A focused laser differential interferometer (FLDI) has been experimentally characterized. The static response was probed using a steady, laminar helium jet, and the dynamic response was investigated using a free ultrasonic acoustic beam. In the case of the jet, the refractive index field was independently measured using a Mach-Zehnder interferometer operating simultaneously alongside FLDI. The experimental data were compared with numerical simulations of the FLDI response based on geometric optics and a ray-tracing algorithm. Close quantitative agreement was found between data and simulation results, validating this approach to modeling FLDI performance. Emphasis was given to quantification of the spatial sensitivity of the system, a key characteristic of FLDI, especially when applied to hypersonic ground testing facilities where strong turbulent flow exists outside the core flow.

I. Nomenclature

FLDI = Focused Laser Differential Interferometry  
MZI = Mach-Zehnder Interferometry  
\( \lambda \) = wavelength (of spatial density disturbances, or of laser emission)  
\((x, y, z)\) = Cartesian co-ordinate system centered at FLDI foci centroid  
\( \Delta x \) = separation of foci centers at \( z = 0 \) (plane of best focus)  
\( d_0 \) = Gaussian diameter of foci at \( z = 0 \)  
\( \Delta \Phi \) = total/integrated phase change measured at photodetector  
\( n \) = refractive index  
\( \Delta n \) = difference in refractive index between two spatial locations  
\( p \) = separation between Wollaston prism and focusing lens  
\( FoV \) = field of view  
\( f \) = driven frequency of ultrasound transducer

II. Introduction

Invented by Smeets and George [11] in 1973 at the French-German Research Institute of Saint-Louis, focused laser differential interferometry (FLDI) is an optical technique which features test and reference beams of opposite polarization, both focused into the test volume with close spatial proximity.

One of the attractive features of FLDI for flow investigation is its high bandwidth. Sensors immersed in the flow tend to have inertial or resonance properties that limit their response at high frequencies, e.g. hot-wire anemometers have finite thermal capacity; pressure transducers have inertia in their active components, as well as resonant frequencies both of the device itself, and of the cavity if recessed from the model surface. Being optically-based, FLDI’s frequency response is limited only by the photo-detection component. This is of particular relevance in hypersonic flows, where frequencies of interest for transition studies (e.g. the second, or Mack, mode) can extend into the MHz range.

Besides this high-frequency capability, two other aspects of FLDI’s response are of particular interest: fine spatial resolution in the vicinity of the beam foci, and reduced sensitivity to optical disturbances in regions of the beams away from the foci. The benefits of these features for use in wind-tunnel measurements are given in more detail in several of the works cited in this section. With all these features in mind, FLDI is a promising technique for making measurements both in the freestream and within boundary layers over models in hypervelocity ground testing facilities.
FLDI has seen a recent surge in interest [2-9], which benefits from modern advances in electronics not available in the 1970s. Modern solid-state photo-detectors and high-bandwidth signal acquisition electronics have made it possible to use FLDI to measure spectral content into the MHz range.


Promising results have been obtained with FLDI, and comparisons made with other flow-sensing methods such as hot-wire anemometry [9]. Advances have been made in the theoretical modeling of FLDI as an optical system, and these models have been compared with experimental results by both Schmidt and Shepherd [7], and Settles and Fulghum [9]. These theoretical models have further been used to develop analytical solutions for particular flow cases, and in the creation of a numerical FLDI simulation code.

Further understanding of FLDI’s performance is required, particularly regarding its application to wind tunnels. Although previous works describe experimental validations, additional quantitative data are needed, where the reference phase object is more precisely known via independent measurements, and more extensively measured by FLDI.

This work intends to be a step towards a more complete characterization and understanding of how FLDI behaves. Section III reviews previous experimental and theoretical efforts toward understanding and applying FLDI. With these in mind, Section IV details the new contributions that we wish to make towards this understanding. Sections V and VI detail the design of the FLDI system and the experimental characterization methods used, with results and analysis following in Sections VII and VIII.

III. Previous Characterization Work

Parziale et al. derived an approximation for the FLDI response for a laser wavelength \( \lambda_0 \):

\[
\Delta \Phi = \frac{2\pi}{\lambda_0} \Delta OPL \approx \frac{2\pi}{\lambda_0} L \Delta n
\]  

This is a fully-averaged approach where the difference in optical path length, \( \Delta OPL \), between the two beams is approximated as \( L \Delta n \), where \( L \) represents a ‘sensitive length’ about the plane of best focus, outside of which there is negligible contribution to the instrument response; \( \Delta n \) is the average difference in refractive index between the two beams over this length. Because their quantity of interest is a fluctuation about some mean local density, the Gladstone-Dale relationship is applied along with a linearization of the photodetector response, to yield:

\[
\frac{\Delta \rho}{\rho_L} = \frac{\lambda_0}{2\pi K \rho_L} \sin^{-1} \left( \frac{V}{V_0} - 1 \right)
\]  

where \( V_0 \) is the linearization voltage of the photodetector, \( \rho_L \) is the mean local density, and \( K \) is the Gladstone-Dale constant. Parziale’s key assumption is that the spatial sensitivity of FLDI is due to the overlap of the two beams except in the vicinity of the plane of best focus.

In Ref. [5], \( L \) is determined via geometrical considerations of the beam sizes and spacing, which are in turn calculated from the ideal parameters of the optical components used, rather than directly measured. In this way, it was determined that \( L = \pm 10 \text{ mm} \) about the focal plane, for a particular FLDI system with foci separation of 350 \( \mu \text{m} \) and diameters less than 100 \( \mu \text{m} \). An experiment was performed to validate the prediction of \( L \) [2, 6], where an uncharacterized subsonic turbulent CO\(_2\) jet was traversed away from the beam focus. In this case, \( L \) was defined as the distance in which the response fell off to \( 1/e \) of the peak value. This distance was found to be close to 10 mm for a jet nozzle diameter on the order of millimeters, which was used as the value of \( L \) in Equation (3). Parziale also considered the response of FLDI to density fluctuations of certain wavelength and propagation direction, and within this framework developed a response coefficient for wavelength, \( \lambda \):

\[
c(\lambda) = \sin \left( \frac{2\pi \Delta x/2}{\lambda} \right)
\]  

where \( \Delta x \) is the spacing between the foci. This coefficient was used to correct the output obtained from Equation (2). Note that Equation (3) predicts reduced FLDI response with increasing disturbance frequency.
Further refinement of the theory behind FLDI’s response was developed by Settles and Fulghum [8, 9]. A mathematical description of the entire optical system was derived using Jones matrices. This theory was used to generate an analytical transfer function for fluctuation wavenumber, in the case of turbulent jets with Gaussian velocity profiles.

