Zeros of ferromagnetic 2-spin systems

Heng Guo* Jingcheng Liu† Pinyan Lu‡

Abstract

We study zeros of the partition functions of ferromagnetic 2-state spin systems in terms of the external field, and obtain new zero-free regions of these systems via a refinement of Asano's and Ruelle's contraction method. The strength of our results is that they do not depend on the maximum degree of the underlying graph. Via Barvinok's method, we also obtain new efficient and deterministic approximate counting algorithms. When the edge interaction is attractive for both spins, our algorithm outperforms all other methods such as Markov chain Monte Carlo and correlation decay.

1 Introduction

Spin systems are widely studied in statistical physics, probability theory, machine learning, and theoretical computer science, sometimes under a different name such as *Markov random field*. An important special case is when there are only 2 spins, and a systematic study of their computational complexity was initiated by Goldberg et al. [GJP03]. In addition to their intrinsic importance, these systems are also great test beds for algorithmic ideas. Many interesting tools and techniques are developed through studying them. By now, we have almost completely settled the anti-ferromagnetic case, whereas a definitive answer to the ferromagnetic case still remains elusive.

Before reviewing the state-of-the-art, we define the 2-state spin system first. In a graph $G = (V, E)$, a *configuration* $\sigma : V \to \{0, 1\}$ assigns one of the two spins "0" and "1" to each vertex. The 2-spin system is specified by the edge interaction matrix, which we normalise to $\begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$, and the external field λ for vertices that are assigned 1. All parameters here are non-negative. For a particular configuration σ, its weight $w(\sigma)$ is a product over all edge interactions and vertex weights, that is

$$w(\sigma) = \beta^{m_0(\sigma)}\gamma^{m_1(\sigma)}\lambda^{n_1(\sigma)}, \tag{1.1}$$

where $m_0(\sigma)$ is the number of $(0, 0)$ edges given by the configuration σ, $m_1(\sigma)$ is the number of $(1, 1)$ edges, and $n_1(\sigma)$ is the number of vertices assigned 1. The *Gibbs measure* (or *Gibbs distribution*) of the system is one where the probability of a configuration is proportional to its weight. The partition function Z_{spin} is the normalising factor of the Gibbs distribution:

$$\begin{aligned} Z_{\text{spin}}(G; \beta, \gamma, \lambda) &= \sum_{\sigma:V\to\{0,1\}} w(\sigma) \\ &= \sum_{\sigma:V\to\{0,1\}} \beta^{m_0(\sigma)}\gamma^{m_1(\sigma)}\lambda^{n_1(\sigma)}. \end{aligned} \tag{1.2}$$

An important special case is the Ising model, where $\beta = \gamma$. We note that in the statistical physics literature, parameters are usually chosen to be the logarithms of our parameters above. Change of variables as such do not affect the complexity of the same system.

Many macroscopic properties of the system can be studied through partition functions, which raises the interest of computing them. Exact computation of Z_{spin} is #**P**-hard for all but trivial cases [Bar82], so the main focus is on approximating Z_{spin}.

The system shows drastically different behaviours depending on whether $\beta\gamma < 1$ or $\beta\gamma > 1$ (the case where $\beta\gamma = 1$ is degenerate). The antiferromagnetic case $\beta\gamma < 1$ is now very well understood by a series of work [Wei06, LLY13, SST14, SS14, GŠV16], where an exact threshold of computational complexity transition is identified and the only remaining case is at the critical point. This threshold corresponds to the uniqueness threshold of Gibbs measures in infinite regular trees (also known as the Bethe lattice).

On the other hand, far less is known for the ferromagnetic case $\beta\gamma > 1$. Due to symmetry, we will assume $\beta \geq \gamma$ throughout this paper as the other case is similar. This assumption means that the edge interaction favours the spin "0". As it turns out, if the external field also favors "0" (namely $\lambda \leq 1$), then the system can be reduced to the ferromagnetic Ising model (under a holographic reduction), and efficient algorithms can be obtained in a number of ways. The real challenge is how far we can allow λ to go beyond 1, and a critical threshold is conjectured to exist.

Unlike antiferromagnetic systems, the tree unique-

*Heng Guo, University of Edinburgh, UK. Email: hguo@inf.ed.ac.uk.
†Jingcheng Liu, California Institute of Technology, USA. Email: liuexp@caltech.edu.
‡Pinyan Lu, Shanghai University of Finance and Economics, China. Email: lu.pinyan@mail.shufe.edu.cn.

ness threshold is not the right answer, as the pioneering algorithm of Jerrum and Sinclair [JS93] is efficient on both sides of the tree uniqueness threshold for ferromagnetic Ising models ($\beta = \gamma$). This algorithm is based on the Markov chain Monte Carlo (MCMC) method. The MCMC method has been adapted to general ferromagnetic 2-spin systems in [GJP03], whose bound was later slightly improved in [LLZ14] to give an efficient approximation algorithm of Z_{spin} if $0 < \lambda \le \lambda_{\text{MCMC}} = \frac{\beta}{\gamma}$, for fixed $\beta \ge \gamma$.

The algorithmic success in the anti-ferromagnetic case is largely thanks to the correlation decay method introduced by Weitz [Wei06]. It is natural to try this method on ferromagnetic systems as well. Non-trivial results have been obtained in [GL18], but these results still fall short from solving the problem in general. In [GL18], the first and the third author raised the following conjecture.

Conjecture 1.1. ([GL18]) *Let β, γ, λ be positive parameters such that $\beta \ge \gamma$ and $\beta\gamma > 1$. If $\lambda \le \lambda_c$ where $\lambda_c := \left(\frac{\beta}{\gamma}\right)^{d_c}$ and $d_c := \frac{\sqrt{\beta\gamma}}{\sqrt{\beta\gamma}-1}$, then a fully polynomial-time approximation scheme (FPTAS) exists for Z_{spin}.*

Conjecture 1.1 is confirmed in [GL18] for the case of $\gamma \le 1$. However, it does not generalise to $\gamma > 1$ because certain key properties in correlation decay fail. Part of the difficulty is that if $\gamma > 1$, the edge interaction is attractive for both spins (albeit more so for one than the other). For any fixed external field, there is a degree threshold beyond which the influence of boundary conditions never diminishes. On the other hand, one should not expect to go beyond λ_c too far. Indeed, Liu et al. [LLZ14] identified another threshold beyond which the problem is as hard as approximately counting independent set in bipartite graphs, which is a notorious open problem in approximate counting and is conjectured to have no efficient algorithm [DGGJ04]. The hardness threshold of [LLZ14] is almost equal to λ_c except for a small integral gap.

In this paper, we obtain new algorithmic result that outperforms both the MCMC and the correlation decay methods in the $\gamma > 1$ regime.

Theorem 1.1. *Let β, γ, λ be positive parameters such that $\beta \ge \gamma$ and $\beta\gamma > 1$. If $\lambda < \lambda^\star$ where $\lambda^\star := \left(\frac{\beta}{\gamma}\right)^{d^\star/2}$ and $d^\star := \frac{\pi}{\tan^{-1}(\sqrt{\beta\gamma-1})}$, then an FPTAS exists for Z_{spin} in bounded degree graphs.*

Theorem 1.1 is a generalisation of the algorithm for the ferromagnetic Ising model ($\beta = \gamma$) by Liu, Sinclair, and Srivastava [LSS19b]. We note that our bound on λ is uniform and does not depend on the maximum degree of the underlying graph. The requirement of bounded degree is only for the efficiency of our algorithm. Without this assumption, our algorithm becomes quasi-polynomial time. This is a typical behavior of deterministic approximate counting algorithms.

To compare $\lambda^\star$ with λ_c, we note that as $\beta\gamma \to 1$, $d^\star$ is asymptotically the square root of d_c. An illustration of comparing λ_{MCMC}, λ_c and $\lambda^\star$ is given in Figure 1.

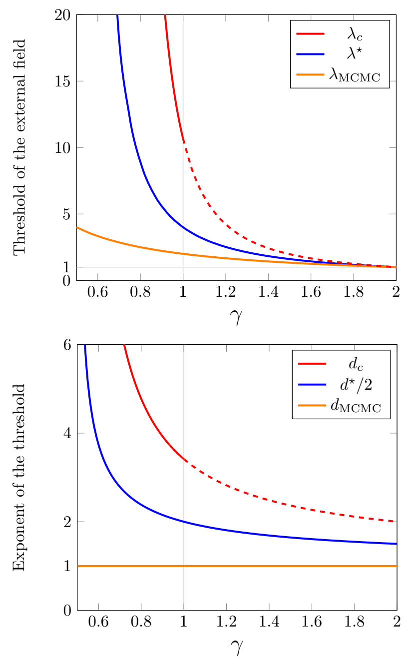


Figure 1: Fix $\beta = 2$ and the range of γ is $(1/2, 2]$. Left: comparison of λ_{MCMC}, λ_c and $\lambda^\star$. Right: comparison of $d_{\text{MCMC}} := 1$, d_c and $d^\star/2$. The dashed red line marks the conjectured threshold for $\gamma \ge 1$.

