

# HOMOTOPY TYPES AND GEOMETRIES BELOW $\text{Spec } \mathbf{Z}$

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**Abstract.** After the first heuristic ideas about “the field of one element”  $\mathbf{F}_1$  and “geometry in characteristics 1” (J. Tits, C. Deninger, M. Kapranov, A. Smirnov et al.), there were developed several general approaches to the construction of “geometries below  $\text{Spec } \mathbf{Z}$ ”. Homotopy theory and the “the brave new algebra” were taking more and more important places in these developments, systematically explored by B. Toën and M. Vaquié, among others.

This article contains a brief survey and some new results on *counting problems* in this context, including various approaches to zeta–functions and generalised scissors congruences.

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## 1. Brief summary and plan of exposition

**1.1. Geometries below  $\text{Spec } \mathbf{Z}$ : a general categorical framework.** Following [ToVa09], Sec. 2.2 – 2.5, we start with a symmetric monoidal category with unit  $(C, \otimes, \mathbf{1})$ .

The category of commutative associative unital monoids  $\text{Comm}(C)$  will play the role of commutative rings; accordingly, the opposite category  $\text{Aff}_C := \text{Comm}(C)^{op}$  will be an analogue of the category of *affine schemes*.

In order to be able to define more general schemes, objects of a category  $\text{Sch}_C$ , we must introduce upon  $\text{Aff}_C$  a *Grothendieck topology* by giving a collection of sieves (covering families) defined for each object of  $\text{Aff}_C$ .

It is shown in [ToVa09] that if  $(C, \otimes, \mathbf{1})$  is complete, cocomplete and closed, then there are several natural topologies upon  $\text{Aff}_C$ , whose names encode the similarities between them and respective topologies on the category of usual affine schemes (spectra of commutative rings), in particular, *Zariski topology*. Starting with monoidal symmetric category of abelian groups  $(\mathbf{Z}\text{-Mod}, \otimes_{\mathbf{Z}}, \mathbf{Z})$  one comes to the usual category of schemes  $\text{Sch}_{\mathbf{Z}}$ .

The version of  $\mathbf{F}_1$ -schemes  $Sch_{\mathbf{F}_1}$  suggested in [ToVa09] is embodied in the final stretch of the similar path starting with  $(Sets, \times, \{*\})$ .

Finally, some pairs of categories  $(Sch_C, Sch_D)$  can be related by two functors going in reverse directions and satisfying certain adjointness properties. Intuitively, one of them describes appropriate *descent data* upon certain objects  $X_C$  of  $Sch_C$  necessary and sufficient to define an object  $X_D$  lying *under*  $X_C$ . One of the most remarkable examples of such descent data from  $Sch_{\mathbf{Z}}$  to  $Sch_{\mathbf{F}_1}$  was developed by J. Borger, cf. [Bo11a], [Bo11b]: roughly speaking, it consists in lifting all Frobenius morphisms to the respective  $\mathbf{Z}$ -scheme. In a weakened form, when only a subset of Frobenius morphisms is lifted, it leads to geometries below  $\text{Spec } \mathbf{Z}$  but not necessarily over  $\mathbf{F}_1$ .

**1.2. Schemes in the brave new algebra.** In a very broad sense, the invasion of homotopy theory to mathematics in general started with radical enrichment of Cantorian intuition about what are natural numbers  $\mathbf{N}$ : they are cardinalities of not just arbitrary finite sets, but rather of sets of connected components  $\pi_0$  of topological spaces.

The multiplication and addition in  $\mathbf{N}$  have then natural lifts to the world of stable homotopy theory, the ring of integers being enriched by passing to sphere spectrum, where it becomes the initial object, in the same way as  $\mathbf{Z}$  itself is an initial object in the category of commutative rings, etc. More details and references are given in the Section 4 of this article.

Moreover, “counting functions”, such as numbers of  $\mathbf{F}_q$ -points of a scheme reduced modulo a prime  $p$ , with  $q = p^a$ , can be generalized to the world of scissors congruences where they become the basis for study of zeta-functions.

**1.3. The structure of the article.** (A). Sec. 2 is dedicated to a categorification in homotopy theory of a class of schemes in characteristics 1 admitting the following intuitive description: *1-Frobenius morphisms acting upon their cohomology have eigenvalues that are roots of unity.*

Here are some more details. The main arithmetic invariant of an algebraic manifold  $\mathcal{V}$  defined over a finite field  $\mathbf{F}_q$ , is its zeta-function  $Z(\mathcal{V}, s)$  counting the numbers of its points  $\text{card } V(\mathbf{F}_{q^n})$  over finite extensions of  $\mathbf{F}_q$ .

Assuming for simplicity that  $V$  is irreducible and smooth, we can identify  $Z(V, s)$  with a rational function of  $q^{-s}$  which is an alternating product of polynomials whose roots are characteristic numbers of the Frobenius endomorphism  $Fr_q$  of  $V$  acting upon étale cohomology  $H_{\text{ét}}^*(V)$  of  $V$ .

In various versions of  $\mathbf{F}_1$ -geometry, the structure consisting of cohomology with action of a Frobenius upon it is conspicuously missing, although it is clearly lurking behind the scene (see e. g. a recent survey and study [LeBr17]).

Our homotopical approach here develops the analogy between Frobenius maps and Morse–Smale diffeomorphisms.

In the remaining part of the article, we focus upon another ”counting formalism” and its well developed homotopical environment.

Namely, on the  $\mathbf{F}_q$ -schemes of finite type, non-necessarily smooth and proper ones, the function  $c : \mathcal{V} \mapsto \text{card } \mathcal{V}(\mathbf{F}_q)$  satisfies the following “scissors identity”: if  $X \subset \mathcal{V}$  is a Zariski closed subscheme,  $Y := \mathcal{V} \setminus X$ , then

$$c(\mathcal{V}) = c(X) + c(Y)$$

and also  $c(\mathcal{V}_1 \times \mathcal{V}_2) = c(\mathcal{V}_1)c(\mathcal{V}_2)$ . In other words,  $c$  becomes a ring homomorphism of the Grothendieck ring  $K_0$  of the category of  $\mathbf{F}_q$ -schemes.

(B). In Sec. 3 we consider a lift to the equivariant Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  of the integral Bost–Connes system, described in [CCMar09] in the context of  $\mathbf{F}_1$ -geometry. We obtain endomorphisms  $\sigma_n$  and additive maps  $\tilde{\rho}_n$  that act on  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  and map to the corresponding maps of the integral Bost–Connes system through the equivariant Euler characteristic. We obtain in this way a noncommutative enrichment  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  of the Grothendieck ring and an Euler characteristic, which is a ring homomorphism to the integral Bost–Connes algebra. After passing to  $\mathbf{Q}$ -coefficients, both algebras become semigroup crossed products by the multiplicative semigroup of positive integers,

$$\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}) \otimes_{\mathbf{Z}} \mathbf{Q} = K_0^{\hat{\mathbf{Z}}}(\mathcal{V})_{\mathbf{Q}} \rtimes \mathbf{N},$$

with  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})_{\mathbf{Q}} = K_0^{\hat{\mathbf{Z}}}(\mathcal{V}) \otimes_{\mathbf{Z}} \mathbf{Q}$  and target of the Euler characteristic the rational Bost–Connes algebra  $\mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$ .

(C). In Sec. 4, we revisit the construction of the previous section, by further lifting it from the level of the Grothendieck ring to the level of spectra. We use the approach based on assemblers, developed in [Za17a, [Za17b].