They obtained $\Delta x$ through an indirect experimental measurement, rather than via geometric optics calculations as done by Parziale. Experimentally, they also compared the response of FLDI to that of hot-wire anemometry in the presence of a sonic air-jet. They found good agreement for $x/D \geq 20$ (where $x$ is the stream-wise distance and $D$ is the jet diameter), however significant discrepancy was observed between the two for $x/D < 20$, the cause of which is unknown.

Settles and Fulghum consider the spatial sensitivity as an averaging effect, which is a function of the relative sizes of the local beam diameter and the fluctuation wavelength. The qualitative shape of the sensitive region is discussed, along with its dependence on the choice of optics. However, they do not directly address the significance of beam overlap and the possible common-mode rejection as postulated by Parziale.

Schmidt and Shepherd [7] developed a theoretical model for an FLDI transfer function. They performed a ray-tracing method that was implemented numerically. The key result is:

$$\Delta \Phi = \frac{2\pi}{\lambda} \int_D I_0(\xi, \eta) \left( \int_{s_1}^{D(\xi, \eta)} n(x_1) ds_1 - \int_{s_2}^{D(\xi, \eta)} n(x_2) ds_2 \right) d\xi d\eta \tag{4}$$

where two equivalent ray paths, one in each beam, are parameterized by $s_1$ and $s_2$ in a co-ordinate frame defined by $\chi = (x, y, z)$, and recombine at some point $(\xi, \eta)$ on the detector face $D$. $I_0(\xi, \eta)$ is the unperturbed intensity distribution over this face. This yields the total phase change $\Delta \Phi$ measured at the detector. Equation (4) represents a more general class of interferometer than FLDI. They further develop a discretization scheme that generates the beam profiles seen in FLDI. The reader is referred to Ref. [7] for further details and assumptions regarding this model.

Separately to this computational scheme, Schmidt and Shepherd developed several analytical wavenumber-dependent transfer functions for different disturbance geometries. The output from the computational model was verified by comparison with the predictions of these analytical functions. Experimental verification was also carried out using an argon jet, where the FLDI foci were traversed to the location in the jet which yielded maximum signal at the photodetector. The corresponding phase shift was then compared with the computational prediction, where the refractive index field input for the code was obtained via a separate numerical simulation of a jet similar to the one used in the experiment. Good agreement was found for both these verification methods.

Schmidt contends that beam overlap is not the mechanism for signal rejection. He shows that spatial sensitivity, as quantified by the analytical transfer functions, is a strong function of both disturbance wavenumber and local beam width. As a consequence, low-wavenumber disturbances may still contribute significantly to FLDI response even at locations far from the focal plane of the instrument, whereas a flow dominated by high-wavenumber disturbances would give a very localized sensitive region. Thus, the ‘sensitive length’ of FLDI is not fixed by beam geometry for a given optical setup, but rather is dependent on the frequency content and spatial extent of the flow under investigation.

It is clear that the phenomenon of spatial sensitivity is central to the appeal of FLDI as a flow-measurement technique. Previous research has recognized this, with experiments showing the existence of the effect, and multiple theoretical approaches have addressed possible mechanisms for it. However, further quantification is still required, which this work hopes to contribute towards.

IV. Objectives

In the previous experimental validation studies outlined in Section [11], the phase object used to probe the FLDI response was either quantitatively uncharacterized, or indirectly characterized via simulation. The spatial positioning of the foci with respect to the phase object was generally not known to a precision comparable with the FLDI characteristic dimensions (typically the foci spacing is on the order of hundreds of $\mu$m, and the foci diameters themselves are tens of $\mu$m). Hence, in this work, we aim to more fully and directly characterize the phase object by simultaneously making measurements using an imaging interferometer whose response is already understood. We employ Mach-Zehnder interferometry (MZI) [11] for this purpose, as our wider research group already has experience in its use, and has developed the post-processing tools necessary for extracting refractive index fields from the raw fringe patterns [11].

The primary goals of this work are to obtain experimentally the response of an FLDI system to a known refractive index field, to use these data to validate Schmidt’s computational model, and to quantify the spatial and temporal
response of the instrument to known index of refraction fields in the vicinity of the focal region. To simplify analysis, we wish to explore two separate cases: the ‘static’ response, where \( n = n(\vec{x}) \) only, and the ‘dynamic’ response, where \( n = n(t) \) only. In the latter case, it is difficult to obtain a temporally-varying but spatially-uniform refractive index field in a laboratory setting, so instead, we aim to generate a field that varies temporally at a known and controllable frequency, in such a way that we can normalize the spatial variation and thereby characterize the frequency response of the FLDI.

The secondary goal of this work is to use Schmidt’s model (or a variant, if the validation experiments show that modifications are required) to explore the FLDI response as a function of the optical system parameters, such as the foci size and spacing. The purpose of this is to better understand how accurately these parameters need to be known, as well as informing the design of new FLDI systems for a particular application.