Our algorithm is based on a recent algorithmic technique developed by Barvinok [Bar16] and extended by Patel and Regts [PR17]. The idea is to view Z_{spin} as a polynomial in λ, and turn zero-free regions of this polynomial in the complex plane into efficient approximation algorithms of the corresponding parameters. The major challenge of applying this algorithmic framework is to obtain sharp zero-free regions along the real axis.

There are two main methods in obtaining zero-free regions. The first one is the recursion method, where one gradually eliminates vertices from the graph, and shows that the zeros are always outside of the desired region. This method has found many successes, see e.g. some work of Sokal [Sok01, SS05]. More recently, it has been successfully applied to solve long-standing conjectures [PR19] and open problems [LSS19a]. However, there are also strong connections between correlation decay and the recursion method. In some sense, both results of [PR19] and [LSS19a] are turning correlation decay analysis into zero-freeness bounds using complex dynamical systems. For ferromagnetic 2-spin systems, because correlation decay fails if $\gamma > 1$ [GL18], it would be surprising to obtain any meaningful result using the recursion method in this case.

In order to bypass the correlation non-decay barrier, we employed the other method, namely the contraction method, pioneered by Asano [Asa70] and Ruelle [Rue71, Rue99]. In a typical application, one starts with a graph of isolated components, and then contract vertices or edges to form the desired graph G. The zero-free regions of isolated components are easy to analyse, but the contractions will spread the zeros across the complex plane. The main effort is to control this spread. In all previous applications of this method that we are aware of, either the unit circle or half planes are used as the starting point. Our idea is to consider circles whose center and radius are carefully chosen (depending on the parameters), and sometimes their complements. The main technical challenge is a detailed analysis for contracting an *arbitrary* number of corresponding regions, which involves repeated Minkowski product of circular regions. We do so by solving a highly non-trivial optimisation problem in complex variables (see (4.9)). It remains to be explored whether this methodology has other applications as well.

Theorem 1.2. *Let β, γ be positive parameters such that $\beta \geq \gamma$ and $\beta\gamma > 1$, and $\lambda^\star$ defined as in Theorem 1.1. Then for any graph with minimum degree at least 2, Z_{spin}, viewed as a polynomial in λ, is zero-free in a constant-sized small neighbourhood of the interval $[0, \lambda']$ for any $\lambda' < \lambda^\star$.*

The minimum degree requirement in Theorem 1.2 comes from some technical difficulty with degree 1 vertices. They do not affect the algorithmic result, Theorem 1.1, because we can preprocess the graph to remove the leaves, and then deal with an instance with non-uniform external fields. In order to do so, we in fact show a stronger multivariate zero-free theorem, see Theorem 4.1.

The main message of our paper is to show that the failure of correlation decay is *not* an essential barrier for efficient algorithms. However, because of some inherent difficulties of the contraction method, as explained in Section 5, our result still falls short of confirming Conjecture 1.1. By now we have three different point of views for approximating Z_{spin}, namely MCMC, correlation decay, and zeros of polynomials. They are just different aspects of the same object, and perhaps settling the complexity of ferromagnetic 2-spin systems requires a more unified view.

Notation. For sets A and B over complex numbers, we denote their *Minkowski* product by $A \cdot B := \{ab : a \in A, b \in B\}$. When it is clear from the context, we will write $A^2 := A \cdot A$ for short. For a complex number c, we will write $c \cdot A := \{ca : a \in A\}$. In particular, $-A = \{-a : a \in A\}$. Throughout this paper, we use $\overline{A}$ for the closure of a set A, and A^{c} for the complement of A. To avoid confusion with the index i, we use $\iota := \sqrt{-1}$ to denote the imaginary unit.

2 Barvinok's algorithm

Recall equations (1.1) and (1.2) that

$$Z_{\text{spin}}(G; \beta, \gamma, \lambda) = \sum_{\sigma : V \to \{0,1\}} \beta^{m_0(\sigma)} \gamma^{m_1(\sigma)} \lambda^{n_1(\sigma)}.$$

We will fix β and γ, and view equation (1.2) as a polynomial in λ. In that case, we write $Z_{\text{spin}}(G; \lambda)$ for short. The main effort of this paper is to show that for a certain region of λ on the complex plane, $Z_{\text{spin}} \neq 0$.

Our interest in the zeros of the partition function is due to the algorithmic approach developed by Barvinok [Bar16, Section 2]. Let the δ-strip of $[0, t]$ be

$$\{z \in \mathbb{C} \mid |\Im z| \leq \delta \text{ and } -\delta \leq \Re z \leq t + \delta\}.$$

Suppose a polynomial $P(z) = \sum_{i=1}^{n} c_i z^i$ of degree n is zero-free in a strip containing $[0, t]$. Barvinok's method roughly states that $P(t)$ can be $(1 \pm \varepsilon)$-approximated using $c_0, \ldots, c_k$ for some $k = O\left(e^{\Theta(1/\delta)} \cdot \log \frac{n}{\varepsilon}\right)$, via truncating the Taylor expansion of the logarithm of the polynomial. In general, computing these coefficients naively will take quasipolynomial-time. However, Patel and Regts [PR17] have provided additional insights on how to compute these coefficients efficiently for a large family of graph polynomials in bounded degree graphs. As explained in [LSS19b], the idea of Patel and Regts [PR17] can be applied to the partition functions of spin systems in much more generality, which includes $Z_{\text{spin}}(G; \lambda)$ that we are interested in. Thus, combining the algorithmic paradigm of Barvinok [Bar16, Section 2] and the idea of Patel and Regts [PR17], we have the following useful lemma.

Lemma 2.1. *Fix β, γ and an integer $\Delta \geq 2$. Let G be a graph of maximum degree Δ. If $Z_{spin}(G; \lambda)$ does not vanish in a δ-strip containing $[0, \lambda']$, then there is an FPTAS for $Z_{spin}(G; \lambda)$ for all $\lambda \in [0, \lambda']$.*

In fact, as it has been observed in [PR17], the algorithm can be extended to a multivariate version of the partition function easily. Let $\boldsymbol{\lambda} \in \mathbb{C}^V$ be a vector that specifies an external field for each vertex. The multivariate partition function is given by

(2.3)
$$Z_{\text{spin}}(G; \beta, \gamma, \boldsymbol{\lambda}) := \sum_{\sigma : V \to \{0,1\}} \beta^{m_0(\sigma)} \gamma^{m_1(\sigma)} \prod_{v \in V} \lambda_v^{[\sigma(v)=1]}.$$

LEMMA 2.2. *Fix β, γ and an integer $\Delta \geq 2$. Let G be a graph of maximum degree Δ and $n = |V|$. If $Z_{spin}(G; \boldsymbol{\lambda})$ does not vanish in a δ-polystrip $\{\boldsymbol{z} \in \mathbb{C}^n \mid \forall i \in [n], |\Im z_i| \leq \delta \text{ and } -\delta \leq \Re z_i \leq \lambda' + \delta\}$, then there is an FPTAS for $Z_{spin}(G; \boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in [0, \lambda']^n$.*

Proof. For any $\boldsymbol{\lambda} \in [0, \lambda']^n$, we consider the univariate polynomial $f(t) = Z_{\text{spin}}(G; t \cdot \boldsymbol{\lambda})$. On the one hand, $f(1) = Z_{\text{spin}}(G; \boldsymbol{\lambda})$ is the quantity what we want to approximate. On the other hand, the fact that $Z_{\text{spin}}(G; \boldsymbol{\lambda})$ does not vanish in a δ-polystrip containing the poly-region $[0, \lambda']^n$ implies that there exists a $\delta' > 0$ (depending on δ and λ'), such that $f(t)$ does not vanish in a δ'-strip containing $[0, 1]$. Hence, applying Lemma 2.1 on $f(t)$ yields our desired FPTAS for $Z_{\text{spin}}(G; \boldsymbol{\lambda})$. □

We note that for any fixed β, γ, λ, and Δ, our FPTAS runs in time bounded by a polynomial in $n = |V|$ and $1/\varepsilon$. However, as is typical for deterministic counting algorithms, the exponent can grow with Δ and other parameters as they approach the threshold.