We recall briefly the general formalism of [Za17a] and present a small modification of the construction of [Za17c] of the assembler and spectrum associated to the

Grothendieck ring of varieties, which will be useful in the following, namely the case of the equivariant Grothendieck ring  $K^{\hat{\mathbf{Z}}}(\mathcal{V})$  considered in the previous section.

We then prove that the lift of the integral Bost–Connes algebra to the level of the Grothendieck ring described in the previous section can further be lifted to the level of spectra.

(D). In Sec. 5 we discuss the construction of quantum statistical mechanical expectation values on the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  based on motivic measures and the expectation values of the Bost–Connes system.

(E). Finally, in Sec. 6 we revisit the construction discussed in the previous section in a setting where, instead of considering varieties with a good  $\hat{\mathbf{Z}}$ -action, we consider a “dynamical” model of  $\mathbf{F}_1$ -structure based on the existence of an endomorphism  $f : X \rightarrow X$  that induces a quasi-unipotent map  $f_*$  in homology. Our purpose here is to show a compatibility between this proposal about  $\mathbf{F}_1$ -structures and the idea of [CCMar09] of  $\mathbf{F}_1$ -geometry encoded in the structure of the integral Bost–Connes algebra, through its relation to cyclotomic fields.

This also returns us to the framework of Sec. 2, thus closing the circle.

## 2. Roots of unity as Weil numbers

**2.1. Local zetas and homotopy.** In this section we are developing the idea, sketched in the subsection 0.2 of [Ma10]: namely, that a  $q = 1$  replacement of the structure  $(Fr_q, H_{\text{ét}}^k(V))$  is a pair  $(f_{*k}, H_k(M, \mathbf{Z}))$  where  $f_{*k}$  is the action of a Morse–Smale diffeomorphism  $f : M \rightarrow M$  of a compact manifold  $M$  upon its homology  $H_k(M, \mathbf{Z})$ .

In particular, in this model characteristic roots of  $f_*$  acting upon (co)homology groups are roots of unity, as might be expected from “Weil numbers in characteristic 1”.

We start with basic definitions and references.

**2.2. Morse–Smale diffeomorphisms.** Below  $M$  always means a compact smooth manifold, and  $f : M \rightarrow M$  its diffeomorphism. A point  $x \in M$  is called non-wandering, if for any neighborhood  $U$  of  $x$ , there is  $n > 0$  such that  $U \cap f^n(U) \neq \emptyset$ .

**2.2.1. Definition.**  *$f$  is called a Morse–Smale diffeomorphism, if*

*(i) The number of non-wandering points of  $f$  is finite.*

(ii)  $f$  is structurally stable that is, any small variation of  $f$  is isotopic to  $f$ .

This short definition appears in [Gr81]; for a more detailed discussion of geometry, see [FrSh81].

**2.3. Action of Morse–Smale diffeomorphisms on homology.** For any compact manifold  $M$ , its homology groups  $H_k(M, \mathbf{Z})$  are finitely generated abelian groups. Any diffeomorphism  $f : M \rightarrow M$  induces automorphisms  $f_{*k}$  of these groups. According to [ShSu75], if  $f$  is Morse–Smale, then each  $f_{*k}$  is quasi-unipotent, that is, its eigenvalues are roots of unity.

However, generally this condition is not sufficient. An additional necessary condition is vanishing of a certain obstruction, described in [FrSh81] and further studied in [Gr81].

Namely, consider the category  $QI$  whose objects are pairs  $(g, H)$ , where  $H$  is a (finitely generated) abelian group, and  $g : H \rightarrow H$  is a quasi-idempotent endomorphism of  $H$  (by definition, this means that eigenvalues of  $g$  are zero or roots of unity.) Morphisms in  $QI$  are self-evident.

The abelian group  $K_0(QI)$  admits a morphism onto its torsion subgroup  $G$ . Denote by  $\varphi : K_0(QI) \rightarrow G$  be such a morphism.

**2.3.1. Theorem ([FrSh81]).** *Let  $f : M \rightarrow M$  be a diffeomorphism such that each  $f_{*k} : H_k(M, \mathbf{Z}) \rightarrow H_k(M, \mathbf{Z})$  is quasi-unipotent. Let  $[f_{*k}]$  be the class in  $K_0(QI)$  of the pair  $(f_{*k}, H_k(M, \mathbf{Z}))$ . Then*

(i) *If  $f$  is Morse–Smale, then  $\chi(f_*) := \sum_k (-1)^k \varphi([f_{*k}]) \in G$  is zero.*

(ii) *If in addition  $M$  is simply connected, and  $\dim M > 5$ , then vanishing of  $\chi(f_*)$  implies that  $f$  is isotopic to a Morse–Smale diffeomorphism.*

**2.4. Passage from  $(f_{*k}, H_k(M, \mathbf{Z}))$  to a scheme over  $F_1$ .** In this subsection, we sketch a final step from homotopy to  $F_1$ -geometry.

One should keep in mind, however, that there might be many divergent paths, starting at this point, because there are several different versions of “geometries over  $F_1$ ”.

We will choose here a version developed in James Borger’s paper [Bo11a] (and continued in [Bo11b]). Roughly speaking, in order to define an affine scheme over  $F_1$ , one should give a (commutative) ring with  $\lambda$ -structure and then treat this  $\lambda$ -structure as “descent data” from the base  $\text{Spec } \mathbf{Z}$  to the base  $\text{Spec } F_1$ .

**2.4.1. From  $(f_{*k}, H_k(M, \mathbf{Z}))$  to a  $\lambda$ -ring.** Consider for simplicity only the case (ii) of Theorem 3.1. Then for each  $k$ ,  $H_k(M, \mathbf{Z})$  is a free  $\mathbf{Z}$ -module of finite rank, and  $f_{*k}$  is its quasi-unipotent automorphism. Fix such a  $k$ .

Introduce upon  $H_k(M, \mathbf{Z})$  the structure of  $\mathbf{Z}[T, T^{-1}]$ -module, where  $T$  acts as  $f_{*k}$ .

We can consider the minimal subcategory  $\mathcal{C}$  of  $\mathbf{Z}[T, T^{-1}]$ -modules, containing  $H_k(M, \mathbf{Z})$  and closed with respect to direct sums, tensor products, and exterior powers, and then produce the Grothendieck  $\lambda$ -ring  $K_0(\mathcal{C})$  using exterior powers to define the relevant  $\lambda$ -structure (cf. [At61], p. 26, and [Le81]).

**2.4.2. Definition.** *The  $F_1$ -scheme, whose Borger’s lift to  $\text{Spec } \mathbf{Z}$  is  $K_0(\mathcal{C})$ , is the representative of  $(f_{*k}, H_k(M, \mathbf{Z}))$  in  $F_1$ -geometry.*

**2.4.3. Remark.** It seems that another short path from  $(f_{*k}, H_k(M, \mathbf{Z}))$  to an  $F_1$ -scheme defined differently might lead to one of Le Bruyn’s spaces in [LeBr17].

**2.5. Cases when eigenvalues of conjectural Frobenius maps are not roots of unity.** Here I will briefly discuss possible extensions of the picture above, leading to various geometries “below  $\text{Spec } \mathbf{Z}$ ” but generally *not* over  $\text{Spec } F_1$ .

The most interesting new virtual zeta-functions of this type were discovered only recently under the generic name “zeta-polynomials”: cf. [Ma16], [JiMaOnSo16], [OnRoSp16].