V. System Design

The FLDI system characterized in this work is shown in Figure 1. As indicated, the co-ordinate system is defined such that \( z \) is in the beam propagation direction, with \( x \) and \( y \) parallel and perpendicular to the optical table, respectively. The system light source is a 200 mW, 532 nm laser. This laser was chosen for its stability as laser power fluctuations would result in FLDI signal noise. The laser light source is first expanded through a diverging lens, then linearly polarized at 45° prior to passing through a Wollaston prism. The Wollaston prism splits the beams by a small angle in the \( x-z \) plane. This results in two orthogonally-polarized beams diverging at the constant prism angle. The two beams are next passed through a focusing lens to focus each beam down to a point in the test section. The foci are separated by some amount, \( \Delta x \), at the point of best focus, which is set by the distance between the Wollaston prism and the focusing lens (distance \( p \) in Figure 1). At the point of best focus, for the system in Table 1, the foci have Gaussian diameters \( d_0 = 80 \mu m \) and are separated by \( \Delta x = 180 \mu m \). These measurements were obtained directly by using a beam profiler. The beams are then passed through a symmetric setup to recombine into a single re-polarized beam and focus onto the photodetector. The optics used for the FLDI system characterized in this work are given in Table 1. Note that in all following discussion, \((x, y, z)\) form a co-ordinate system with its origin at the foci centroid, i.e. at the point midway between the two foci, in the plane of best focus.

<table>
<thead>
<tr>
<th>Description</th>
<th>Specification</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laser</td>
<td>Spectra-Physics EXLSR-532-200-CDRH</td>
<td>1</td>
</tr>
<tr>
<td>Diverging lens</td>
<td>Thorlabs LD4797 (fused silica, ( f = -6 \text{ mm} ))</td>
<td>1</td>
</tr>
<tr>
<td>Linear polarizer</td>
<td>Thorlabs LPVISC100-MP2</td>
<td>2</td>
</tr>
<tr>
<td>Wollaston prism</td>
<td>United Crystals (2 arcmin, 20 × 20 mm)</td>
<td>2</td>
</tr>
<tr>
<td>Focusing lens</td>
<td>Thorlabs AC508-300 (( f = 300 \text{ mm} ))</td>
<td>2</td>
</tr>
<tr>
<td>Photodetector</td>
<td>Thorlabs PDA36A2</td>
<td>1</td>
</tr>
<tr>
<td>Beam profiler</td>
<td>Thorlabs BP209-VIS</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 1  FLDI optical component specifications**

![Fig. 1  Schematic plan view of FLDI. Origin of \((x, y, z)\) is at foci centroid.](image)

The location of the second Wollaston prism and polarizer is set such that the interferometer is in an infinite fringe
configuration. This means that as the prism and polarizer are translated along the x–axis the image on the photodetector brightens and darkens uniformly. The FLDI is initially configured to have an intermediate brightness level, i.e. a photodetector voltage output approximately midway between the maximum and minimum voltages corresponding to fully in-phase and antiphase, respectively.

The Mach-Zehnder interferometer uses the same type of laser as the FLDI system described above, with a 90/10 beam splitter to dump 90% of the power; this is done to protect the camera used to capture the interferogram. The beam expander, BE-1 in Figure 2, brings the beam to a diameter of 25 mm. An arrangement of beam-splitters and mirrors creates and subsequently recombines the measurement and reference beams. The recombined beam passes through a 750 mm focal length lens, F-2, and two additional turning mirrors to focus on the camera. The camera used to capture the interferogram is a Phantom v710. The lens focal length and the lens and camera placement were chosen to give a field of view in the test section roughly 5 mm high by 8 mm wide. The MZI is constructed such that the reference and measurement beams have the same path lengths. Thus, when the measurement beam is undisturbed the two beams do not interfere and no fringes are shown on the interferogram; this is referred to as the infinite fringe configuration. However, unlike the FLDI system, the Mach-Zehnder interferometer is not used in this configuration. To create fringes, the second beam-splitter is tilted; the greater the tilt angle, the greater the number of fringes. The orientation of the fringes is controlled by tilting the turning mirror in the reference beam path. For this work, the MZI was set up with horizontal fringes and a fringe density of about 10.8 fringes/mm. When the measurement beam is perturbed by a phase object the horizontal fringes are deflected. This fringe deflection can be used to calculate the refractive index field as described in Coronel et al. [11].

Fig. 2 Schematic diagram of FLDI and MZI simultaneously observing the test section. MZI beams shown in green. Polarized FLDI beams shown in false color for clarity. Annotations: L = laser, BE = beam expander, BS = beam-splitter, M = flat mirror, F = focusing lens, P = polarizer, WP = Wollaston prism, PD = photodetector, CCD = charge-coupled device (camera), TS = test section.

VI. Characterization Methods

Here we present two different methods of system characterization that address these responses: for the static response, we use a steady laminar jet; for the dynamic response, we use an ultrasonic acoustic beam radiating into quiescent air.

A. Laminar Jet

A laminar jet was used to characterize the static phase response of the FLDI, with the jet axis perpendicular to the optical table, i.e. in the y-direction of the previously-defined FLDI co-ordinate system. A round laminar jet was
chosen because the index of refraction of the axisymmetric flow field can be calculated via Abel inversion from the Mach-Zehnder interferograms. Additionally, the axisymmetry allows the MZI to observe the jet orthogonally to the FLDI beam direction (see Figure 2).