3 The contraction method

We use the contraction method to show zero-freeness for a δ-strip containing part of the non-negative real line. The contraction method is an important technique of locating the zeros of graph polynomials [Asa70, Rue71]. It was first introduced by Asano [Asa70] as an alternative way of proving the celebrated Lee-Yang circle theorem [LY52].

The contraction method has two main ingredients. Firstly we want to relate zeros of a univariate polynomial with those of its polar form. For a polynomial $P(z) = \sum_{i=0}^{d'} a_i z^i$ of degree $d' \leq d$, its d-th polar form with variables $\boldsymbol{z} = (z_1, \ldots, z_d)$ is

$$\widehat{P}(\boldsymbol{z}) := \sum_{I \subseteq [d]} \frac{a_{|I|}}{\binom{d}{|I|}} z_I,$$

where $a_i = 0$ if $i > d'$, $[d]$ denotes $\{1, 2, \ldots, d\}$, and for an index set I, $z_I = \prod_{i \in I} z_i$. The polar form $\widehat{P}(\boldsymbol{z})$ is the unique multi-linear symmetric polynomial of degree at most d' such that $\widehat{P}(z, z, \ldots, z) = P(z)$. When $d' < d$, we view $P(z)$ as a degenerate case, and it has zeros at ∞ with multiplicity $d - d'$.

Let $\mathcal{C}$ be a region in $\mathbb{C}$. We say a polynomial $P(\boldsymbol{z})$ in $d \geq 1$ variables is *$\mathcal{C}$-stable* if $P(\boldsymbol{z}) \neq 0$ whenever $z_1, \ldots, z_d \in \mathcal{C}$. We call $\mathcal{C}$ *a circular region* if it is a disk, a half plane (a disk whose center is at infinity), or the complement of a disk in $\mathbb{C}$.[1]

The Grace-Szegő-Walsh coincidence theorem [Gra02, Sze22, Wal22] has the following immediate consequence.

PROPOSITION 3.1. *If $\mathcal{C}$ is a circular region and $P(z)$ is a non-degenerate univariate polynomial of degree d, then $P(z)$ is $\mathcal{C}$-stable if and only if its d-th polar form $\widehat{P}(\boldsymbol{z})$ is $\mathcal{C}$-stable.*

The next ingredient is the Asano contraction [Asa70, Rue71]. We will use a slightly different version than the standard one.

LEMMA 3.1. *Let K be a subset of the complex plane $\mathbb{C}$ which does not contain 0, and $d \geq 1$ be an integer. If the complex polynomial*

$$P(\boldsymbol{z}) := \sum_{I \subseteq [d]} c_I \prod_{i \in I} z_i$$

can vanish only when $z_i \in K$ for some $i \in [d]$, then

$$Q(z) := c_\emptyset + c_{[d]} z$$

can vanish only when $z \in \mathcal{K}_d := (-1)^{d+1} K \cdot K \cdots K$ (d times).

Proof. If $c_{[d]} = 0$, since $0 \notin K$, $P(0, 0, \ldots, 0) = c_\emptyset \neq 0$. Thus, $Q(z) = c_\emptyset \neq 0$ for any z.

Otherwise $c_{[d]} \neq 0$. Consider the univariate polynomial $\widetilde{P}(z) = P(z, z, \ldots, z)$ of degree d and let $\zeta_1, \ldots, \zeta_d$ be its roots. Clearly $\zeta_i \in K$ for all $i \in [d]$ because of the assumption. Thus, by Vieta's formula,

$$\prod_{i=1}^{d} \zeta_i = (-1)^d \frac{c_\emptyset}{c_{[d]}}.$$

It implies that $-\frac{c_\emptyset}{c_{[d]}} \in \mathcal{K}_d$. □

Some form of Lemma 3.1 was first discovered by Asano [Asa70] to provide a simple and alternative proof for the celebrated Lee-Yang circle theorem [LY52], where one chooses K to be the unit disk or its complement. The contraction method was further extended by Ruelle [Rue71] and applied to subgraph counting polynomials [Rue99], where one chooses K to be half planes. This choice has also found some algorithmic success recently [GLLZ19]. As we will see in the next section, our choices are much more intricate, including both disks and their complements, and the center and radius are carefully calculated so that the result is optimal for the contraction method.

[1] Including complements of disks is slightly more general than what is usually stated, but this definition suits our purposes better and Proposition 3.1 still holds with this definition. See for example [RS02, Section 3, Theorem 3.41b].

4 Analyzing the contraction

We begin with an overview of our approach, and outline the considerations in choosing the set K. The first step of the contraction method is to consider the partition function on a single edge, which is just a quadratic polynomial:

$$Z_{\text{spin}}(G;\lambda) = \gamma\lambda^2 + 2\lambda + \beta. \tag{4.4}$$

Due to the ferromagnetic assumption $\beta\gamma > 1$, the equation $\gamma x^2 + 2x + \beta = 0$ has two complex roots:

$$\zeta_1 = \frac{-1+\sqrt{1-\beta\gamma}}{\gamma}, \quad \zeta_2 = \frac{-1-\sqrt{1-\beta\gamma}}{\gamma}. \tag{4.5}$$

In particular $|\zeta_1| = |\zeta_2| = \sqrt{\frac{\beta}{\gamma}}$. Then, we observe that for any closed circular region K containing ζ_1 and ζ_2, the multivariate partition function $\gamma\lambda_1\lambda_2 + \lambda_1 + \lambda_2 + \beta$ can only vanish if either $\lambda_1 \in K$ or $\lambda_2 \in K$.

To see this, we note that the polar form of equation (4.4) is the same as the multivariate partition function on a single edge. Thus the observation follows directly from Proposition 3.1. As it turns out, the only constraint in choosing K is that it contains ζ_1 and ζ_2, and that it is a closed circular region.

Next, we apply Asano contraction in the form of Lemma 3.1, and keep track of the location of the zeros as we contract vertices. More specifically, we will consider a sequence of graphs $G_0, G_1, \cdots, G_n$, with G_0 being a disjoint union of singleton edges, and $G_n = G$ being the graph we are interested in. By applying Lemma 3.1 repeatedly, we will show that the following property is maintained throughout the contraction: the multivariate partition function $Z_{\text{spin}}(G_i;\boldsymbol{\lambda})$ can only vanish if for some vertex j, $\lambda_j \in \mathcal{K}_d$ where d is the degree of vertex j in the graph G_i. Therefore, to translate this back to the univariate partition function (in which all the λ_i's are equal), one naturally wants to choose K so as to maximize the region $\left(\bigcup_d \mathcal{K}_d\right)^{\mathsf{c}}$. As it will become clear, for the purpose of our application, it suffices to maximize the minimum of the intercept of $\mathcal{K}_d$ on the positive real line for $d \geq 2$.

In the following, we first describe our choice of the region K, and give a bound on the minimum intercept of $\mathcal{K}_d$ on the positive real line for all $d \geq 2$. Combining this bound with the contraction method outlined as above, we will prove the main result of this section in Theorem 4.1.

We will choose K to be a closed circular region based on (the sign of) the following quantity:

$$\Phi := \log\sqrt{\frac{\beta}{\gamma}} - \tan^{-1}\left(\sqrt{\beta\gamma-1}\right)\cdot\sqrt{\beta\gamma-1}. \tag{4.6}$$

The main case is when $\Phi < 0$, which includes the case of $\gamma > 1$. However, when γ is sufficiently close to $1/\beta$, $\Phi \geq 0$ and we need a different solution.

4.1 $\Phi < 0$

In this case we choose the circular region to be the open disk centered at some real $c \geq 0$ with radius $r > 0$, denoted by $\mathcal{D}(c,r)$. Namely, $\mathcal{D}(c,r) = \{z \in \mathbb{C} \mid |z-c| < r\}$. Let $\mathcal{C}(c,r) := \partial\mathcal{D}(c,r)$ be the circle centered at c with radius r. The region K in Proposition 3.1 and Lemma 3.1 will be chosen as the complement of the disk $\mathcal{D}(c,r)^{\mathsf{c}}$. An illustration of K and $\mathcal{K}_i$ for $i = 2,3,4$ is given in Figure 2.

