Supplemental material for "A characterization of quantum chaos by two-point correlation function"

I. THE NUMBER VARIANCE OF THE UNFOLDED SPECTRUM

In this section we present another indicator of the random matrix behavior of the exponents obtained from the singular values of the correlation functions. The number variance $\Sigma^2(L)$ is the variance of the number of unfolded exponents in the section of spectrum of width L, where L unfolded exponents are expected on average. It is equal to L for the uncorrelated case, while it grows logalithmically as a function of L for random matrix ensembles [6]. Here, we compare the values of $\Sigma^2(L)$ against those calculated for the eigenvalues of the N (for the SYK model) or N_{site} (for the XXZ spin chain) dimensional random matrices. The spectra are unfolded so that the average spacing is unity and are shifted so that the average becomes zero, then the variance of number of the values in the unfolded and shifted spectra within $[-L/2, L/2]$ are obtained.

For comparison to the SYK model, in Fig. 11, the largest $N/2$ eigenvalues of $N/2$-dimensional symmetric real (GOE) and hermitian complex (GUE) random matrices are used for calculating the number variance. We observe that for $N = 22$ and $K = 0.0001$, the results at $t = 10$ and 100 strongly resemble that for the GUE random matrix, including the finite-scale fluctuations. The result for $N = 16$ and $K = 0.0001$ resembles that for the GOE random matrix at long times as expected. On the other hand, for $K = 10$, both for $N = 22$ and for $N = 16$ we observe larger variances than expected for the uncorrelated case ("Poisson") in the plots, presumably because the exponents coalesce with each other.

In Fig. 12, we compare the number variance between the exponents obtained from the singular values of eq. 4 for the XXZ model and the eigenvalues of GOE random matrices with $N_{\text{site}} = 12$ and $N_{\text{site}} = 8$. In both cases, half of the spectrum is unfolded and shifted for the computation of $\Sigma^2(L)$. For $W = 0.5$ in the chaotic regime, the increase of $\Sigma^2(L)$ as a function of L is slowed down as t is increased and almost converged for $t \gtrsim 10$ near the GOE result. For $W = 4$ in the many-body localized regime, the increase is closer to linear and to the result for the Poisson distribution, $\Sigma^2(L) = L$.

II. INVERSE PARTICIPATION RATIO FOR THE SINGULAR VECTOR

The inverse participation ratio (IPR) for a wavefunction expressed on a basis $\{|j\rangle\}$, $|\psi\rangle = \sum_j \psi_j |j\rangle$, is defined as

$$\text{IPR}(|\psi\rangle) = \frac{\sum_j |\psi_j|^4}{\left(\sum_j |\psi_j|^2\right)^2}, \tag{6}$$

in which the denominator is unity when the wavefunction is normalized ($\langle j|j\rangle = \sum_j |\psi_j|^2 = 1$). In this definition, when the dimension of the Hilbert space is D, the IPR takes its smallest value, $1/D$, when all D components of the wavefunction has the equal absolute value and thus most delocalized in the basis, $|\psi_j| = 1/\sqrt{D}$. A more localized wavefunction has a larger IPR, and in the most localized case with only one component having nonzero absolute value, $|\psi_{j_0}| = 1$, the IPR equals unity. Thus the IPR parametrizes the localization of the wavefunction in the given basis and can be regarded as the inverse of the number of components having significant amplitude.

The IPR can also be calculated for the (left and right) singular vectors of the correlation matrix that we compute in our work. We have checked that the values from the left and right singular vectors are indistinguishable. Therefore, we have used the IPR for the right singular vectors in the following. In Fig. 13 we plot the average of the IPR over all the right singular vectors obtained for the correlation function matrix for all energy eigenstates as a function of the time.

For the SYK model (eq. (3)), the average of the IPR is small through the time evolution and decreases as $\sim 1/N$ as N is increased as shown in the top panel of Fig. 13. For the almost integrable Hamiltonian with the large $K = 100$, the value is almost independent of the time. For the maximally chaotic case with very small K, for $N = 14$ and 18 the value slightly increases during $1 \lesssim t \lesssim 10$ and stabilizes, while it has a peak for $N = 16 \equiv 0 \bmod 8$, presumably correlated with the change of the symmetry class (see Fig. 2). The small IPR does not guarantee the random matrix behavior, and the behavior observed here is consistent with the results of other quantities presented in the main text.

For the XXZ model (eq. (4)), for large W corresponding to the many-body localized region, the IPR is large and not significantly changed as a function of t or N_{site}. The singular vectors of the matrix $G_{ij}^{(\phi)}(t)$ are also strongly localized, as in the bottom panel of Fig. 13. On the other hand, for $W = 0.5$, the IPR decreases as a function of t before stabilizing at a value that is almost in proportion to $1/N_{\text{site}}$, indicating the delocalized nature of the singular vectors.