The jet assembly is shown in Figure 3a, and consists of a small reservoir and a nozzle made from 3D-printed ABS-M30. Bottled helium is fed into the reservoir and the flow rate is controlled using a two-stage pressure regulator and is measured using a rotameter. The rotameter was used to ensure the flow rate remained steady during and constant between experiments. Helium was chosen as the gas because its refractive index is different enough from air such that the MZI can resolve the jet. Other common gases have similar $\Delta n$ with respect to air (e.g. CO$_2$), but He is lighter than air so the plume will not affect the measurement volume by returning via a ‘geyser’ effect. The reservoir is filled with a volume of 20-40 mesh size (400–800 $\mu$m diameter) corn-cob abrasive material. The corn-cob is held in place with woven-wire steel cloth with 230 $\mu$m square openings. The corn-cob is used to straighten the flow entering the nozzle and ensure the flow is uniform across the nozzle entrance plane. The nozzle profile is given by Equation (5), where $C_1$ is the vertical difference between the nozzle inlet and throat, and $C_2$ is the length of the nozzle. The nozzle used in these experiments has an inlet diameter of 12.7 mm and a nominal outlet diameter of 0.5 mm. A secondary check was performed to ensure the jet was laminar: by occulting the reference beam, and placing a knife-edge near the focus located beyond focusing lens F-2 (Figure 2), a schlieren system was created. With this, we observed video footage of the jet, and selected a fixed flow-rate that was steady and laminar. This flow-rate was approximately 36 cm$^3$ s$^{-1}$, which corresponds to a Reynolds number of approximately 390 based on nozzle diameter.

$$y = C_1 \cos \left( \frac{\pi x}{C_2} \right)$$

(a) The jet assembly is shown in cross-section, mounted on translation stages. Selected volumes where axisymmetric jet data are collected using MZI are indicated with a dashed box. Orientation of FLDI foci are indicated with green circles.

(b) A typical traverse pattern performed by the motorized translation stages. The x-increment is 0.1 mm.

**Fig. 3** Schematics of the laminar jet experiment setup.

The laminar jet was mounted on three translational stages. The bottom stage has a 25 mm range and a Vernier scale allowing for 0.01 mm increments. This stage was used to translate the jet in 5 mm increments along the z axis. The other two stages were Thorlabs MT1-Z8 motorized translation stages. One stage was orientated to translate in the x-axis and the other stage orthogonally in the z-axis. The stages have a 12 mm total range and minimum bidirectional repeatability of 1.5 $\mu$m. A typical traverse pattern used in the experiments is shown in Figure 3b. For each of the z locations of the bottom stage, the motorized stages were used to translate the jet a total of 8 mm in 0.1 mm increments across the FLDI foci in the x-direction and a total of 12 mm in 1 mm increments in the z-direction. This resulted in a total z range of 37 mm, containing, but not necessarily centered on, the plane of best focus. Each of these automated traverse
patterns had more than 50% overlap with the corresponding traverse at the next position on the manual translation stage, and furthermore, the entire domain was covered multiple times over the experimental campaign, allowing several independent measurements to be made at every \((x, z)\) co-ordinate. Data was acquired at each position for 2 seconds.

Data acquisition was performed with a Yokogawa DL850 oscilloscope. During an automated \(x-z\) traverse, the photodetector voltage was recorded continuously, as well as the TTL pulses of the motorized stages. These pulses, coupled with knowledge of the trajectory, were used to extract voltage time-series during the dwell time at each position. Only the second half of each time-series was preserved, to lessen any residual motion in the jet from the stage movement. The sample rate was such that even this truncated series still had a record length on the order of \(10^3-10^4\). This was then used to calculate an average voltage at each location.

Referring to Figure \ref{fig:interferometer}, both interferometers are stationary with respect to the optical table. The test phase object (in this case, the laminar jet) moves. This means that the fringe image of the jet moves in the MZI field-of-view (FoV). In order to simultaneously obtain positioning and scaling information for the FoV, a small drill bit \([#71, \sigma 0.026^º]\), was inserted with a tight fit in the jet nozzle. The jet assembly was then traversed, and the drill bit imaged over a range of positions, with the reference arm of the MZI blocked. In this way, a series of fringeless images of the drill bit were obtained at known locations. An edge-finding algorithm was then applied to extrapolate the location of the jet nozzle orifice; furthermore, the drill bit diameter was accurately measured and used to obtain the scale of the images.

The magnification of the MZI FoV was chosen such that the jet diameter was at least several fringe widths. Given that the interferograms had approximately 50–100 fringes, and that the jet nozzle diameter is 0.1 mm, the scale factor was approximately \(6 \times 10^{-6}\) m/px. The camera gives \(1280 \times 800\) px images, resulting in a field of view about 7.7 mm wide by 4.8 mm high. The jet assembly traverses considerably larger distances than this, which means the jet was not visible to MZI at every \((x, z)\). However, because the jet flow is nominally steady, we were able to take a large number of interferograms while the jet was visible. Using the database of jet origin locations described above, each unwrapped phase map from the raw interferograms was subject to Abel inversion about the vertical axis of symmetry passing through the jet origin. This gives a field of refractive index. The jet axes of many such refractive index fields were aligned to give a smoother average field. Because Abel inversion is inherently axisymmetric, it only requires one half of the unwrapped phase map, with respect to the axis of symmetry. Because the jet was often not in the center of the image, we chose to take the larger half for a given image. This meant that the averaged field is able to have a radius that exceeds half the width of the field of view, giving more data in the far-field than if we just used a single centered jet image.

The \(y\)-positioning was performed with a height dial gauge, accurate to 0.0254 mm. The sharp and straight edge of the gauge was used to occult the FLDI beams at multiple \(z\) locations, while observing the projection of the diverged beams through laser glasses. By finding the midpoint of the two heights corresponding to the onset and completion of occultation, the height of the beam centerline from the optical table could be obtained. Excellent consistency was obtained, with a standard deviation of \(70\) \(\mu\)m (for reference, this is approximately the FLDI foci diameter). The gauge was also used to measure the height of the jet nozzle orifice from the optical table, and images of the gauge at a known height were obtained via MZI in the same way as the drill bit. In this way, the relative heights of the jet, the FLDI foci, and the MZI FoV were calculated with high precision.