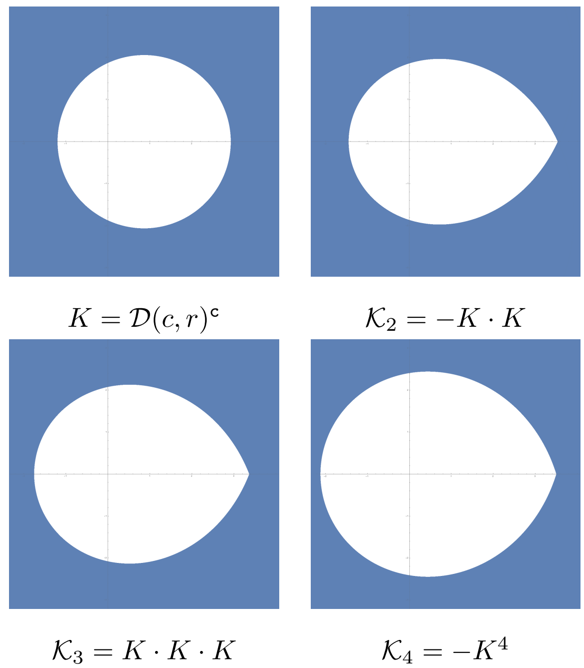


Figure 2: Our region $K = \mathcal{D}(c,r)^{\mathsf{c}}$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ in the case of $\beta = 3$ and $\gamma = \frac{4}{3}$. Here, the intercept of $\mathcal{K}_d$ on the positive real line is minimised at $d = 3$ for all $d \geq 2$.

As explained in the beginning of the section, we will ensure that ζ_i lies on the boundary of the disk $\mathcal{D}(c,r)$ we are choosing. Namely, once c is fixed, $r = r(c,\beta,\gamma)$ will be chosen to satisfy the following equation

$$\frac{\beta\gamma-1}{\gamma^2} + \left(c + \frac{1}{\gamma}\right)^2 = r^2. \tag{4.7}$$

Eventually, we will choose c to be

$$c^\star := \frac{-\beta\log\sqrt{\frac{\beta}{\gamma}}}{\Phi}. \tag{4.8}$$

We remark that most of the argument in this subsection does not require $\Phi < 0$, but only requires that $0 \leq c < r$

is a positive real number. The condition $\Phi < 0$ is only needed in the end, where we have to choose c.

For some integer d, we want to argue that $\mathcal{K}_d$ where $K = \mathcal{D}(c,r)^{\mathsf{c}}$ does not contain a neighbourhood of $[0,\lambda]$ for some $\lambda > 0$. Consider the following program:

(4.9)

$$\begin{aligned} \min \quad & \prod_{i=1}^{d} r_i \\ \text{subject to} \quad & \sum_{i=1}^{d} \theta_i = \begin{cases} 0 \mod 2\pi & \text{if } d \text{ is odd;} \\ \pi \mod 2\pi & \text{if } d \text{ is even;} \end{cases} \\ & \forall i \in [d],\ r_i \geq 0 \text{ and } 0 \leq \theta_i < 2\pi; \\ & \forall i \in [d],\ \left| r_i e^{\iota\theta_i} - c \right| \geq r. \end{aligned}$$

The last constraint ensures that $z_i := r_i e^{\iota\theta_i} \in \mathcal{D}(c,r)^{\mathsf{c}}$ and the objective is to minimise the smallest positive real value in $\mathcal{K}_d$. An illustration is given in Figure 3.

The usefulness follows from the next lemma. Let the optimal value of (4.9) be $\lambda_d^\star$. Thus $[0,\lambda_d^\star) \cap \mathcal{K}_d = \emptyset$. The lemma below follows from the fact that the complex plane is a normal Hausdorff space and both $\mathcal{K}_d$ and $[0,\lambda]$ are closed sets for any $\lambda > 0$.

LEMMA 4.1. *For any $d \geq 2$ and any $\lambda < \lambda_d^\star$, there is a δ-strip containing $[0,\lambda]$ that does not intersect $\mathcal{K}_d$ for some small δ.*

It remains to solve the program (4.9). Suppose the minimum is achieved by some $\boldsymbol{z} = \{r_i e^{\iota\theta_i}\}_{i\in[d]}$. First assume that there are at least two z_i in the right half plane, say z_1 and z_2. In other words, $\theta_i \in [0,\pi/2) \cup (3\pi/2, 2\pi)$, for $i = 1,2$. We replace θ_1 and θ_2 by $\theta_1' = \theta_1 + \pi \mod 2\pi$ and $\theta_2' = \theta_2 + \pi \mod 2\pi$. The effect of this substitution is

$$\theta_1 + \theta_2 = \theta_1' + \theta_2' \mod 2\pi, \quad \theta_1', \theta_2' \in [\pi/2, 3\pi/2].$$

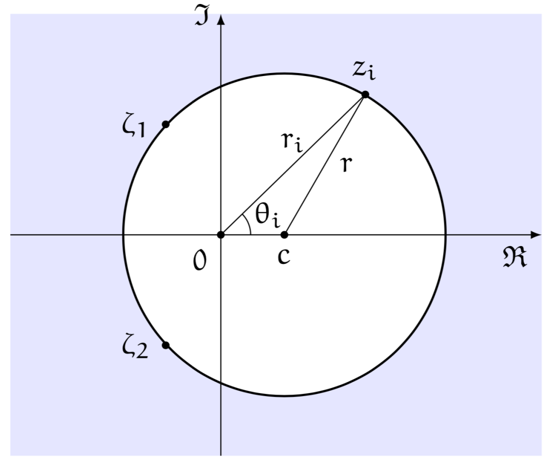
Moreover, for $i \in \{1,2\}$, if $r_i e^{\iota\theta_i} \in \mathcal{D}(c,r)^{\mathsf{c}}$, then $r_i e^{\iota\theta_i'} \in \mathcal{D}(c,r)^{\mathsf{c}}$ as well. This is because that the center of $\mathcal{D}(c,r)$ is a positive real number. Therefore, we may assume that there is at most one z_i such that $\theta_i \in [0,\pi/2) \cup (3\pi/2, 2\pi)$.

Next observe that if we shrink r_i until z_i is on the circle $\mathcal{C}(c,r)$, then the optimal value only improves. Thus we may assume that all z_i are on the circle $\mathcal{C}(c,r)$. As a consequence, r_i is determined by θ_i for all $i \in [d]$. Indeed, by the cosine law and equation (4.7),

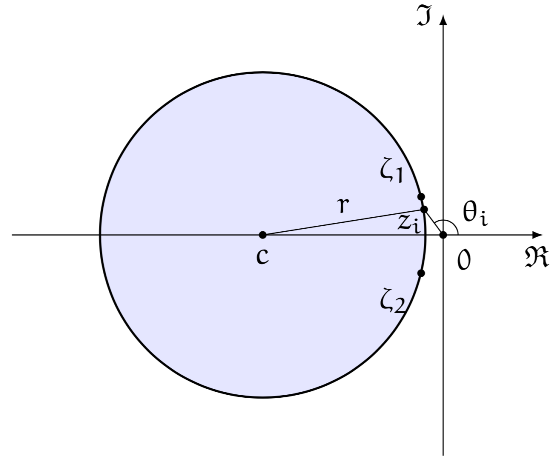
$$r_i^2 + c^2 - 2cr_i \cos\theta_i = r^2 = \frac{\beta\gamma - 1}{\gamma^2} + \left(c + \frac{1}{\gamma}\right)^2,$$

which implies that

$$r_i^2 - 2cr_i \cos\theta_i - \frac{\beta + 2c}{\gamma} = 0.$$



Parameters: $\beta = 3$, $\gamma = 4/3$. In this case $\Phi < 0$ and $c > 0$.



Parameters: $\beta = 4$, $\gamma = 1/2$. In this case $\Phi > 0$ and $c < 0$.

Figure 3: Illustrations for the programs (4.9) and (4.15). Feasible regions are colored blue.

Since one of the solutions is negative, solving r_i we have that $r_i = f(\theta_i)$, where

$$f(x) := c\cos x + \sqrt{c^2\cos^2 x + \frac{\beta + 2c}{\gamma}}. \tag{4.10}$$

The next lemma states that we can further assume that all z_i on the left half plane to be the same.