In [Ma16], it was described how to produce such polynomials from period polynomials of any cusp form over  $SGL(2, \mathbf{Z})$  which is an eigenform for all Hecke operators: this passage is a kind of “discrete Mellin transform”. It was also proved that zeroes of period polynomials lie on the unit circle of the complex plane, but generally are *not* roots of unity.

Both this construction and the results about zeroes were generalised in [JiMaOnSo16], [OnRoSp16] to the case of cusp newforms of even weight for  $\Gamma_0(N)$ . It turned out that, with appropriate scaling, zeroes of period polynomials lie on the circle  $\{z \mid |z| = \frac{1}{\sqrt{N}}\}$ .

**2.5.1. Problem.** Make explicit geometries under  $\text{Spec } \mathbf{Z}$  in which one can accommodate the respective “motives”.

### 3. The Bost–Connes system and the Grothendieck ring

**3.1.  $\hat{\mathbf{Z}}$ -equivariant Grothendieck ring ([Lo99]).** We recall the following definitions from [Lo99]. Let  $G$  be an affine algebraic group acting upon an algebraic

variety  $X$ . We say that this action is *good*, if each  $G$ -orbit is contained in an affine open subset of  $X$ .

The Grothendieck ring  $K_0^G(\mathcal{V})$  is generated by isomorphism classes  $[X]$  of pairs, consisting of varieties with good  $G$ -action. Upon these pairs the inclusion-exclusion relations are imposed:  $[X] = [Y] + [X \setminus Y]$  where  $Y \hookrightarrow X$  and  $X \setminus Y \hookrightarrow X$  are  $G$ -equivariant embeddings. Multiplication in  $K_0^G(\mathcal{V})$  is induced by the diagonal  $G$ -action on the product.

In the main special case considered in [Lo99],

$$G = \hat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}.$$

One defines  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  as the Grothendieck ring of varieties with an action of  $\hat{\mathbf{Z}}$  that factors through a good action of some finite quotient  $\mathbf{Z}/n\mathbf{Z}$ .

In this section, we first consider varieties defined over the field  $\mathbf{C}$  of complex numbers and classes in the Grothendieck ring correspondingly taken in  $K_0(\mathcal{V}_{\mathbf{C}})$  and the equivariant  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}})$ . We then consider the case of varieties defined over  $\mathbf{Q}$  with the equivariant Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$ . In the first case the target of the equivariant Euler characteristic consists of the abelian part  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  of the integral Bost-Connes algebra, while in the second case it is a subring spanned by the range projectors of the Bost-Connes algebra.

As observed in [Lo99], there is an Euler characteristic ring homomorphism

$$\chi^{\hat{\mathbf{Z}}} : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathbf{C}),$$

to the Grothendieck ring of finite dimensional representations. Since the character group is  $\mathrm{Hom}(\hat{\mathbf{Z}}, \mathbf{G}_m) = \mathbf{Q}/\mathbf{Z}$ , the latter is identified with the group ring

$$K_0^{\hat{\mathbf{Z}}}(\mathbf{C}) = \mathbf{Z}[\mathbf{Q}/\mathbf{Z}].$$

**3.2. Equivariant Euler characteristics.** Since we are considering varieties with a good action of  $\hat{\mathbf{Z}}$  that factors through some finite quotient, the Euler characteristic above can be obtained as in [Ve73], for an action of a finite group  $G$ , by replacing the alternating sum of dimensions of the cohomology groups with a sum in the ring  $R(G)$  of representations of  $G$  of the classes of cohomology groups, viewed as  $G$ -modules.

As observed in [Gu-Za17], one can also define an equivariant Euler characteristic, for good actions on varieties of a finite group  $G$ , as a ring homomorphism  $\chi^G : K_0^G(\mathcal{V}) \rightarrow A(G)$ , where  $A(G)$  is the Burnside ring of  $G$ , the Grothendieck ring of the category of finite  $G$ -sets. In this case the equivariant Euler characteristic is defined as  $\chi^G(X) = \sum_{k \geq 0} [X_k]$  where  $X$  has a simplicial decomposition with  $k$ -skeleton  $X_k$ , such that  $G$  acts by simplicial maps which map each  $k$ -simplex either identically to itself or to another simplex, so that it makes sense to consider the classes  $[X_k]$  in  $A(G)$ . It is shown in [Gu-Za17] that the result is independent of the choice of such a simplicial decomposition. It is also shown that any invariant with values in a commutative ring  $R$ , defined on varieties with a good  $G$ -action homeomorphic to locally closed unions of cells in finite CW-complexes with  $G$  acting by cell maps, which satisfies inclusion-exclusion (on  $G$ -invariant decompositions) and multiplicativity on products is necessarily a composition

$$K_0^G(\mathcal{V}) \rightarrow A(G) \rightarrow R$$

of  $\chi^G$  with a ring homomorphism  $\varphi : A(G) \rightarrow R$ . In particular, this is the case for the Euler characteristic  $K_0^G(\mathcal{V}) \rightarrow R(G)$  obtained by composing  $\chi^G : K_0^G(\mathcal{V}) \rightarrow A(G)$  with the natural ring homomorphism  $\varphi : A(G) \rightarrow R(G)$  that sends  $G$ -sets to their space of functions.

When considering the profinite group  $\hat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}$ , the Burnside rings  $A(\mathbf{Z}/n\mathbf{Z})$  form a projective system with limit

$$\hat{A}(\hat{\mathbf{Z}}) = \varprojlim_n A(\mathbf{Z}/n\mathbf{Z}),$$

the completed Burnside ring of  $\hat{\mathbf{Z}}$ , which is the Grothendieck ring of almost finite  $\hat{\mathbf{Z}}$ -spaces, namely those  $\hat{\mathbf{Z}}$ -spaces that are discrete and essentially finite, that is, such that for every open subgroup  $H$  the set of points fixed by all elements of  $H$  is finite, see Section 2 of [DrSi88]. The Burnside ring  $A(\hat{\mathbf{Z}})$  of finite  $\hat{\mathbf{Z}}$ -spaces sits as a dense subring of  $\hat{A}(\hat{\mathbf{Z}})$ . Moreover, there is an identification of this completed Burnside ring with the Witt ring  $\hat{A}(\hat{\mathbf{Z}}) = W(\hat{\mathbf{Z}})$ , see Corollary 1 of [DrSi88].

**3.3. Lifting the integral Bost–Connes system.** Consider now the endomorphisms  $\sigma_n : \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  given by  $\sigma_n(e(r)) = e(nr)$  on the standard basis  $\{e(r) : r \in \mathbf{Q}/\mathbf{Z}\}$  of  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ . The integral Bost–Connes algebra  $\mathcal{A}_{\mathbf{Z}}$  introduced in [CCMar09] is generated by the group ring  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  together with elements  $\tilde{\mu}_n$  and  $\mu_n^*$  satisfying the relations

$$\tilde{\mu}_{nm} = \tilde{\mu}_n \tilde{\mu}_m, \quad \mu_{nm}^* = \mu_n^* \mu_m^*, \quad \mu_n^* \tilde{\mu}_n = n, \quad \tilde{\mu}_n \mu_n^* = \mu_n^* \tilde{\mu}_n, \quad (3.1)$$

where the first two relations hold for arbitrary  $n, m \in \mathbf{N}$ , the third for arbitrary  $n \in \mathbf{N}$  and the fourth for  $n, m \in \mathbf{N}$  satisfying  $(n, m) = 1$ , and the relations

$$x\tilde{\mu}_n = \tilde{\mu}_n\sigma_n(x) \quad \mu_n^*x = \sigma_n(x)\mu_n^*, \quad \tilde{\mu}_n x \mu_n^* = \tilde{\rho}_n(x), \quad (3.2)$$

for any  $x \in \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ , where  $\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r')$ .