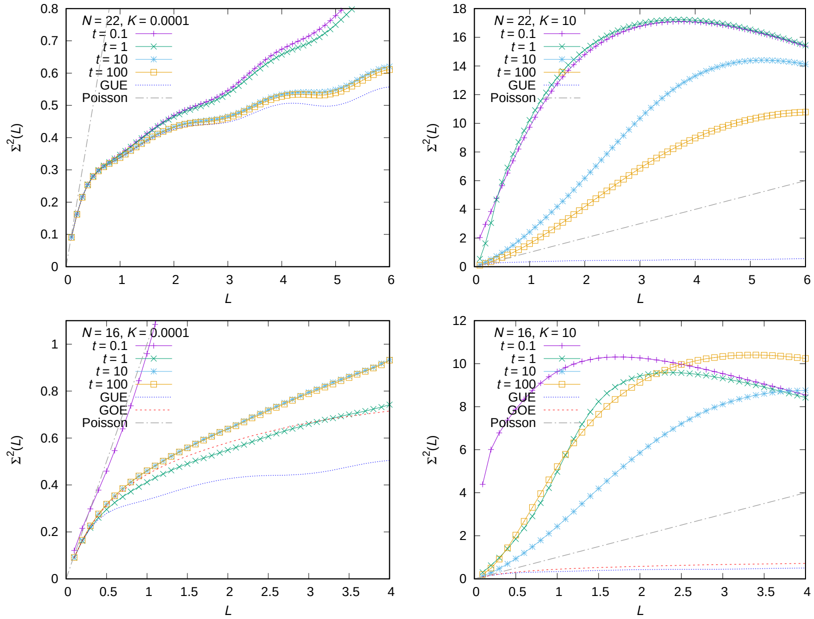

FIG. 11. SYK model, the number variance $\Sigma^2(L)$ plotted against the range L of the unfolded spectrum for $G^{(\phi)}_{ij} = \langle\phi|\psi_i(t)\psi_j(0)|\phi\rangle$ with $N = 22$ (top) and $N = 16$ (bottom). 1000 samples have been used. The data for N-dimensional GOE and GUE matrices are obtained using 10^7 samples.

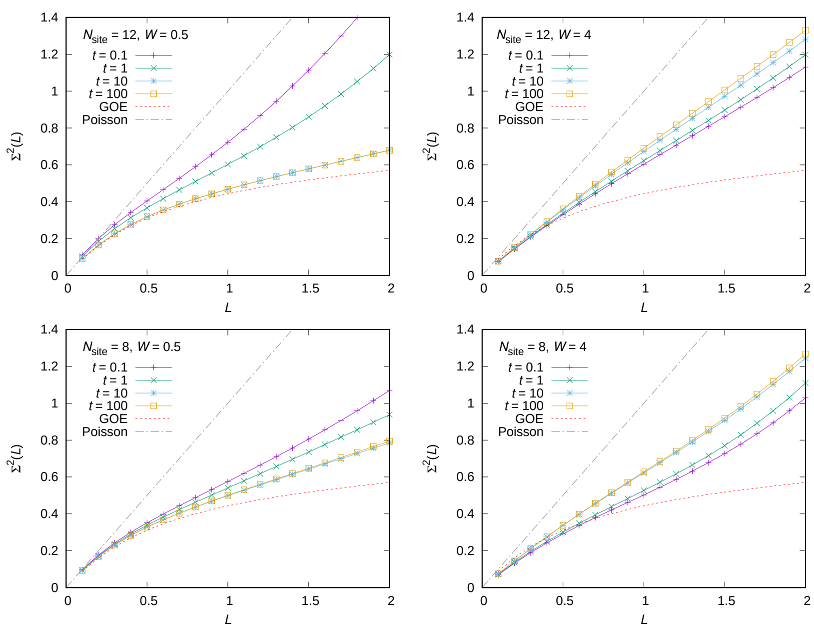

FIG. 12. XXZ model, the number variance $\Sigma^2(L)$ plotted against the range L of the unfolded spectrum for $G^{(\phi)}_{ij} = \langle\phi|\sigma_{+,i}(t)\sigma_{-,j}(0)|\phi\rangle$ with $N_{\text{site}} = 12$ (top) and $N_{\text{site}} = 8$ (bottom). At least 20000 samples have been used. The data for N_{site}-dimensional GOE matrices are obtained using 10^7 samples.

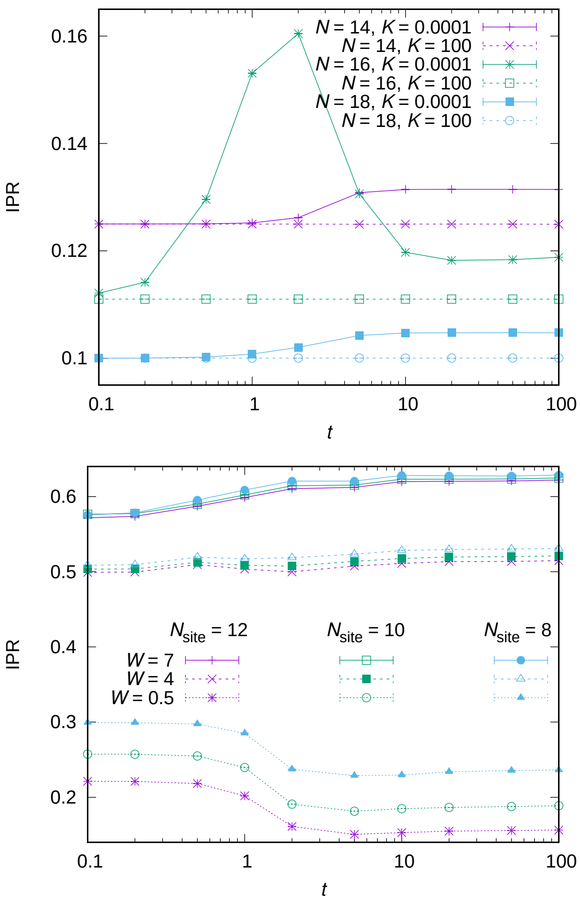

FIG. 13. The inverse participation ratio calculated for the right singular vectors of the correlation functions. Top: eq. (3) for the SYK model. Bottom: eq. (4) for the XXZ model.