fact shown by Marriott and Watrous [MW05] that any language in QIP can be verified by a three-message public-coin quantum interactive proof system of perfect completeness with exponentially small error in soundness (i.e., $\text{QIP} \subseteq \text{QMAM}(1, 2^{-p})$ for any $p \in \text{poly}$).

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# Appendix

## A    Proof of Lemma 14

*Proof. Completeness:* Assume the input $x$ is in $L$. Let $P_1, \ldots, P_k$ be the honest quantum provers in the original proof system with a priori shared state $|\Phi\rangle$ in $(\mathsf{P}_1, \ldots, \mathsf{P}_k)$. Let $|\psi_2\rangle$ be the quantum state in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{P}_1, \ldots, \mathsf{P}_k)$ just after the second turn in the original proof system. We construct honest provers $R_1, \ldots, R_k$ for the public-coin system. In addition to $\mathsf{V}$ and $\mathsf{M}_1$, $R_1$ prepares $\mathsf{P}_1$ in his private space. Similarly, in addition to $\mathsf{M}_i$, $R_i$ prepares $\mathsf{P}_i$ in his private space, for $2 \leq i \leq k$. $R_1, \ldots, R_k$ initially share $|\psi_2\rangle$ in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k, \mathsf{P}_1, \ldots, \mathsf{P}_k)$. At the first turn of the constructed proof system, $R_1$ sends $\mathsf{V}$ to $W$, while each $R_i$, $2 \leq i \leq k$ send nothing to $W$. At the third turn, if $b = 0$ each $R_i$ applies $P_i^2$ to the qubits in $(\mathsf{M}_i, \mathsf{P}_i)$ and then sends $\mathsf{M}_i$ to $W$, while if $b = 1$, each $R_i$ does nothing and sends $\mathsf{M}_i$ to $W$. It is obvious that the provers $R_1, \ldots, R_k$ can convince $W$ with probability at least $c$ if $b = 0$, and with certainty if $b = 1$. Hence, $W$ accepts every input $x \in L$ with probability at least $\frac{1+c}{2}$.

*Soundness:* Now suppose that $x$ is not in $L$. Let $R'_1, \ldots, R'_k$ be arbitrary provers for the constructed proof system, and let $|\psi\rangle$ be an arbitrary quantum state that represents the state just after the first turn in the constructed system. Suppose that at the third turn each $R'_i$ applies $X_i$ if $b = 0$ and $Y_i$ if $b = 1$, for $1 \leq i \leq k$ and write $\widetilde{X} = X_1 \otimes \cdots \otimes X_k$ and $\widetilde{Y} = Y_1 \otimes \cdots \otimes Y_k$. Note that $\widetilde{X}$ and $\widetilde{Y}$ are unitary transformations that do not act over the qubits in $\mathsf{V}$. Let $|\alpha\rangle = \frac{1}{\|\Pi_{\mathrm{acc}} V^2 \widetilde{X}|\psi\rangle\|} \Pi_{\mathrm{acc}} V^2 \widetilde{X} |\psi\rangle$ and $|\beta\rangle = \frac{1}{\|\Pi_{\mathrm{init}}(V^1)^\dagger \widetilde{Y}|\psi\rangle\|} \Pi_{\mathrm{init}}(V^1)^\dagger \widetilde{Y} |\psi\rangle$, where $\Pi_{\mathrm{acc}}$ is the projection onto accepting states in the original proof system and $\Pi_{\mathrm{init}}$ is the projection onto states in which all the qubits in $\mathsf{V}$ are in state $|0\rangle$.

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**Verifier's Protocol in One-Round System (Direct Construction)**

1. Choose $b \in \{0, 1\}$ uniformly at random. Send $b$ only to the first $k$ provers, and send nothing to the $(k+1)$-st prover.

2. Receive $\mathsf{M}_i$ from the $i$th prover, for $1 \le i \le k$, and $\mathsf{V}$ from the $(k+1)$-st prover.

   2.1 If $b = 0$, apply $V^2$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Accept if the content of $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$ is an accepting state in the original proof system, and reject otherwise.