Various external methods of locating the relative \((x, z)\) locations of the jet and the FLDI foci were posited, however no sufficiently precise practical method was found. Ultimately, we decided to use the anticipated symmetry of the FLDI response itself to determine the location of the origin (i.e. the location where the foci centroid lies on the jet axis). A similar approach was taken by Schmidt with his experimental jet validation \cite{Schmidt}. Finally, a method of converting photodetector voltage, \(V\), to the integrated phase difference, \(\Delta \Phi\), between the FLDI beams was developed. As mentioned in Section \ref{sec:imageprocessing}, perturbing the second prism-polarizer pair in the \(x\) direction causes the infinite fringe to change in brightness. The perturbation described essentially creates an initial phase difference, \(\Delta \Phi_0\), so that the photodetector response can be idealized as:

\[
V = A \cos (\Delta \Phi - \Delta \Phi_0) + D \tag{6}
\]

We determined the constants \(A\) and \(D\) by traversing the prism-polarizer pair enough that an entire phase cycle was traced out on the oscilloscope. We then set an intermediate position and recorded this voltage, \(V_{\text{int}}\). Then \(A = (V_{\text{max}} - V_{\text{min}})/2\) and \(D = (V_{\text{max}} + V_{\text{min}})/2\), which allowed \(\Delta \Phi_0\) to be calculated from \(V_{\text{int}}\). Note that this method does not rely on operating in the ‘linear region’ of the sinusoid as done in the works discussed in Section \ref{sec:imageprocessing}, although we generally chose \(V_{\text{int}}\) close to \(D\).
B. Free Ultrasonic Acoustic Beam

A commercial electrostatic ultrasonic transducer [Senscomp Series 600], resonant frequency 50 kHz, was used to generate a dynamic phase object in the FLDI test section. The transducer was oriented with its surface normal to the x-direction of the FLDI co-ordinate system, such that a propagating wavefront would reach one FLDI focus before the other, thereby generating a response. This particular transducer was chosen because it is designed for operation in air, and it is a broadband device, meaning it can be driven at frequencies significantly different from its resonance frequency while still emitting significant acoustic power.

The manufacturer’s intended operating mode for the transducer is to drive it using a square pulse train with repetition rate far below the resonant frequency, yielding a free response that ‘rings down’ at this resonant frequency. However, we wish to explore the frequency dependence of FLDI’s response, so we instead drive the transducer using a continuous sinusoidal signal, provided by a signal generator [Stanford Research Systems DS335] then amplified [A.A. Lab Systems A-301HS]. The manufacturer data shows a moderately flat transmission response above resonance out to 100 kHz, though with more rapid drop-off below resonance, hence in this work we use frequencies between 60–100 kHz. At 50 kHz the beam pattern is multi-lobed, with the bulk of the power within ±15° of the axis. However, it is not known how this pattern changes with frequency. Unfortunately, the density perturbations induced by the transducer are much too weak to be visualized by MZI.

The transducer face is 38 mm in diameter, which is large compared to the dimensions of the FLDI beams in the sensitive region. Preliminary experiments showed that the location of maximum FLDI response shifted with frequency, which we attributed to changes in the positions and powers of the beam lobes with frequency. In order to alleviate this, we applied a paper mask with an aperture of approximately 3 mm to the transducer face, with the idea of making a more point-like acoustic source. The transducer face was located approximately 10 mm from the optical axis, with the acoustic axis close to y = 0, i.e. passing through the FLDI foci.

The transducer was mounted on an optical rail that allowed it to be traversed ±50 mm about the focal plane, with a precision of 1 mm. At each traverse location and frequency, the AC-coupled FLDI photodetector signal was recorded at 1 MS/s over a record length of 5 s.

VII. Results

A. Laminar Jet

The main experimental campaign was carried out with the FLDI foci 5.37 mm above the jet orifice. The traversals in this x–z plane were the most thorough, with each (x, z) location being visited between 5 to 17 times. The averaged ΔΦ surface obtained is shown in Figure 4 from two different viewpoints. The form of the surface is as expected: in the far-field, where both beams traverse nominally uniform air, there is no phase shift. When one beam begins to interact with the jet, ΔΦ increases because this beam is now integrating a significantly lower refractive index than air. A maximum is reached, then ΔΦ decreases again, until reaching zero at the point where both beams are symmetrically arranged about the jet axis — at this position, the beams integrate the same Δn field. As the jet continues to move in the x-direction, the same ΔΦ signal is obtained, except with opposite sign. In the z-direction, the decrease in signal amplitude is especially apparent in Figure 4B. The foci centroid is within the traverse range of the experiment, since the zero-crossing between the positive and negative peaks (the x-coordinate of the centroid) and the global extrema of ΔΦ (the z-coordinate of the centroid) are both visible. Note the spikes that are visible in Figure 4B: because the signal changes very rapidly near x = 0, very small alignment errors can cause an individual ΔΦ data point to be assigned to a neighboring bin during the averaging process, skewing the cumulative average.

In addition to the expected drop-off in response as the jet is moved away from z = 0, we also note that the extrema diverge in x. This is shown in Figure 4B, where slices at select values of z are taken from the data in Figure 4A. Although the symmetry about the centroid (i.e. the zero-crossing location) stays fixed, the peak and trough move further from the centroid as z increases.

The averaged refractive index field of the jet is shown in Figure 4A. Note that it is presented as Δn (with respect to the far-field value). This is done because to calculate n absolutely, a value for the ambient room air n0 is required. However, this is unnecessary, because Equation (2) integrates a cumulative difference in n between the two beams; hence adding a constant n0 will not change the calculated ΔΦ. Hecht [12] gives n for air and helium at 0 °C, 1 atm, and 589.29 nm, which are comparable to our conditions. These values yield Δn = nHe – nair = –25.7 × 10⁻⁵. This is slightly higher than the maximum values we obtain of around –19.0 × 10⁻⁵, but we would only expect this full difference in regions of pure helium, i.e. near the jet centerline. The contour map and the overlaid radial cross-section
Fig. 4 Averaged $\Delta \Phi$ profile obtained from traversing the jet in an $x$–$z$ plane 5.37 mm below the FLDI foci. $\Delta x = 0.1$ mm, $\Delta z = 1$ mm.
show that although the data are still a little noisy even after averaging, both the spatial extent and magnitude of the $\Delta n$ field is in line with expectations, given the gases used and the size of the jet orifice.