The maps  $\tilde{\rho}_n$  and the endomorphisms  $\sigma_n$  satisfy the compatibility conditions, for all  $x, y \in \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  (see Proposition 4.4 of [CCMar09])

$$\tilde{\rho}_n(\sigma_n(x)y) = x\tilde{\rho}_n(y), \quad \sigma_n(\tilde{\rho}_m(x)) = (n, m) \cdot \tilde{\rho}_{m'}(\sigma_{n'}(x)), \quad (3.3)$$

where  $(n, m) = \gcd\{n, m\}$  and  $n' = n/(n, m)$  and  $m' = m/(n, m)$ .

**3.3.1. Lemma.** *The endomorphisms  $\sigma_n : \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  lift to endomorphisms  $\sigma_n : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}})$  such that the following diagram commutes*

$$\begin{array}{ccc} K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}}) & \xrightarrow{\chi^{\hat{\mathbf{Z}}}} & \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \\ \downarrow \sigma_n & & \downarrow \sigma_n \\ K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}}) & \xrightarrow{\chi^{\hat{\mathbf{Z}}}} & \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]. \end{array}$$

*These endomorphisms define a semigroup action of the multiplicative semigroup  $\mathbf{N}$  on the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}})$ .*

**Proof.** Let  $X$  be a variety with a good  $\hat{\mathbf{Z}}$ -action, which factors through some finite quotient  $\mathbf{Z}/N\mathbf{Z}$ . Let  $\alpha : \hat{\mathbf{Z}} \times X \rightarrow X$  denote the action. The endomorphisms  $\sigma_n : \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  given by  $\sigma_n(e(r)) = e(nr)$  have an equivalent description as the action on the group of roots of unity of all orders given by raising to the  $n$ -th power  $\sigma_n : \zeta \mapsto \zeta^n$ . One can then obtain an action  $\alpha_n : \hat{\mathbf{Z}} \times X \rightarrow X$  given by  $\alpha_n = \alpha \circ \sigma_n$ . Thus, we assign to a pair  $(X, \alpha)$  of a variety with a good  $\hat{\mathbf{Z}}$ -action the pair  $(X, \alpha_n)$  of the same variety with the action  $\alpha_n$ . This assignment respects isomorphism classes and is compatible with the relations, hence it determines endomorphisms of  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}})$ , with  $\sigma_{nm} = \sigma_n \circ \sigma_m$ , namely a semigroup action of the multiplicative semigroup  $\mathbf{N}$  on the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{C}})$ . ■

The maps  $\tilde{\rho}_n : \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  of the form

$$\tilde{\rho}_n(e(r)) = \sum_{nr'=r} e(r') \quad (3.4)$$

are not ring homomorphisms but only morphisms of abelian groups. After tensoring with  $\mathbf{Q}$ , one obtains the group algebra  $\mathbf{Q}[\mathbf{Q}/\mathbf{Z}] = \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \otimes_{\mathbf{Z}} \mathbf{Q}$  on which the  $\tilde{\rho}_n$  induce endomorphisms of the form  $\rho_n(e(r)) = n^{-1} \sum_{nr'=r} e(r')$  satisfying the relations  $\sigma_n \rho_n(x) = x$  and  $\rho_n \sigma_n(x) = \pi_n x$ , for  $x \in \mathbf{Q}[\mathbf{Q}/\mathbf{Z}]$  and the idempotent  $\pi_n = n^{-1} \sum_{ns=0} e(s)$ . The arithmetic Bost–Connes algebra is the crossed product  $\mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$  generated by  $\mathbf{Q}[\mathbf{Q}/\mathbf{Z}]$  and  $\mu_n, \mu_n^*$  with the crossed-product action of  $\mathbf{N}$  implemented by  $\mu_n x \mu_n^* = \rho_n(x)$ , see [CCMar09]

Once one considers varieties defined over  $\mathbf{Q}$ , the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  can be characterized as follows.

**3.3.2. Lemma.** *The Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  can be identified with the subring of  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  generated by the elements  $n\pi_n = \sum_{ns=0} e(s)$ .*

**Proof.** As noted in [Lo99], the element  $\sum_{ns=0} e(s)$  in  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  is the image of the irreducible representation of  $\mathbf{Z}/n\mathbf{Z}$  given by the cyclotomic field  $\mathbf{Q}(\zeta_n)$  seen as a  $\mathbf{Q}$ -vector space, and these representations give a basis of  $K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$ . ■

**3.3.3. Remark.** According to the Corollary 4.5 in [CCMar09], the range of the maps  $\tilde{\rho}_n$  in (3.4) is an ideal in  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ . This follows from the relation  $\tilde{\rho}_n(\sigma_n(x)y) = x\tilde{\rho}_n(y)$  of (3.3). If  $r'$  an element of the set  $E_n(r) = \{r' \in \mathbf{Q}/\mathbf{Z} : nr' = r\}$ , we have  $\tilde{\rho}_n(e(r)) = e(r') \sum_{ns=0} e(s)$ .

**3.3.4. Lemma.** *There are endomorphisms  $\sigma_n : K_0^{\hat{\mathbf{Z}}}(\mathbf{Q}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  induced by the endomorphisms  $\sigma_n : \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ . They lift to endomorphisms  $\sigma_n : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  given by  $\sigma_n(X, \alpha) = (X, \alpha \circ \sigma_n)$  as in Lemma 3.3.1.*

**Proof.** By identifying  $K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  with a subring of  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  as in Lemma 3.3.2. we see that the endomorphisms  $\sigma_n : \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  induce endomorphisms of  $K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  by the relations  $\sigma_n(\tilde{\rho}_m(x)) = (n, m) \cdot \tilde{\rho}_{m'}(\sigma_{n'}(x))$  as in (3.3.3). Since we have  $n\pi_n = \tilde{\rho}_n(1)$ , we see that the endomorphisms  $\sigma_n$  map the subring  $K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  to itself. The lift to  $\sigma_n : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  is obtained by the same argument as in Lemma 3.3.1. Namely, the map  $\sigma_n(X, \alpha) = (X, \alpha \circ \sigma_n)$  defines an endomorphism of  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  which satisfies  $\sigma_n \circ \chi^{\hat{\mathbf{Z}}} = \chi^{\hat{\mathbf{Z}}} \circ \sigma_n$ , with  $\chi^{\hat{\mathbf{Z}}} : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  the Euler characteristic. ■

In order to lift the maps  $\tilde{\rho}_n$  to the level of the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$ , we put

$$\mathbf{V}_n := [Z_n, \gamma_n] \in K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}), \quad (3.5)$$

where  $Z_n$  is a zero-dimensional variety over  $\mathbf{Q}$  with  $\#Z_n(\bar{\mathbf{Q}}) = n$ . Any such variety with a smooth model over  $\text{Spec } \mathbf{Z}$  can be identified with  $Z_n = \text{Spec } (\mathbf{Q}^n)$ . It is endowed with the natural action  $\gamma_n : \hat{\mathbf{Z}} \times Z_n \rightarrow Z_n$  that factors through  $\mathbf{Z}/n\mathbf{Z}$ .