   2.2 If $b = 1$, apply $(V^1)^\dagger$ to the qubits in $(\mathsf{V}, \mathsf{M}_1, \ldots, \mathsf{M}_k)$. Accept if all the qubits in $\mathsf{V}$ are in state $|0\rangle$, and reject otherwise.

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Figure 5: Verifier's protocol to reduce the number of turns to two (direct construction).

Then, with a similar argument to that in the proof of Lemma 12, the probability $p_{\mathrm{acc}}$ that $W$ accepts $x$ when communicating with $R'_1, \ldots, R'_{k+1}$ is bounded by

$$
p_{\mathrm{acc}} \le \frac{1}{2} \left( 1 + F\big( \widetilde{X}^\dagger (V^2)^\dagger |\alpha\rangle\langle\alpha| V^2 \widetilde{X}, \widetilde{Y}^\dagger V^1 |\beta\rangle\langle\beta| (V^1)^\dagger \widetilde{Y} \big) \right)
$$
$$
= \frac{1}{2} \left( 1 + F\big( |\alpha\rangle\langle\alpha|, V^2 \widetilde{X}\widetilde{Y}^\dagger V^1 |\beta\rangle\langle\beta| (V^1)^\dagger \widetilde{Y}\widetilde{X}^\dagger (V^2)^\dagger \big) \right).
$$

Since $\Pi_{\mathrm{init}}|\beta\rangle = |\beta\rangle$ is a legal quantum state which could appear just after the first turn in the original proof system, $V^1, \big(\widetilde{X}\widetilde{Y}^\dagger\big), V^2$ form a legal sequence of transformations in the original proof system, and $\Pi_{\mathrm{acc}}|\alpha\rangle = |\alpha\rangle$, again a similar argument to that in the proof of Lemma 12 shows that $F\big( |\alpha\rangle\langle\alpha|, V^2 \widetilde{X}\widetilde{Y}^\dagger V^1 |\beta\rangle\langle\beta| (V^1)^\dagger \widetilde{Y}\widetilde{X}^\dagger (V^2)^\dagger \big) \le \sqrt{s}$.

Hence the probability $p_{\mathrm{acc}}$ that $W$ accepts $x$ is bounded by $p_{\mathrm{acc}} \le \frac{1}{2} + \frac{\sqrt{s}}{2}$, as desired. $\qquad\square$

# B   Direct Proof of Modifying Three-Turn Systems to Two-Turn Systems

For completeness, here we give a direct proof of the fact that any $k$-prover three-turn system can be converted into a $(k+1)$-prover two-turn system.

**Theorem 16.** *For any $c, s$ satisfying $c^2 > s$, $\mathrm{QMIP}(k, 3, c, s) \subseteq \mathrm{QMIP}\left( k+1, 2, \frac{1+c}{2}, \frac{1+\sqrt{s}}{2} \right)$.*

*Proof.* The proof is very similar to that of Lemma 14. Indeed, our starting point is the same, but this time we move to a two-turn proof system, instead of a three-turn public-coin system, by adding an extra prover. As in Lemma 15, we first broadcast a random bit $b \in \{0, 1\}$ to all but the extra prover, and ask the extra prover to send us a register $\mathsf{V}$ and the other provers to send us registers $\mathsf{M}_i$. We then proceed as in Step 3 of the proof system given in Lemma 14: a detailed description is given in Figure 5.

*Completeness:* This follows immediately from the completeness of the proof system in Lemma 14: in Lemma 14 the first prover sends both $\mathsf{V}$ (before receiving the bit $b$) and $\mathsf{M}_1$ (after); here we can imagine that before the protocol starts the first prover gives register $\mathsf{V}$ to the extra $(k+1)$-st prover, who sends it to $\mathsf{V}$.

*Soundness:* This also follows from the soundness of the proof system in Lemma 14: by combining the actions of the first prover and the extra $(k+1)$-st prover (and thus making the provers only stronger), we can construct a set of provers that would succeed in the proof system of Lemma 14 with the same probability as they succeed here. $\qquad\square$