LEMMA 4.2. *Let $0 \leq c < r$. Suppose all $i \in [k]$, $z_i = r_i e^{\iota\theta_i}$ is on $\mathcal{C}(c,r)$ and $\theta_i \in [\pi/2, 3\pi/2]$. Let $\widehat{z} = \widehat{r}e^{\iota\widehat{\theta}}$ be on $\mathcal{C}(c,r)$ such that $\widehat{\theta} = \frac{1}{k}\sum_{i=1}^{k}\theta_i$. Then, $\prod_{i=1}^{k} r_i \geq \widehat{r}^k$.*

Proof. We just need to show that if $x \in [\pi/2, 3\pi/2]$, then $g(x) := \log f(x)$ is a convex function and Jensen's inequality applies, where $f(x)$ is defined in equation (4.10). This can be verified by a straightforward calculation that

$$g''(x) = -\cos x \cdot \frac{c\sqrt{\gamma}(\beta+2c+c^2\gamma)}{(\beta+2c+c^2\gamma\cos^2 x)^{3/2}} \geq 0,$$

as $x \in [\pi/2, 3\pi/2]$. ∎

We still need to handle the possibility that one of z_i, say z_1, is on the right half plane.

Lemma 4.3. *Let $0 \leq c < r$. Let $d \geq 2$ be an integer and k be another integer whose parity is the opposite from that of d. Let z_1 and $\widehat{z}$ be two complex numbers on $\mathcal{C}(c, r)$. Suppose that $z_1 = r_1 e^{\iota\theta_1}$ where $\theta_1 \in [0, \pi/2) \cup (3\pi/2, 2\pi)$ and $\widehat{z} = \widehat{r}e^{\iota\widehat{\theta}}$ where $\widehat{\theta} \in [\pi/2, 3\pi/2]$. If $\theta_1 + (d-1)\widehat{\theta} = k\pi$ is fixed, then the minimum of $\widehat{r}^{d-1} r_1$ is attained either when $\theta_1 = \pi/2$ or $\theta_1 = 0$.*

Proof. As $\pi = -\pi \mod 2\pi$, by taking the complex conjugate if necessary, we may assume that $\theta_1 \in [0, \pi/2]$. Then, as θ_1 increases, $\widehat{\theta}$ decreases. If $\widehat{\theta} \in (\pi, 3\pi/2]$, then as θ_1 increases, both r_1 and $\widehat{r}$ decreases and the lemma holds. So we only need to handle the case that $\widehat{\theta} \in [\pi/2, \pi]$.

As $\theta_1 + (d-1)\widehat{\theta} = k\pi$, $\widehat{\theta} = \frac{k\pi - \theta_1}{d-1}$. Using equation (4.10), we then can write $(d-1)\log \widehat{r} + \log r_1$ as a function in θ_1, denoted $\tau(\theta_1)$. The minimum of $\widehat{r}^{d-1} r_1$ is attained as long as the minimum of $\tau(\theta_1)$ is attained. A straightforward calculation yields

$$\tau'(\theta_1) = c\sqrt{\gamma}\left(\frac{\sin(\widehat{\theta})}{\sqrt{2c+\beta+c^2\gamma\cos^2(\widehat{\theta})}} - \frac{\sin(\theta_1)}{\sqrt{2c+\beta+c^2\gamma\cos^2(\theta_1)}}\right). \tag{4.11}$$

Note that $\frac{x}{\sqrt{2c+\beta+c^2\gamma(1-x^2)}}$ is an increasing function for $0 < x < 1$.

If $\theta_1 + \widehat{\theta} \geq \pi$, then $\sin(\widehat{\theta}) \leq \sin(\theta_1)$ and τ is a decreasing function in θ_1. In this case, if we increase θ_1, the decrease of $\widehat{\theta}$ is smaller, and thus the assumption that $\theta_1 + \widehat{\theta} \geq \pi$ is maintained. We can keep increasing θ_1 until it hits $\pi/2$.

Otherwise $\theta_1 + \widehat{\theta} < \pi$ and τ is increasing. Similar to the case above, we can keep decreasing θ_1 until it hits 0. The lemma follows from the two cases above. ∎

Now we can argue when the minimum of the program (4.9) is achieved.

Lemma 4.4. *Let $0 \leq c < r$. For any $d \geq 2$, the minimum of the program (4.9) is achieved when all z_i's are equal and $\theta_i = \frac{d-1}{d} \cdot \pi$ for all i.*

Proof. As argued above, we may assume that either all z_i's are on the left half plane or only z_1 is on the right half plane. In the former case, by Lemma 4.2, we may assume that all z_i's in the left half plane are equal. In the latter case, by Lemma 4.4, We can assume that either $\theta_1 = \pi/2$ or $\theta_1 = 0$:

- if $\theta_1 = \pi/2$, then we invoke Lemma 4.2 again to reduce to the case where all z_i's are equal;
- if $\theta_1 = 0$, then by Lemma 4.2, we can assume that all other z_i's are equal. As $\pi = -\pi \mod 2\pi$, by taking the complex conjugate if necessary, we can also assume that $\theta_i \in [\pi/2, \pi]$ for all $i \geq 2$. Then because of the constraint on $\sum_{i=2}^{d} \theta_i$, there must exist a positive integer k whose parity is opposite to that of d and that $\theta_i = \frac{k\pi}{d-1}$ for all $i \geq 2$. It is a simple geometric fact that if $\theta_i \in [\pi/2, \pi]$, r_i decreases as θ_i increases as z_i lies on $\mathcal{C}(c, r)$. On the other hand, $\theta_i \leq \pi$ implies that $k \leq d$. Because k has the opposite parity against d, to achieve the minimum in (4.9), $k = d-1$ and $\theta_i = \pi$ for all $i \geq 2$.

  As $d \geq 2$, consider $r_1 r_2$. Since $\theta_1 = 0$ and $\theta_2 = \pi$, $r_1 = r + c$ and $r_2 = r - c$, and $r_1 r_2 = r^2 - c^2$. We can replace both of them by $z_1' = z_2' = r' e^{\iota\theta'}$ where $r' = \sqrt{r^2 - c^2}$ and $\theta' = \frac{\pi}{2}$. It is straightforward to verify that z_1' and z_2' are on the circle $\mathcal{C}(c, r)$, and $r_1 r_2 = r_1' r_2'$. Thus we are reduced to the setting of Lemma 4.2, and applying it makes all z_i's equal.

To summarize, in all cases, we can assume that all z_i's are equal.

Similar to the complicated case above, we now assume that there is an integer k such that $\theta_i = \frac{k\pi}{d}$ for all $i \in [d]$, $k \leq d$, and k has the opposite parity against d. Once again, the larger k the smaller $\prod_{i \in [d]} r_i$. Thus, the minimum is achieved when $k = d-1$ and $\theta_i = \frac{d-1}{d} \cdot \pi$ for all i. The lemma holds. ∎

By Lemma 4.4 and equation (4.10),

$$\begin{aligned}
\lambda_d^\star &= \left(f\left(\pi - \frac{\pi}{d}\right)\right)^d \\
&= \left(c\cos\left(\pi - \frac{\pi}{d}\right) + \sqrt{c^2\cos^2\left(\pi - \frac{\pi}{d}\right) + \frac{\beta+2c}{\gamma}}\right)^d \\
&= \left(-c\cos\frac{\pi}{d} + \sqrt{c^2\cos^2\frac{\pi}{d} + \frac{\beta+2c}{\gamma}}\right)^d.
\end{aligned} \tag{4.12}$$

Still, as we want to deal with vertices of all degrees, we need to determine when the expression in equation (4.12) attains its minimum when d varies. We will view $\lambda_d^\star$ as a function of d with the expression in equation (4.12), and relax d to be a continuous variable taking values in $[2,\infty)$. With this in mind, let $h(d) := \log \lambda_d^\star$. We take derivatives of $h(d)$:

$$h'(d) = \frac{\log \lambda_d^\star}{d} - \frac{c\pi\sqrt{\gamma}\sin(\pi/d)}{d\sqrt{\beta+2c+c^2\gamma\cos^2(\pi/d)}}; \tag{4.13}$$

$$h''(d) = \cos\frac{\pi}{d}\cdot\frac{c\pi^2\sqrt{\gamma}(\beta+2c+c^2\gamma)}{d^3\left(\beta+2c+c^2\gamma\cos^2(\pi/d)\right)^{3/2}}. \tag{4.14}$$

As $d \geq 2$, $h''(d) \geq 0$ and $h(d)$ is a convex function. Thus, the minimum of $h(d)$ (and therefore that of $\lambda_d^\star$) is attained at the solution to $h'(d) = 0$. Call the solution $d^\star$ and let $\lambda^\star := \lambda_{d^\star}^\star$. To summarize the argument above, we have the following lemma.