Given a variety  $X$  with a good  $\hat{\mathbf{Z}}$ -action  $\alpha : \hat{\mathbf{Z}} \times X \rightarrow X$ , let

$$\Phi_n(\alpha) : \hat{\mathbf{Z}} \times X \times Z_n \rightarrow X \times Z_n$$

be given by

$$\begin{aligned} \Phi_n(\alpha)(\zeta, x, a_i) &= (x, \gamma_n(\zeta, a_i)) \quad \text{for } i = 1, \dots, \\ &\text{and } (\alpha(\zeta, x), \gamma_n(\zeta, a_n)) \quad \text{for } i = n. \end{aligned} \quad (3.6)$$

Notice that (3.6) is just a form of the *Verschiebung* map: for  $\zeta$  the generator of  $\mathbf{Z}/n\mathbf{Z}$  we have

$$\begin{aligned} \Phi_n(\alpha)(\zeta, x, a_i) &= (x, a_{i+1}) \quad \text{for } i = 1, \dots, n-1, \\ &\text{and } (\alpha(\zeta, x), a_1) \quad \text{for } i = n. \end{aligned}$$

**3.3.5. Proposition.** *The maps*

$$\tilde{\rho}_n[X, \alpha] := [X \times Z_n, \Phi_n(\alpha)] \quad (3.7)$$

define a homomorphism of the Grothendieck group  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  that satisfies

$$\sigma_n \circ \tilde{\rho}_n[X, \alpha] = [X, \alpha]^{\oplus n} \quad (3.8)$$

and

$$\tilde{\rho}_n \circ \sigma_n[X, \alpha] = \tilde{\rho}_n[X, \alpha \circ \sigma_n] = [X, \alpha] \cdot [Z_n, \gamma_n]. \quad (3.9)$$

**Proof.** Given a variety  $X$  with a good  $\hat{\mathbf{Z}}$ -action  $\alpha : \hat{\mathbf{Z}} \times X \rightarrow X$ , consider the product  $[X, \alpha] \cdot \mathbf{V}_n$  in  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$ . This class has a representative  $[X \times Z_n, (\alpha \times \gamma_n) \circ \Delta]$ , where  $\Delta : \hat{\mathbf{Z}} \rightarrow \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}$  is the diagonal. We have

$$\sigma_n \circ \tilde{\rho}_n[X, \alpha] = \sigma_n[X \times Z_n, \Phi_n(\alpha)] = [X \times Z_n, \Phi_n(\alpha) \circ \sigma_n].$$

We also have

$$\Phi_n(\alpha) \circ \sigma_n = (\alpha \times 1) \circ \Delta,$$

since

$$\Phi_n(\alpha) \circ \sigma_n(\zeta, x, a_i) = \Phi_n(\alpha)(\zeta^n, x, a_i) = (\Phi_n(\alpha)(\zeta))^n(x, a_i),$$

where we write  $\Phi_n(\alpha)(\zeta) : X \times Z_n \rightarrow X \times Z_n$  for the action of  $\zeta \in \hat{\mathbf{Z}}$ , with  $\Phi_n(\alpha)(\zeta_1 \cdots \zeta_n) = \Phi_n(\alpha)(\zeta_1) \circ \cdots \circ \Phi_n(\alpha)(\zeta_n)$ , and the  $n$ -fold composition gives

$$\Phi_n(\alpha)(\zeta) \circ \cdots \circ \Phi_n(\alpha)(\zeta)(x, a_i) = (\alpha(\zeta, x), a_i).$$

This shows (3.8).

The second relation is obtained similarly. We have

$$\tilde{\rho}_n \circ \sigma_n[X, \alpha] = \tilde{\rho}_n[X, \alpha \circ \sigma_n] = [X \times Z_n, \Phi_n(\alpha \circ \sigma_n)],$$

where

$$\Phi_n(\alpha)(\zeta, x, a_i) = (x, a_{i+1}) \quad \text{for } i = 1, \dots, n-1,$$

$$\text{and } (\alpha(\zeta^n, x), a_1) \quad \text{for } i = n.$$

Now  $\alpha(\zeta^n, x) = \alpha(\zeta)^n(x)$ , hence we have  $\Phi_n(\alpha \circ \sigma_n)(\zeta)(x, a_i) = \Phi_n(\alpha(\zeta)^n)(x, a_i)$ . The usual relations  $V_n(F_n(a)b) = aV_n(b)$  between Frobenius  $F_n$  and Verschiebung  $V_n$  (see Proposition 2.2 of [CC14]) holds in this case as well in the form  $\Phi_n(\alpha(\zeta)^n) = \alpha(\zeta)\Phi_n(1)$  where  $\Phi_n(1)(x, a_i) = (x, a_{i+1})$  is the cyclic permutation action of  $\mathbf{Z}/n\mathbf{Z}$  on  $Z_n$ . Thus, we obtain  $\Phi_n(\alpha \circ \sigma_n) = (\alpha \times \gamma_n) \circ \Delta$ . This gives (3.9). ■

The relation (3.8) corresponds to  $\sigma_n \circ \tilde{\rho}_n(x) = nx$ , and (3.9) to  $\tilde{\rho}_n \circ \sigma_n(x) = n\pi_n x$ , for  $x \in \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ . They are geometric manifestations of the same relation between the maps  $\sigma_n$  and  $\tilde{\rho}_n$  of the integral Bost–Connes system and the Frobenius and Verschiebung described in [CC14]. For other occurrences of the same relation see also [MaRe17], [MaTa17].

**3.4. A non-commutative extension of the Grothendieck ring.** Let  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  be the non-commutative ring generated by  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  and elements  $\tilde{\mu}_n, \mu_n^*$  for  $n \in \mathbf{N}$  satisfying the relations (3.1) for all  $n, m \in \mathbf{N}$ , and (3.2) for all  $x = [X, \alpha] \in K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  and all  $n \in \mathbf{N}$ .

**3.4.1. Lemma.** *The Euler characteristic  $\chi^{\hat{\mathbf{Z}}} : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathbf{Q}) \hookrightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  extends to a ring homomorphism  $\chi : \mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow \mathcal{A}_{\mathbf{Z}}$  to the integral Bost–Connes algebra. After tensoring with  $\mathbf{Q}$ , we obtain a homomorphism of semigroup crossed product rings*

$$\chi : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})_{\mathbf{Q}} \rtimes \mathbf{N} \rightarrow \mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N},$$

where  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q} = K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})_{\mathbf{Q}} \rtimes \mathbf{N}$  with  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})_{\mathbf{Q}} = K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q}$ , and  $\mathcal{A}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathcal{A}_{\mathbf{Q}} = \mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$  is the rational Bost–Connes algebra.