These two raw datasets, from FLDI and MZI, along with the locating and scaling information discussed in Section VI, are then used to investigate our chosen theoretical model for FLDI, which is a completely new implementation of Schmidt’s code. The FLDI beams are discretized according to Schmidt’s scheme, with the Cartesian origin of the computational domain at the foci centroid. The experimentally-measured values of $\Delta x$ and $d_0$ are used to define the beam geometry. The jet origin is then fixed at a constant negative $y_{jet}$, i.e. below the foci, as per the experiment. The axisymmetric rotation of the MZI 2D $\Delta n$ field about the vertical axis through the jet origin generates a cylindrical region in the domain, where jet data is available. Outside of this region, it is assumed that $\Delta n = 0$, i.e. that the far-field refractive index is attained by the edge of the MZI FoV. For every discretization point on the beams that lies within this cylindrical region, rectangular bivariate spline interpolation is used to calculate the refractive index value at the point. The integrations of Equation (4) are then performed using Simpson’s rule, to yield a simulated FLDI $\Delta \Phi$ value from the experimental MZI data. A single simulation consists of fixing $x, z$ values used in the experimental campaign.

In Figure 7a we simulate a single $x$-traverse in the plane of best focus. The experimental location of $x = 0$ was found by choosing the midpoint of the $x$-locations of the peak and trough of the data; no further adjustments have been made. The $\Delta \Phi$ are the averaged raw values converted directly from the photodetector voltage measurements. We observe excellent qualitative and quantitative agreement, particularly on the left-hand side of the signal. The simulated FLDI peaks are, by definition, symmetrical about $x = 0$, but asymmetry is observed in the experimental peaks, with a difference in amplitude of about $0.1 \text{ rad}$ between the positive and negative peaks.

We do not have a conclusive answer for this asymmetry, though it could be due to either asymmetry in the FLDI system (e.g. foci of slightly different diameters) or in the jet (e.g. the jet axis being off-vertical). In the former case, the beam profiler software is designed for viewing a single beam; when viewing two beams, it would lock onto the larger of the two, and calculate its Gaussian diameter and centroid location. We made the assumption that the beams were of equal diameter, then adjusted the rotation of the beam profiler to a position where it fluctuated between locking onto both beams, that is, where the software considered the two beams to be of equal size. We took this rotational position to be where the planar face of the profiler was orthogonal with the beams. Of course, if there were asymmetries in the beams, then this approach is flawed: either the beams could have inherently different diameters, or they could be skewed such that the point of best focus in each beam occurs at a different $z$-location. In the case of jet asymmetry, we did a series of tests where the jet assembly was mounted on a rotational stage, then MZI interferograms acquired at $10^\circ$ increments. The jet axis orientation was estimated by finding the axis with the highest mirror symmetry, i.e. where ‘folding’ the image halves on each other yielded the lowest residual difference. This experiment gave an approximately constant jet angle of between $0^\circ$–$3^\circ$ from the vertical, at all viewing orientations. If there was an inherent angle to the jet, we would expect the projected angle, as viewed in the fixed MZI frame, to trace out a sinusoid as the jet assembly was rotated. Hence, we concluded that this tilt was either due to a slight air draft in the room, or some bias in the MZI setup. To mitigate the former, an enclosure was constructed around the jet assembly.

Fig. 5 The divergence from $x = 0$ of the extrema of $\Delta \Phi$ as $z$ increases. Experimental FLDI data.
Fig. 6 Average $\Delta n$ field of the axisymmetric He jet for the main experimental campaign. The axis of symmetry is the left edge of the image. The dashed line shows the $y$-plane in which the FLDI foci lie. Overlaid with the solid line is $\Delta n$ as a function of radius for this height.

An additional set of FLDI and MZI measurements were made with the jet assembly raised up, such that the jet origin was very close to the foci, only 0.15 mm below the FLDI plane. This was done in order to probe the core of the jet before it diverges or significantly mixes with the surrounding air. The corresponding comparison of experimental and simulated FLDI output is given in Figure 7b. Again, we have excellent agreement, and qualitatively the two positions give the same response, the main difference being that in the second case, the peaks are narrower and closer together; this is expected since the jet is narrower closer to the nozzle. However, in this case, the experimental signal is much more symmetric, i.e. the peaks are of similar amplitude. No changes were made to the FLDI setup between these experiments; the only difference was that the jet assembly was raised (and possibly rotated), which may have subtly changed the orientation of the jet. This result leads us to believe that the asymmetry is more likely to originate from the jet rather than from the FLDI foci.

The above results show that the computational model can satisfactorily predict the response in the plane of best focus. Next, we examined the spatial sensitivity as quantified by the drop-off in $\Delta n$ as $z$ increases. At every $z$ location, the maximum and minimum value of $\Delta n$ is extracted. Comparison is then made with simulation in Figure 8. It was found that the simulation with $d_0 = 80 \mu m$ (the experimental value) predicts a far more gradual drop-off than seen in the experimental data. Various smaller values of $d_0$ were then simulated, still using the same MZI input data for the jet. The rate of drop-off increases as $d_0$ decreases, and for this set of data, the experimental drop-off is bounded by $10 \mu m < d_0 < 20 \mu m$. Note that all the curves meet at $z = 0$, i.e. the magnitude of the simulated signal in the plane of best focus (e.g. Figure 7) is not affected by $d_0$.

This result lead us to explore further the effects of changing $d_0$ and $\Delta x$ in the simulations. Figure 9 shows simulations in the plane of best focus. As before, $d_0$ has no effect on the gross shape of the signal in this plane, but the zoomed inset in Figure 9 illustrates that the signal becomes less smooth as $d_0$ decreases. Recall from Figure 4 that the input data from MZI has small-scale noise. As $d_0$ becomes smaller, there is less integration over the beam area of this noise. In contrast, $\Delta x$ has a large effect on the signal: as $\Delta x$ increases, the signal magnitude increases, and the peaks also broaden. This also makes sense, as when the beams are further apart, they traverse regions of the refractive index field that have larger differences from each other, which is reflected in $\Delta n$, being an integrated measure of these differences.