Lemma 4.5. *Let $0 \leq c < r$. For any $d \geq 2$, $\lambda_d^\star \geq \lambda^\star$.*

The only remaining task is to find out how large $\lambda^\star$ is and this depends on the value of c. Recall ζ_1 and ζ_2 in equation (4.5). Since both ζ_1 and ζ_2 must be in K, our idea is to ensure that the minimum is attained precisely at the first time when ζ_1 and ζ_2 *could* reach the positive real line under Minkowski product of $d^\star$ times. To do so, we will choose c such that $\pi - \pi/d^\star = \arg\zeta_1$. In other words, we want that $\tan(\pi/d^\star) = \sqrt{\beta\gamma-1}$. Let $d^\star := \frac{\pi}{\tan^{-1}\left(\sqrt{\beta\gamma-1}\right)}$, where we take the principle branch of $\tan^{-1}(\cdot) \in (-\pi/2,\pi/2)$. In this case, $\lambda^\star = |\zeta_1|^{d^\star} = \left(\frac{\beta}{\gamma}\right)^{d^\star/2}$.

Lemma 4.6. *If $\Phi < 0$, then we can choose $c = c^\star > 0$ in equation (4.8) so that $\lambda^\star = \left(\frac{\beta}{\gamma}\right)^{d^\star/2}$, where $d^\star = \frac{\pi}{\tan^{-1}\left(\sqrt{\beta\gamma-1}\right)}$.*

Proof. Denote the right hand side of equation (4.13) by $\rho(c,d)$. Then

$$\rho(c,d^\star) = \log\sqrt{\frac{\beta}{\gamma}} - \tan^{-1}\left(\sqrt{\beta\gamma-1}\right)\cdot\frac{c\sqrt{\beta\gamma-1}}{\beta+c}.$$

It is straightforward to verify that $\rho(c^\star,d^\star) = 0$.

Since $h''(d) \geq 0$, $h'(d) = 0$ has at most one zero in d for any fixed c. Once we chose $c = c^\star$, $d^\star$ is the unique zero of $h'(d)$. The lemma follows. ∎

4.2 $\Phi > 0$ When $\Phi > 0$, the argument is almost the same as or even simpler than that in Section 4.1. The main issue is that following Lemma 4.6 would yield $c < 0$ and some geometry changes. We consider instead a disk $\mathcal{D}(c,r)$ with $c < -\beta < 0$. Eventually we also choose $c = c^\star$ according to equation (4.8), although now $c^\star < -\beta < 0$ as $\Phi > 0$. The radius r is still chosen according to equation (4.7) such that ζ_1,ζ_2 are on $\mathcal{C}(c,r)$. The main change is that now we choose region $K = \overline{\mathcal{D}(c,r)}$. Namely, K is the closure of $\mathcal{D}(c,r)$ instead of its complement. An illustration of K and $\mathcal{K}_i$ for $i = 2,3,4,5$ is given in Figure 4.

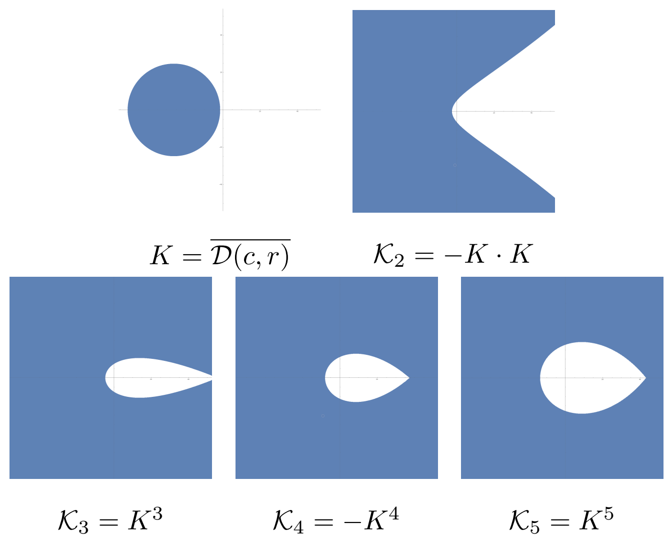


Figure 4: Our region $K = \overline{\mathcal{D}(c,r)}$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ and $\mathcal{K}_5$ in the case of $\beta = 4$ and $\gamma = \frac{1}{2}$. Here, the intercept of $\mathcal{K}_d$ on the positive real line is minimised at $d = 4$ for all $d \geq 2$.

Then, the program (4.9) becomes

$$\begin{aligned} &\min && \prod_{i=1}^{d} r_i \\ &\text{subject to} && \sum_{i=1}^{d}\theta_i = \begin{cases} 0 & \bmod 2\pi \quad \text{if } d \text{ is odd;} \\ \pi & \bmod 2\pi \quad \text{if } d \text{ is even;} \end{cases} \\ &&& \forall i\in[d],\ r_i \geq 0 \text{ and } 0 \leq \theta_i < 2\pi; \\ &&& \forall i\in[d],\ \left|r_i e^{\iota\theta_i} - c\right| \leq r. \end{aligned} \tag{4.15}$$

Still denote the optima of (4.15) by $\lambda_d^\star$ and it is easy to verify that Lemma 4.1 holds in this setting. An illustration can be found in Figure 3.

As $c < -\beta$, it is easy to verify that $c < -r$ using *equation* (4.7), and $0 \notin K$. So for any z_i, we can shrink it until it hits the right boundary of $\mathcal{C}(c,r)$. In this case, similar to equation (4.10), $r_i = \widetilde{f}(\theta_i)$, where

$$\widetilde{f}(x) := c\cos x - \sqrt{c^2\cos^2 x + \frac{\beta+2c}{\gamma}}. \tag{4.16}$$

The sign changed because now there are two positive solutions and we should choose the smaller one.

Moreover, notice that due to the constraints in (4.15), $\theta_i \in [\pi/2, 3\pi/2]$ and is further constrained into a range so that $\widetilde{f}(\cdot)$ is real, namely

$$c^2\gamma\cos^2\theta_i + \beta + 2c > 0. \tag{4.17}$$

In particular, since $\theta_i = \pi$ satisfies the constraint of (4.15), $c^2\gamma + \beta + 2c > 0$. The analogue of Lemma 4.2 also holds.

Lemma 4.7. *Let $c < -r < 0$. Suppose all $i \in [k]$, $z_i = r_i e^{\iota\theta_i}$ where $r_i = \widetilde{f}(\theta_i)$ and equation (4.17) holds. Let $\widehat{z} = \widehat{r}e^{\iota\widehat{\theta}}$ be on $\mathcal{C}(c, r)$ such that $\widehat{\theta} = \frac{1}{k}\sum_{i=1}^k \theta_i$. Then, $\prod_{i=1}^k r_i \ge \widehat{r}^k$.*

Proof. The proof goes through similar calculations to that of Lemma 4.2. Let $\widetilde{g} = \log\widetilde{f}$. Then for x such that equation (4.17) holds,

$$\widetilde{g}''(x) = \cos x \cdot \frac{c\sqrt{\gamma}(\beta + 2c + c^2\gamma)}{(\beta + 2c + c^2\gamma\cos^2 x)^{3/2}} \ge 0,$$

as $\beta + 2c + c^2\gamma > 0$. ∎

Since in this case all z_i's are on the left half plane, there is no need to consider z_i's on the right half plane like Lemma 4.3. We directly go to the analogue of Lemma 4.4.

Lemma 4.8. *Let $c < -r < 0$. For any $d \ge 2$, the minimum of the program (4.15) is achieved when all z_i's are equal and $\theta_i = \frac{d-1}{d}\cdot\pi$ for all i.*

Proof. We first invoke Lemma 4.7 to assume that all z_i's are equal. Therefore there exists k of opposite parity against d such that $\theta_i = \frac{k\pi}{d}$. We may assume that $\theta_i \in [\pi/2, \pi]$ by taking conjugates if necessary. Then, r_i is a decreasing function in θ_i, and the minimum of $\prod_{i=1}^d r_i$ is achieved when $k = d - 1$. ∎

Some calculations need to be changed due to the sign change in equation (4.16). By Lemma 4.8 and equation (4.16),

$$\lambda_d^\star = \left(-c\cos\frac{\pi}{d} - \sqrt{c^2\cos^2\frac{\pi}{d} + \frac{\beta + 2c}{\gamma}}\right)^d. \tag{4.18}$$

Let $\widetilde{h}(d) := \log\lambda_d^\star$ where $\lambda_d^\star$ is given as the expression in equation (4.18). We take derivatives of $\widetilde{h}(d)$:

$$\widetilde{h}'(d) = \frac{\log\lambda_d^\star}{d} + \frac{c\pi\sqrt{\gamma}\sin(\pi/d)}{d\sqrt{\beta + 2c + c^2\gamma\cos^2(\pi/d)}}; \tag{4.19}$$

$$\widetilde{h}''(d) = -\frac{c\pi^2\sqrt{\gamma}(\beta + 2c + c^2\gamma)\cdot\cos\frac{\pi}{d}}{d^3\left(\beta + 2c + c^2\gamma\cos^2(\pi/d)\right)^{3/2}}. \tag{4.20}$$

As $d \ge 2$ and $\beta + 2c + c^2\gamma > 0$, $\widetilde{h}''(d) \ge 0$ and $\widetilde{h}(d)$ is still a convex function. Thus, the minimum of $\widetilde{h}(d)$ is attained at the solution to $\widetilde{h}'(d) = 0$. With a little abuse of notation, call the solution $d^\star$ and let $\lambda^\star := \lambda_{d^\star}^\star$.