**Proof.** We define the map  $\chi$  as  $\chi^{\hat{\mathbf{Z}}}$  on elements of  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  and the identity on the extra generators  $\chi(\tilde{\mu}_n) = \tilde{\mu}_n$  and  $\chi(\mu_n^*) = \mu_n^*$ . By Lemma 3.3.4 and Proposition 3.3.5, this map is compatible with the relations in  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  and in  $\mathcal{A}_{\mathbf{Z}}$ . After tensoring with  $\mathbf{Q}$ , the algebra  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q}$  can be identified with a semigroup crossed product by taking as generators the elements of  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  and  $\mu_n = n^{-1}\tilde{\mu}_n$  and  $\mu_n^*$ , which satisfy the relations

$$\mu_n^* \mu_n = 1, \quad \mu_{nm} = \mu_n \mu_m, \quad \mu_{nm}^* = \mu_n^* \mu_m^*, \quad \forall n, m \in \mathbf{N}, \quad \mu_n \mu_m^* = \mu_m^* \mu_n \text{ if } (n, m) = 1$$

$$\mu_n x \mu_n^* = \rho_n(x) \quad \text{with} \quad \rho_n(x) = \frac{1}{n} \tilde{\rho}_n(x),$$

with  $\sigma_n \rho_n(x) = x$ , for all  $x = [X, \alpha] \in K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$ . The semigroup action in the crossed product  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})_{\mathbf{Q}} \rtimes \mathbf{N}$  is given by  $x \mapsto \rho_n(x) = \mu_n x \mu_n^*$ . The target algebra is the rational Bost–Connes algebra  $\mathcal{A}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$ . Again the map given by  $\chi^{\hat{\mathbf{Z}}}$  on elements of  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  and by  $\chi(\mu_n) = \mu_n$  and  $\chi(\mu_n^*) = \mu_n^*$  determines a homomorphism of crossed-product algebras. ■

## 4. From Rings to Spectra

**4.1. Spectra.** We give a brief review of spectra, with the purpose of recalling a construction of Segal [Se74] that constructs spectra from  $\Gamma$ –spaces. We then review the notion of assembler categories [Za17a] and how they can be used to construct a  $\Gamma$ –space and an associated spectrum whose  $\pi_0$  realises certain abstract scissor–congruence relations.

The construction of spectra from  $\Gamma$ –spaces was first developed in the context of the Bousfield–Friedlander spectra, see Definition 2.1 of [BousFr78].

In this setting, one considers the simplicial category  $\Delta$ , which has an object  $[n]$  for each  $n \in \mathbf{N}$  given by the finite totally ordered set  $[n] = \{0 < 1 < \dots < n - 1\}$ , with morphisms the face and degeneracy maps  $\delta_i^n$  and  $\sigma_i^n$  satisfying the simplicial relations.

A simplicial object is a contravariant functor  $S : \Delta^{op} \rightarrow \mathcal{C}$  from  $\Delta$  to a given category  $\mathcal{C}$ . It is determined by a sequence of objects  $X(n)$  of  $\mathcal{C}$  with morphisms corresponding to faces and degeneracies. We denote by  $\Delta(\mathcal{C})$  the resulting category

of simplicial objects in  $\mathcal{C}$ . In particular, a simplicial set is a simplicial object in the category of sets and we will use the notation  $\underline{\Delta} = \Delta(\text{Sets})$  for the category of simplicial sets.

Similarly, a bisimplicial object is a functor  $BS : \Delta^{op} \times \Delta^{op} \rightarrow \mathcal{C}$ , or equivalently a simplicial object in the category of simplicial objects  $\Delta(\mathcal{C})$ . The diagonal of a bisimplicial object  $BS$  is the simplicial object obtained by precomposition of  $BS$  with the diagonal functor  $\Delta^{op} \rightarrow \Delta^{op} \times \Delta^{op}$ .

The category  $\mathcal{S}$  of Bousfield–Friedlander spectra has objects  $X$  given by sequences of simplicial sets  $X = \{X_n\}_{n \geq 0}$  endowed with structure maps  $\varphi_n^X : S^1 \wedge X_n \rightarrow X_{n+1}$  for all  $n \geq 0$ , and morphisms given by maps  $f_n : X_n \rightarrow Y_n$  with commutative diagrams

$$\begin{array}{ccc} S^1 \wedge X_n & \xrightarrow{\varphi_n^X} & X_{n+1} \\ \downarrow 1_{S^1} \wedge f_n & & \downarrow f_{n+1} \\ S^1 \wedge Y_n & \xrightarrow{\varphi_n^Y} & Y_{n+1}. \end{array}$$

The sphere spectrum  $\mathbf{S}$  has  $\mathbf{S}_n = S^1 \wedge \cdots \wedge S^1$ , the  $n$ -fold smash product, and  $\varphi_n$  the identity map.

Let  $\gamma_n^X : X_n \rightarrow \Omega X_{n+1}$  be the maps induced by the adjoints of the structure maps. An  $\Omega$ -spectrum is a spectrum where the maps  $\gamma_n^X$  are weak equivalences for all  $n$ .

The homotopy groups  $\pi_k(X)$  of spectra are given by

$$\pi_k(X) = \varinjlim_n \pi_{k+n}(X_n)$$

over the maps  $\pi_{k+n}(X_n) \rightarrow \pi_{k+n}(\Omega X_{n+1}) \simeq \pi_{n+k+1}(X_{n+1})$ , induced the  $\gamma_n^X$ . A spectrum is  $n$ -connected if  $\pi_k(X) = 0$  for all  $k \leq n$  and connective if it is  $-1$ -connected. A spectrum  $X$  is cofibrant if all the structure maps  $\varphi_n^X : S^1 \wedge X_n \rightarrow X_{n+1}$  are cofibrations.

However, a problem with the Bousfield–Friedlander spectra is that they do not have a homotopically good smash product. Constructions of categories of spectra with smash products were developed in the '90s, in particular, the  $S$ -modules model of [EKMM97] and the symmetric spectra model of [HSS00]. In a more modern approach, it is therefore preferable to work with symmetric spectra for the Segal

construction. Indeed, the  $\Gamma$ -spaces, SW-categories and Waldhausen categories that occur in relation to the spectra underlying the Grothendieck ring of varieties and its variants naturally give rise to symmetric spectra.

A symmetric spectrum consists of a sequence of pointed spaces (pointed simplicial sets)  $X = \{X_n\}_{n \geq 0}$  together with a left action of the symmetric group  $S_n$  on  $X_n$  for all  $n \geq 0$  and structure maps given by based maps  $\varphi_n^X : S^1 \wedge X_n \rightarrow X_{n+1}$  for all  $n \geq 0$ , with the condition that, for all  $n, m \geq 0$  the composition  $\varphi_{n+m-1}^X \circ \cdots \circ \varphi_n^X$

$$\varphi_{n+m-1}^X \circ \cdots \circ \varphi_n^X : S^m \wedge X_n \rightarrow S^{m-1} \wedge X_{n+1} \rightarrow \cdots \rightarrow S^1 \wedge X_{n+m-1} \rightarrow X_{n+m}$$

is  $S_n \times S_m$ -equivariant.

A morphism of symmetric spectra is a collection of  $S_n$ -equivariant based maps  $f_n : X_n \rightarrow Y_n$  such that  $f_{n+1} \circ \varphi_n^X = \varphi_n^Y \circ (f_n \wedge Id_{S^1})$ , for all  $n \geq 0$ . A symmetric spectrum is ring spectrum if it is also endowed with  $S_n \times S_m$ -equivariant multiplication maps

$$M_{n,m} : X_n \wedge X_m \rightarrow X_{n+m}$$

and unit maps  $\iota_0 : S^0 \rightarrow X_0$  and  $\iota_1 : S^1 \rightarrow X_1$ . They must satisfy the associativity laws consisting of commutative squares

$$\begin{array}{ccc} X_n \wedge X_m \wedge X_r & \xrightarrow{Id \wedge M_{m,r}} & X_n \wedge X_{m+r} \\ M_{n,m} \wedge Id \downarrow & & \downarrow M_{n,m+r} \\ X_{n+m} \wedge X_r & \xrightarrow{M_{n+m,r}} & X_{n+m+r} \end{array}$$

the unit relations

$$M_{n,0} \circ (Id \wedge \iota_0) = Id : X_n \simeq X_n \wedge S^0 \rightarrow X_n \wedge X_0 \rightarrow X_n$$

and similarly  $M_{0,n} \circ (\iota_0 \wedge Id) = Id$  for all  $n \geq 0$ , as well as  $\chi_{n,1} \circ (M_{n,1} \circ (Id \wedge \iota_1)) = (M_{1,n} \circ (\iota_1 \wedge Id)) \circ \tau$  with  $\tau : X_n \wedge S^1 \rightarrow S^1 \wedge X_n$  and  $\chi_{n,m} \in S_{n+m}$  the shuffle permutation moving the first  $n$  elements past the last  $m$ . Commutativity of a symmetric ring spectrum is expressed by the commutativity of the diagrams

$$\begin{array}{ccc} X_n \wedge X_m & \longrightarrow & X_m \wedge X_n \\ M_{n,m} \downarrow & & \downarrow M_{m,n} \\ X_{n+m} & \xrightarrow{\chi_{n,m}} & X_{m+n} \end{array}$$

with the twist as the first map. For a detailed introduction to symmetric spectra we refer the reader to [Schw12].