Figure 10 illustrates the strong effect $d_0$ has on spatial sensitivity: for $d_0 = 10 \mu m$, the signal has roughly halved 25 mm from the focus, and has decayed almost to nothing by $z = 200$ mm, whereas for $d_0 = 500 \mu m$, the signal has hardly decreased at all even at $z = 500$ mm. This is also expected: under the assumption of Gaussian beam propagation in the model, a smaller $d_0$ requires the beams to converge and diverge more rapidly with $z$. This is alluded to in Settles...
Fig. 7 Comparison of experimental and simulated FLDI responses to an x-traverse of the He jet at two positions below the foci, both at $z = 0$, i.e., in the plane of best focus. The experimental error bounds are quantified by the standard deviation of the average measurement at each location.
and Fulghum [9], where they discuss how the ‘elongation of the region of best sensitivity in z depends on the lens f/number’, as this number determines how rapidly the beam converges to its best focus.

The close match of the peak spacing and amplitude between experiment and simulation shows that the measured $\Delta x$ is correct. However, $d_0$ is substantially mismatched. We believe this can be explained by the non-ideality of the experimental beam profile. The beam expander used in the FLDI (BE-2 in Figure 2) expands the beam such that the beam diameter is larger than the first 20 mm square Wollaston prism at the prism location. This results in the expanded beam being cut off, and the beams propagating through the remainder of the FLDI are now square in shape rather than the assumed Gaussian profile, although the energy distribution in the beams should still be approximately Gaussian near the center. $\Delta x$ is correct because this is simply the distance between the centroids of the two foci. This leads us to propose that perhaps an effective $d_0$ can still be found for non-circular beams.

Following the two experimental campaigns using the translation stage setup described above (which had a total z range of 37 mm bounding the plane of best focus), a third setup was created, where a single mechanized stage oriented in the x direction was mounted to a linear rail. This gave a z range of more than 200 mm in one direction — the jet assembly could be moved right up to the focusing lens (F-1 in Figure 2). This was a less precise setup: the linear rail only offers precision of 1 mm, with a single x-traverse being performed at each z position. However, this was satisfactory for the purpose of seeing whether the long-range behavior remained consistent with the simulated predictions. Also note that due to the different mounting arrangement, $y_{jet} = 8.90$ mm in this configuration. A new set of MZI interferograms were acquired for this height, to be used in the corresponding simulations.

The results of this longer z-range campaign are shown in Figure 11. In Figure 11a, $\Delta \Phi$ is presented as contours rather than as a surface as in Figure 4. Although noisier than the data for that campaign, the divergence of the peaks with z is clearly visible. Also, the asymmetry in amplitude between the peaks and troughs is again absent. It is worth noting that the noise in the signal is not only due to random noise in the laser or photodetector: it was clear from both the FLDI and MZI data that the jet was less steady. For these data, we are substantially higher in the jet, at almost 18 nozzle diameters from the origin. It is likely that there is substantial unsteadiness from mixing.

Again, in Figure 11b we see that the experimental drop-off is bounded by $10 \mu m < d_0 < 20 \mu m$, which are the same bounds as in Figure 8. This consistency between the short-range and long-range datasets, despite being at significantly different positions in the jet, lends credence to the idea that even a non-ideal laser beam can be assigned an effective Gaussian diameter, such that the simulation will give accurate predictions.
Fig. 9  The effects of changing $d_0$ and $\Delta x$ in the simulated FLDI, at $z = 0$.

Fig. 10  Effects in both $x$ and $z$ of changing $d_0$ in the simulated FLDI, for fixed $\Delta x = 180 \mu m$. 
(a) Contours of experimentally-measured $\Delta \Phi$. The dashed lines indicate the location of the FLDI foci centroid.

(b) Experimental and simulated sensitivity decrease as the jet moves further from the plane of best focus.

Fig. 11 Summary of results for the long $z$-range experiments.

Following the above comparisons between the experimental data and the numerical model, the effect on FLDI response of varying the jet diameter was explored numerically. An idealized jet model that varies only radially was created by fitting a function of the form:

$$\Delta n = (A + Bx) \tanh(Cx + D) + E$$

(7)

to the experimental MZI $\Delta n$ field, at a fixed height near the experimental foci location. This fit is illustrated in Figure 12a. By changing the fitting coefficient $C$ only, we can simulate a family of jets with varying radius. This radius is quantified as the location where the function has maximum curvature. Using the analytically-obtained second derivative of Equation (7) to calculate these radii, Figure 12b shows the range of jet radii that were generated.

(a) The base fitted function with the corresponding raw data.

(b) A family of jets with effective radii of given ratio to the radius of the base jet, $R_0$.

Fig. 12 Fitting a hyperbolic tangent function to the experimental $\Delta n$ field and using the function to generate simulated jets of varying radius.

The FLDI responses were computed over a range of $(x, z)$ locations for each of these simulated jets, with the results summarized in Figures 13 and 14. For the drop-off in $\Delta \Phi$ with $z$, Gaussian curves were fitted to each set of simulated data points, and the width as quantified by the standard deviation $\sigma$ was used as a measure of this decrease.
In Figure 13a we plot only the response for \( x > 0 \) for clarity, since the response is symmetric. We observe that the response broadens and the peak moves outward, while also slightly increasing in height. It is found that the peak position has a linear relationship with the non-dimensional jet radius \( R/R_0 \) (Figure 13b). This suggests that a similarity solution exists that could collapse all these responses for a given jet shape, although such a normalization would also have to account for the changing heights. Similarly, Figure 14b shows that the size of the sensitive region is also linearly related to the size of the jet.

Note also the comparison with the simulation directly from the raw MZI data. Although the fit shown in Figure 12a is very good, even the slight variations lead to substantial differences in the FLDI signal: the peaks are significantly mismatched, and the small-scale fluctuations in the \( \Delta n \) field yield larger fluctuations in the response. This serves to illustrate how sensitive FLDI is to apparently minor flow features.