Lemma 4.9. *Let $c < -r < 0$. For any $d \ge 2$, $\lambda_d^\star \ge \lambda^\star$.*

We still need to choose c so that $d^\star = \frac{\pi}{\tan^{-1}\left(\sqrt{\beta\gamma - 1}\right)}$.

Lemma 4.10. *If $\Phi > 0$, then we can choose $c = c^\star < -\beta < 0$ in equation (4.8) so that $\lambda^\star = \left(\frac{\beta}{\gamma}\right)^{d^\star/2}$, where $d^\star = \frac{\pi}{\tan^{-1}\left(\sqrt{\beta\gamma - 1}\right)}$.*

Proof. Denote the right hand side of equation (4.19) by $\widetilde{\rho}(c, d)$. Then

$$\begin{aligned}\widetilde{\rho}(c, d^\star) &= \log\sqrt{\frac{\beta}{\gamma}} + \frac{\pi}{d^\star}\cdot\sqrt{\beta\gamma - 1}\cdot\frac{c}{|\beta + c|} \\ &= \log\sqrt{\frac{\beta}{\gamma}} - \tan^{-1}\left(\sqrt{\beta\gamma - 1}\right)\cdot\sqrt{\beta\gamma - 1}\cdot\frac{c}{\beta + c}.\end{aligned}$$

It is straightforward to verify that $\rho(c^\star, d^\star) = 0$.

Since $\widetilde{h}''(d) \ge 0$, $\widetilde{h}'(d) = 0$ has at most one zero in d for any fixed c. Once we chose $c = c^\star$, $d^\star$ is the unique zero of $\widetilde{h}'(d)$. The lemma follows. ∎

4.3 $\Phi = 0$ In fact, the arguments in Section 4.1 and Section 4.2 can be viewed as moving c from 0 to ∞, then "wrapping around" to $-\infty$, and eventually to $-\beta$. The threshold case of $\Phi = 0$ requires us to take $c = \infty$, in which case K becomes the closed half plane $\{z \mid \Re z \le -\frac{1}{\gamma}\}$. The program becomes

$$\begin{aligned} \min \quad & \prod_{i=1}^d r_i \\ \text{subject to} \quad & \sum_{i=1}^d \theta_i = \begin{cases} 0 \mod 2\pi & \text{if } d \text{ is odd;} \\ \pi \mod 2\pi & \text{if } d \text{ is even;} \end{cases} \\ & \forall i \in [d],\ r_i \ge 0 \text{ and } \theta_i \in (\pi/2, 3\pi/2); \\ & \forall i \in [d],\ -r_i \le \frac{1}{\gamma}. \end{aligned} \tag{4.21}$$

Still denote the optima of (4.21) by $\lambda_d^\star$ and it is easy to verify that Lemma 4.1 holds in this setting.

Once again, we can assume that all $z_i's$ are on the boundary, namely that $\Re z = -\frac{1}{\gamma}$. In this case

$$r_i = -\frac{1}{\gamma\cos\theta_i}. \tag{4.22}$$

It is easy to check that $\log r_i = -\log(-\cos\theta_i) - \cos\gamma$ is a convex function. By the same argument as in Lemma 4.8,

$$\lambda_d^\star = \frac{1}{\gamma^d \cos^d(\pi/d)}. \tag{4.23}$$

LEMMA 4.11. *If $\Phi = 0$, then choosing $K = \{z \mid \Re z \le -\frac{1}{\gamma}\}$ ensures that for any $d \ge 2$, $\lambda_d^\star \ge \lambda^\star = \left(\frac{\beta}{\gamma}\right)^{d^\star/2}$, where $d^\star = \frac{\pi}{\tan^{-1}(\sqrt{\beta\gamma-1})}$.*

Proof. We just need to verify that $\log\lambda_d^\star = -d\log\gamma - d\log\cos(\pi/d)$ (with $\lambda_d^\star$ in equation (4.23)) takes its minimum at $d = d^\star$. In this case,

$$(\log\lambda_d^\star)' = -\log\gamma - \log\cos(\pi/d) - \frac{\pi\tan(\pi/d)}{d};$$
$$(\log\lambda_d^\star)'' = \frac{\pi^2}{d^3\cos^2(\pi/d)} \ge 0.$$

The lemma follows from $(\log\lambda_{d^\star}^\star)' = 0$, which can be verified using $\Phi = 0$, $\cos(\pi/d^\star) = 1/\sqrt{\beta\gamma}$, and $\tan(\pi/d^\star) = \sqrt{\beta\gamma-1}$. ∎

4.4 Proof of Theorem 1.1 and Theorem 1.2 In order to avoid considering infinitely many degrees, we observe the following.

LEMMA 4.12. *Let $\beta > \gamma$ be the parameters. For any of our chosen K, if $z \in K$, then $|z| > 1$.*

Proof. The assumption $\beta > \gamma$ implies that $\lambda^\star > 1$. Assume otherwise that $\exists z \in K$ such that $|z| \le 1$. Observe that our chosen K is either centered around the real axis, or being a half plane (in the case of $\Phi = 0$). Therefore, there must be a real number $z \in K \cap \mathbb{R}$ such that $|z| \le 1$. Then, either z^2 or z^3 will be a positive real number less or equal to 1. In other words, either $\mathcal{K}_2$ or $\mathcal{K}_3$ will intersect the positive real interval $(0, 1]$. On the other hand, Lemmas 4.6, 4.10 and 4.11 all say that the minimum intercept of $\mathcal{K}_d$ on the positive real line for all $d \ge 2$ is $\lambda^\star > 1$. This is a contradiction. ∎

Our method in fact shows a multivariate version of Theorem 1.2. Recall the definition of the multivariate partition function in equation (2.3).

THEOREM 4.1. *Let β, γ be positive parameters such that $\beta \ge \gamma$ and $\beta\gamma > 1$, and $\lambda^\star$ defined as in Theorem 1.1. There exists a $\delta > 0$ such that for any $\lambda' < \lambda^\star$ and any graph $G = (V, E)$ such that $\deg_G(v) \ge 2$ for all $v \in V$, $Z_{spin}(G; \boldsymbol{\lambda})$ does not vanish in a δ-polystrip containing the poly-region $[0, \lambda']^n$ where $n = |V|$.*

Proof. First we claim that for any $\lambda' < \lambda^\star$, we can choose a δ-strip $\mathcal{N}$ containing $[0, \lambda']$ for $\lambda' < \lambda^\star$ so that it does not intersect $\mathcal{K}_d$ for any $d \ge 2$, and the δ-polystrip $\mathcal{N}^n$ is what we choose in the theorem. Lemmas 4.1, 4.5, 4.6 and 4.9 to 4.11 together imply that for any single d, there is a δ_d-strip covering $[0, \lambda']$ that does not intersect $\mathcal{K}_d$. If $\beta > \gamma$, by Lemma 4.12, $|z| > 1$ for any $z \in K$. For sufficiently large d, for any $z \in \mathcal{K}_d$, $|z| > \lambda^\star$. Thus, we only need to take δ to be the minimum one among finitely many δ_d's. If $\beta = \gamma$, then K is the unit circle, $\mathcal{K}_d = K$ for any $d \ge 2$, and $\lambda^\star = 1$. In this case, clearly the claim holds.