**4.2.  $\Gamma$ -spaces.** We recall the setting of  $\Gamma$ -spaces used in the Segal's construction of spectra from categorical data. The notion of  $\Gamma$ -spaces and its relation to connective spectra formalises the intuition that spectra are a natural homotopy-theoretic generalisation of abelian groups.

Let  $\Gamma^0$  denote the category of finite pointed sets, with objects

$$\underline{n} = \{0, 1, 2, \dots, n\}$$

and morphisms  $f \in \Gamma^0(\underline{n}, \underline{m})$  given by functions

$$f : \{0, 1, 2, \dots, n\} \rightarrow \{0, 1, 2, \dots, m\}, \quad \text{with } f(0) = 0.$$

Let  $\Gamma$  denote the opposite category.

A pointed category  $\mathcal{C}$  is a category with a chosen object that is both initial and final. A pointed functor  $F : \Gamma^0 \rightarrow \mathcal{C}$  is called a  $\Gamma$ -object in  $\mathcal{C}$ .

Given a pointed category  $\mathcal{C}$ , the category  $\Gamma\mathcal{C}$  has objects the pointed functors  $F : \Gamma^0 \rightarrow \mathcal{C}$  and morphisms the natural transformations between these functors.

$\Gamma$ -spaces are objects of the category  $\Gamma\mathcal{C}$ , in the case where  $\mathcal{C} = \underline{\Delta}_*$  is the category of pointed simplicial sets.

Given a  $\Gamma$ -space  $F : \Gamma^0 \rightarrow \underline{\Delta}_*$ , the morphisms  $f_j : \underline{n} \rightarrow \underline{1}$  that map the  $j$ -th element to 1 and the rest to 0 induce, for each  $n \geq 1$ , a morphism

$$F(\underline{n}) \rightarrow \prod_{j=1}^n F(\underline{1}). \quad (4.1)$$

The *special*  $\Gamma$ -spaces (or Segal  $\Gamma$ -spaces) are  $\Gamma$ -spaces  $F$  as above, where all the maps (4.1) are weak equivalences. For special  $\Gamma$ -spaces the weak equivalence  $F(\underline{2}) \simeq F(\underline{1}) \times F(\underline{1})$  induces a monoid

$$\pi_0(F(\underline{1})) \times \pi_0(F(\underline{1})) \rightarrow \pi_0(F(\underline{2})) \rightarrow \pi_0(F(\underline{1})).$$

Such a  $\Gamma$ -space is called *very special* when this monoid is an abelian group.

The  $\Gamma$ -space  $\mathbf{S} : \Gamma^0 \rightarrow \underline{\Delta}_*$  is given by the inclusion of the category  $\Gamma^0$  into  $\underline{\Delta}_*$  mapping a finite pointed set to the corresponding discrete pointed simplicial set. As shown in [Se74] (Barratt–Priddy–Quillen theorem), the associated spectrum is the sphere spectrum, which we also denote by  $\mathbf{S}$ .

The category  $\Gamma^0$  of finite pointed sets has a smash product functor  $\wedge : \Gamma^0 \times \Gamma^0 \rightarrow \Gamma^0$ , with  $(\underline{n}, \underline{m}) \mapsto \underline{n} \wedge \underline{m}$ , which extends to a smash product of arbitrary pointed (simplicial) sets.

The smash product of  $\Gamma$ -spaces constructed in [Ly99] is obtained by first associating to a pair  $F, F' : \Gamma^0 \rightarrow \underline{\Delta}_*$  of  $\Gamma$ -spaces a bi- $\Gamma$ -space  $F \tilde{\wedge} F' : \Gamma^0 \times \Gamma^0 \rightarrow \underline{\Delta}_*$

$$(F \tilde{\wedge} F')(\underline{n}, \underline{m}) = F(\underline{n}) \wedge F'(\underline{m})$$

and then defining

$$(F \wedge F')(\underline{n}) = \operatorname{colim}_{\underline{k} \wedge \underline{\ell} \rightarrow \underline{n}} (F \tilde{\wedge} F')(\underline{k}, \underline{\ell}),$$

where  $\underline{k} \wedge \underline{\ell}$  is the smash product  $\wedge : \Gamma^0 \times \Gamma^0 \rightarrow \Gamma^0$ . It is shown in [Ly99] that, up to natural isomorphism, this smash product is associative and commutative and with unit given by the  $\Gamma$ -space  $\mathbf{S}$ , and that the category of  $\Gamma$ -spaces is symmetric monoidal with respect to this product.

**4.3. From  $\Gamma$ -spaces to connective spectra.** The construction of [Se74], and more generally [BousFr78], assigns a connective spectrum to a  $\Gamma$ -space in such a way as to obtain an equivalence between the homotopy category of  $\Gamma$ -spaces and the homotopy category of connective spectra. The construction of spectra from  $\Gamma$ -spaces can be performed in the modern setting of symmetric spectra, rather than in the original Bousfield–Friedlander formulation of [BousFr78], see Chapter I, Section 7.4 of [Schw12].

If  $X$  is a simplicial set, one denotes by  $X_*$  the pointed simplicial set obtained by adding a disjoint base point. Given a  $\Gamma$ -space  $F : \Gamma^0 \rightarrow \underline{\Delta}_*$  and a pointed simplicial set  $X$ , one obtains a new  $\Gamma$ -space  $X \wedge F$ , which maps  $\underline{n} \in \Gamma^0$  to  $X \wedge F(\underline{n})$  in  $\underline{\Delta}_*$ .

Recall that, given a functor  $F : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$ , the coend  $\int^{C \in \mathcal{C}} F(C, C)$  is the initial cowedge, where a cowedge to an object  $X$  in  $\mathcal{C}$  is a family of morphisms  $h_A : A \rightarrow X$ , for each  $A \in \mathcal{C}$ , such that, for all morphisms  $f : A \rightarrow B$  in  $\mathcal{C}$  the

following diagrams commute

$$\begin{array}{ccc}
 F(B, A) & \xrightarrow{F(f, A)} & F(A, A) \\
 \downarrow F(B, f) & & \downarrow h_A \\
 F(B, B) & \xrightarrow{h_B} & X.
 \end{array}$$

The key step in the construction of a connective spectrum associated to a  $\Gamma$ -space consists of extending a  $\Gamma$ -space  $F$  to an endofunctor of the category of pointed simplicial sets. This endofunctor is defined in [BousFr78] (see also [Schw99]) as the functor (still denoted by  $F$ ) that maps a pointed simplicial set  $K \in \underline{\Delta}_*$  to the coend

$$F : K \mapsto \int^{\underline{n} \in \Gamma^{op}} K^n \wedge F(\underline{n}),$$

with natural assembly maps  $K \wedge F(K') \rightarrow F(K \wedge K')$ .