![Graph](image1)

(a) The one-sided response curves for each simulated jet. Original response from experimental data overlaid for comparison.

![Graph](image2)

(b) Location in x of peak FLDI response as a function of jet radius.

**Fig. 13** Effect of varying the radius of similar jets in the plane of best focus, as a function of x.

![Graph](image3)

(a) The drop-off in z, with Gaussian curves fitted.

![Graph](image4)

(b) Fitted width \( \sigma \) of drop-off Gaussian as a function of jet radius.

**Fig. 14** Effect on spatial sensitivity of varying the radius of similar jets.
B. Free Ultrasonic Acoustic Beam

Data from the ultrasound experiment were obtained over a traverse range of approximately 100 mm centered on the focal plane at \( z = 0 \). The distances in \( x \) and \( y \) of the acoustic source from the FLDI beam axis were kept constant as detailed in Section VII, as were the driving voltage amplitude and bias. Because the dependence of acoustic beam power and shape on frequency is unknown, the FLDI response was normalized using the maximum response at each frequency. This was done by computing the Welch power spectrum for each dataset, then extracting the spectral power density of the peak at the driving frequency. In order to locate the plane of best focus, the mean \( z \)-position where the peak spectral power density occurred over all trial frequencies was calculated.

These normalized data are shown in the left-hand plot of Figure 15. The general trend is that higher frequencies are more spatially filtered than lower frequencies, which is the qualitative result predicted by all the authors in Section III. In order to more quantitatively compare these data to the theoretical models, we consider the analytical transfer functions of Schmidt and Shepherd. As previously discussed, these were derived from the same underlying theory that dictates Schmidt’s numerical model, but analytical solutions can be obtained for particular disturbance geometries. The solutions computed by for sinusoidal disturbances of varying spatial extents share a common term:

\[
H(k) \propto \exp\left(-\frac{w^2 k^2}{8}\right)
\]  

where \( H(k) \) is the FLDI system transfer function, \( w \) is the Gaussian beam radius, and \( k \) is wavenumber. Except for very close to the foci, \( w \) grows linearly with \( z \), and \( f \) is of course proportional to \( k \). Hence we can approximate that \( H \propto \exp(-z^2 f^2) \), or equivalently, \( \log H \propto -z^2 f^2 \). Taking the normalized power spectral densities as surrogates for FLDI system response, we apply this transformation: in the right-hand plot of Figure 15 \( \log H \) is plotted against \( z^2 \), with added linear regressions. Although noisy, the data do seem consistent with this model.

![Figure 15](image.png)

**Fig. 15** Results of the ultrasonic beam experiments. At left is the drop-off in FLDI response moving away from the approximate location of the foci at \( z = 0 \). At right are the same data, transformed according to the analytical transfer function model.

VIII. Conclusions and Future Work

The goals of this work were to: obtain precise spatial and temporal characterization data for an FLDI system, use these data to validate a computational model of FLDI, then use the validated model to explore the sensitivity of FLDI’s response to various input parameters. The spatial response of FLDI was obtained using a steady laminar helium jet that was precisely positioned at a range of locations relative to the FLDI foci. The refractive index field of this jet was independently characterized using Mach-Zehnder interferometry. Preliminary temporal characterization was obtained using a free ultrasonic acoustic beam generated by an ultrasound transducer traversed along the sensitive region in the vicinity of the foci.

The results show that the simple ray-tracing scheme given by Equation (4) along with the numerical discretization scheme detailed in Schmidt and Shepherd [2] give accurate quantitative predictions for the static response of an FLDI...
system. This is in spite of the experimental system having numerous non-idealities, chiefly non-circular beam profiles. A key result is that in the plane of best focus, the simulation is not sensitive to the value of the Gaussian beam diameter, $d_0$; away from this plane, an effective $d_0$ can be found that appears to hold even when probing different locations in the jet. We have explored the effects of $d_0$, $\Delta x$, and jet diameter on the FLDI response using the numerical model, with the results matching well with physical intuition. Aspects of the spatial sensitivity property of FLDI, central to its applicability to wind tunnels, have been quantified, both near the focal plane, and over the entire range of the focused beams. However, this was only done using a static phase object of a particular geometry; actual wind tunnel applications involve side-wall boundary layers or free jet shear layers interacting with regions of the beams far from the focal plane. Further quantification using characterized flows that more closely resemble these cases still needs to be performed. For example, a test could repeat the same jet traverses presented here, with the addition of a shear layer analogue operating simultaneously near to the field lenses, and the differences quantified.

While the scope of the experiments regarding the dynamic response was more limited, the results appear to agree with analytical models that also stem from Equation (4). We would like to expand this aspect of the work to obtain more conclusive data. Constructing a more sophisticated aperture or waveguide for the ultrasonic transducer could assist in creating a more point-like source. Another concept is to build a resonant cavity where the density field can be independently computed.

This work serves primarily as a validation of the model of an FLDI system. With this increased confidence, we hope to use the model to inform design decisions when constructing FLDI systems for a particular application. Further, as already mentioned, Equation (4) is not specific to FLDI. The code models FLDI in particular by defining the beam geometry in such a way. But by modifying this scheme, we can explore variants on FLDI, for both practical and theoretical purposes. For example, Settles and Fulghum [9] propose a ‘multiplexed’ FLDI with several pairs of foci but only one laser and set of optics. Another concept mentioned in this work but still unanswered is the importance of beam overlap to spatial sensitivity. A contrived FLDI variant where the beams still converge to a foci pair with the same dimensions, while never overlapping anywhere in the domain, could be tested — both experimentally and numerically — to shed light on this question.

Acknowledgments

The authors would like to gratefully acknowledge Prof. J. E. Shepherd for all his guidance and interest in this project, as well as providing the laboratory space to perform the experimental work. This work was partially supported by the Office of Naval Research award N00014-16-1-2503 with Dr. Eric Marineau and the Air Force Research Laboratory award STTR FA8651-17-C-0071 with Dr. Daniel Reasor.

References