We consider a sequence of graphs $G_0, G_1, \ldots, G_n = G$. For $G_0 = (V_0, E_0)$, we replace each vertex $v \in V$ by $d = \deg_G(v)$ copies, denoted $v_1, v_2, \ldots, v_d$, and connect them according to E so that G_0 is a disjoint union of isolated edges. Then

$$Z_{\text{spin}}(G_0; \lambda) = \prod_{e \in E} \left(\gamma\lambda^2 + 2\lambda + \beta\right).$$

The only zeros of Z_{spin} are ζ_1 and ζ_2, both of which are in the circular region K chosen according to Lemmas 4.6, 4.10 and 4.11. Now consider the multivariate polynomial

$$Z_{\text{spin}}(G_0; \boldsymbol{z}^0) := \prod_{(u_i, v_j) \in E_0} \left(\gamma z_{u_i} z_{v_j} + z_{u_i} + z_{v_j} + \beta\right),$$

where $\boldsymbol{z}^0 = \{z_v\}_{v \in V_0}$. The notation is justified by the fact that $Z_{\text{spin}}(G_0; \boldsymbol{z}^0)$ is the multivariate partition function equation (2.3) for G_0 when we view $\{\lambda_v\}_{v \in V_0}$ as variables. By Proposition 3.1, $Z_{\text{spin}}(G_0; \boldsymbol{z}^0)$ does not vanish if $z_v \notin K$ for all $v \in V_0$.

Fix an arbitrary ordering of vertices in V, say $v^1, \ldots, v^n$. For $1 \le i \le n$, we construct $G_i = (V_i, E_i)$ by contracting $v_1^i, \ldots, v_{d_i}^i$ in G_{i-1}, where $d_i = \deg_G(v_i)$. In other words, we replace all of $v_1^i, \ldots, v_{d_i}^i$ by v^i, and connect v^i to all vertices adjacent to $\{v_j^i\}_{j \in [d_i]}$. Clearly $G_n = G$. Note that for all $1 \le i \le n$,

$$V_i = \{v^1, v^2, \ldots, v^i\} \cup \bigcup_{i < j \le n} \{v_1^j, \ldots, v_{d_j}^j\}$$

and

$$\begin{aligned} &\deg_{G_i}(v^j) = \deg_G(v^j) = d_j && \text{for } j \le i; \\ &\deg_{G_i}(v_k^j) = 1 && \text{for } j > i \text{ and } 1 \le k \le d_j. \end{aligned}$$

In particular, $\deg_{G_i}(v) = \deg_{G_{i-1}}(v)$ for all $v \in V_i \cap V_{i-1}$.

Moreover, the corresponding polynomial $Z_{\text{spin}}(G_i; \boldsymbol{z}^i)$, where $\boldsymbol{z}^i = \{z_v\}_{v \in V_i}$, is obtained by the operation in Lemma 3.1 applied to $\{z_{v_j^i}\}_{j \in [d_i]}$.

Namely, we replace all appearances of $\prod_{j=1}^{d_i} z_{v_j^i}$ by z_{v^i}, and remove all other terms involving at least one of $z_{v_j^i}$. Inductively, the obtained polynomial is still the multivariate partition function equation (2.3) for G_i. Thus, $Z_{\mathrm{spin}}(G_n;\boldsymbol{\lambda})$ defined this way coincides with $Z_{\mathrm{spin}}(G;\boldsymbol{\lambda})$ in equation (2.3).

We claim that $Z_{\mathrm{spin}}(G_i;\boldsymbol{z}^i) \neq 0$ if for all $v \in V_i$, $z_v \notin \mathcal{K}_d$ where $d = \deg_{G_i}(v)$.

We show the claim by induction. For the base case, notice that $\deg_{G_0}(v) = 1$ for all $v \in V_0$ and $\mathcal{K}_1 = K$. Thus the base case follows from Proposition 3.1.

For the induction step, let $\boldsymbol{x}$ be a vector such that for all $v \in V_i$, $x_v \notin \mathcal{K}_d$ where $d = \deg_{G_i}(v)$. Let $\boldsymbol{x}'$ be $\boldsymbol{x}$ restricted to $V_{i-1} \cap V_i = V_{i-1} \setminus \{v_1^i, \dots, v_{d_i}^i\} = V_i \setminus \{v^i\}$. To apply Lemma 3.1, consider the polynomials

$$P(z_{v_1^i}, \dots z_{v_{d_i}^i}) := Z_{\mathrm{spin}}(G_{i-1};\boldsymbol{x}', z_{v_1^i}, \dots z_{v_{d_i}^i});$$
$$Q(z_{v^i}) := Z_{\mathrm{spin}}(G_i;\boldsymbol{x}', z_{v^i}).$$

In other words, P is the polynomial $Z_{\mathrm{spin}}(G_{i-1};\boldsymbol{z}^{i-1})$ with all variables except $\{z_{v_j^i}\}_{j\in[d_i]}$ fixed to $\boldsymbol{x}'$, and Q is the polynomial $Z_{\mathrm{spin}}(G_i;\boldsymbol{z}^i)$ with all variables except z_{v^i} fixed to $\boldsymbol{x}'$. Since for all $v \in V_{i-1} \cap V_i$, $x_v \notin \mathcal{K}_d$ where $d = \deg_{G_i}(v) = \deg_{G_{i-1}}(v)$, the induction hypothesis implies that P does not vanish if for all $j \in [d_i]$, $z_{v_j^i} \notin K$ as $\deg_{G_{i-1}}(v_j^i) = 1$. By Lemma 3.1, Q does not vanish if $z_{v^i} \notin \mathcal{K}_{d_i}$. Notice that $d_i = \deg_{G_i}(v^i)$ and $x_{v^i} \notin \mathcal{K}_{d_i}$. Thus $Z_{\mathrm{spin}}(G_i;\boldsymbol{x}) = Q(x_{v^i}) \neq 0$. The induction step holds.

Our choice of $\mathcal{N}$ already ensures that any $\mathcal{N} \cap \mathcal{K}_d = \emptyset$ for all $d \geq 2$. The theorem follows from the claim for $i = n$. □

Theorem 1.2 is a simple corollary of Theorem 4.1. To prove Theorem 1.1, we need to take some special care of degree 1 vertices.

Proof of Theorem 1.1. Let $G = (V, E)$ be a graph and $v \in V$ such that $\deg_G(v) = 1$. Let $\boldsymbol{\lambda}$ be the (not necessarily uniform) vertex weights. Let the unique neighbour of v in G be u. The "pruning" operation is the following. Construct $G' = G[V \setminus \{v\}]$ and $\lambda'_w = \lambda_w$ if $w \neq u$ and $\lambda'_u = \lambda_u \cdot \frac{\lambda_v\gamma+1}{\lambda_v+\beta}$. Then $Z_{\mathrm{spin}}(G;\boldsymbol{\lambda}) = (\beta + \lambda_v) \cdot Z_{\mathrm{spin}}(G';\boldsymbol{\lambda}')$.

Notice that if $\lambda_v \leq \frac{\beta-1}{\gamma-1}$, then $\frac{\lambda_v\gamma+1}{\lambda_v+\beta} \leq 1$. Moreover, $\lambda^\star \leq \lambda_c$ and $\lambda_c \leq \frac{\beta-1}{\gamma-1}$ [GL18, Lemma 3.2]. Thus, in the assumed range of parameters, we can keep pruning leaves until there is none, and all λ_v after pruning still satisfies that $\lambda_v < \lambda' < \lambda^\star$. When there are no degree 1 vertices, we apply Lemma 2.2 and Theorem 4.1. □

As explained in Section 2, the running time of our algorithm in Theorem 1.1 depends on δ. However, due to Lemma 4.12, we only need to consider finitely many degrees when choosing δ. It implies that our δ does not depend on the maximum degree Δ of the underlying graph, which is different from some previous work, such as [LSS19a]. The overall running time of our algorithm is $O\left(\left(\frac{n}{\varepsilon}\right)^{O(\log\Delta)}\right)$ for any fixed parameters β, γ, and λ.

5 Concluding remarks

The main limit of our approach is that the roots ζ_1, ζ_2 to the single edge case are fixed. Any circular region we choose in Proposition 3.1 and subsequently in Lemma 3.1 must contain ζ_1 and ζ_2. If the degree d of a vertex is very close to $\tan^{-1}\left(\sqrt{\beta\gamma-1}\right)$, then ζ_1 will be mapped to very close to the real axis after the contraction. Thus, our best hope is to make sure that this is the worst case, and that is exactly what we do in Lemmas 4.6, 4.10 and 4.11. This seems to be an inherent difficulty to the contraction method on ferromagnetic 2-spin systems.

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