The spectrum associated to  $F$ , which we denote by  $F(\mathbf{S})$ , is then given by the sequence of pointed simplicial sets  $F(\mathbf{S})_n = F(S^n = S^1 \wedge \cdots \wedge S^1)$ , with the maps  $S^1 \wedge F(S^n) \rightarrow F(S^{n+1})$ .

The smash product of  $\Gamma$ -spaces is compatible with the smash product of spectra: in [Ly99] it is shown that if  $F, F' : \Gamma^0 \rightarrow \underline{\Delta}_*$  are  $\Gamma$ -spaces with  $F(\mathbf{S})$  and  $F'(\mathbf{S})$  the corresponding spectra, then there is a map of spectra

$$F(\mathbf{S}) \wedge F'(\mathbf{S}) \rightarrow (F \wedge F')(\mathbf{S})$$

which is natural in  $(F, F')$ , and a weak equivalence if one of the factors is cofibrant.

This gives rise to a notion of ring spectra (see [Schw99]) defined as the monoids in the symmetric monoidal category of  $\Gamma$ -spaces with the smash product of [Ly99] recalled above. One refers to these as  $\Gamma$ -rings. Namely, a  $\Gamma$ -ring is a  $\Gamma$ -space  $F$  endowed with unit and multiplication maps  $\mathbf{S} \rightarrow F$  and  $F \wedge F \rightarrow F$  with associativity and unit properties (Sec VII.3 of [MacL71]). The associated connective spectrum of a commutative  $\Gamma$ -ring is a commutative symmetric ring spectrum. However, not all connective commutative symmetric ring spectra come from a commutative  $\Gamma$ -ring, see [La09]. For a comparative view of the settings of  $\Gamma$ -rings and symmetric ring spectra, see the discussion in Section 2 of [Schw99].

If  $G$  is an abelian group, there is an associated  $\Gamma$ -space  $HG$  given on objects by

$$HG(\underline{n}) = G \otimes \mathbf{Z}[\underline{n}] \simeq G^n,$$

where  $\mathbf{Z}[\underline{n}]$  is the free abelian group on the finite set  $\underline{n}$ . If  $f : \underline{n} \rightarrow \underline{m}$  is a morphism in  $\Gamma^0$ , then the associated morphism  $H(f) : HG(\underline{n}) \rightarrow HG(\underline{m})$  maps an  $n$ -tuple  $(g_1, \dots, g_n)$  (with  $g_0 = 0$ ) in  $G^n$  to the  $m$ -tuple  $(\sum_{j \in f^{-1}(1)} g_j, \dots, \sum_{j \in f^{-1}(n)} g_j)$ . This Eilenberg–MacLane  $\Gamma$ -space  $HG$  maps to the Eilenberg–MacLane spectrum of  $G$ , which we still denote by  $HG$ . If  $R$  is a simplicial ring, then  $HR$  is an  $\mathbf{S}$ -algebra with multiplication  $HR \wedge HR \rightarrow H(R \otimes R) \rightarrow HR$  and unit  $\mathbf{S} \rightarrow H\mathbf{Z} \rightarrow HR$ .

**4.4. Assemblers, spectra, and the Grothendieck ring.** We pass now to a brief survey of the construction of a spectrum associated to the Grothendieck ring of varieties developed in [Z17a] and [Z17c].

I. Zakharevich developed in [Z17a] and [Z17b] a very general formalism for scissor–congruence relations. The abstract form of scissor–congruence consists of categorical data called *assemblers*, which in turn determine a homotopy–theoretic *spectrum*, whose homotopy groups embody scissor–congruence relations. This formalism is applied in [Z17c] in the framework producing an assembler and a spectrum whose  $\pi_0$  recovers the Grothendieck ring of varieties. This is used to obtain a characterisation of the kernel of multiplication by the Lefschetz motive, which provides a general explanation for the observations of [Bor14], [Mart16] on the fact that the Lefschetz motive is a zero divisor in the Grothendieck ring of varieties.

A *sieve* in a category  $\mathcal{C}$  is a full subcategory  $\mathcal{C}'$  that is closed under precomposition by morphisms in  $\mathcal{C}$ . A *Grothendieck topology* on a category  $\mathcal{C}$  consists of the assignment of a collection  $\mathcal{J}(X)$  of sieves in the over category  $\mathcal{C}/X$ , for each object  $X$  in  $\mathcal{C}$ , with the following properties:

- (i) the over category  $\mathcal{C}/X$  is in the collection  $\mathcal{J}(X)$ ;
- (ii) the pullback of a sieve in  $\mathcal{J}(X)$  under a morphism  $f : Y \rightarrow X$  is a sieve in  $\mathcal{J}(Y)$ ;
- (iii) given  $\mathcal{C}' \in \mathcal{J}(X)$  and a sieve  $\mathcal{T}$  in  $\mathcal{C}/X$ , if for every  $f : Y \rightarrow X$  in  $\mathcal{C}'$  the pullback  $f^*\mathcal{T}$  is in  $\mathcal{J}(Y)$  then  $\mathcal{T}$  is in  $\mathcal{J}(X)$ .

Let  $\mathcal{C}$  be a category with a Grothendieck topology. A collection of morphisms  $\{f_i : X_i \rightarrow X\}_{i \in I}$  in  $\mathcal{C}$  is a *covering family* if the full subcategory of  $\mathcal{C}/X$  that contains all the morphisms of  $\mathcal{C}$  that factor through the  $f_i$ ,

$$\{g : Y \rightarrow X \mid \exists i \in I \ h : Y \rightarrow X_i \text{ such that } f_i \circ h = g\},$$

is in the sieve collection  $\mathcal{J}(X)$ .

In a category  $\mathcal{C}$  with an initial object  $\emptyset$  two morphisms  $f : Y \rightarrow X$  and  $g : W \rightarrow X$  are called *disjoint* if the pullback  $Y \times_X W$  exists and is equal to  $\emptyset$ . A collection  $\{f_i : X_i \rightarrow X\}_{i \in I}$  in  $\mathcal{C}$  is disjoint if  $f_i$  and  $f_j$  are disjoint for all  $i \neq j \in I$ .

An *assembler category*  $\mathcal{C}$  is a small category endowed with a Grothendieck topology, which has an initial object  $\emptyset$  (with the empty family as covering family), and where all morphisms are monomorphisms, with the property that any two finite disjoint covering families of  $X$  in  $\mathcal{C}$  have a common refinement that is also a finite disjoint covering family.

A morphism of assemblers is a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  that is continuous for the Grothendieck topologies and preserves the initial object and the disjointness property, that is, if two morphisms are disjoint in  $\mathcal{C}$  their images are disjoint in  $\mathcal{C}'$ .

For  $X$  a finite set, the coproduct of assemblers  $\bigvee_{x \in X} \mathcal{C}_x$  is a category whose objects are the initial object  $\emptyset$  and all the non-initial objects of the assemblers  $\mathcal{C}_x$ . Morphisms of non-initial objects are induced by those of  $\mathcal{C}_x$ .

The abstract scissor congruences consist of pairs of an assembler  $\mathcal{C}$  and a sieve  $\mathcal{D}$  in  $\mathcal{C}$ . Given such a pair, one has an associated assembler  $\mathcal{C} \setminus \mathcal{D}$  defined as the full subcategory of  $\mathcal{C}$  that contains all the objects that are not non-initial objects of  $\mathcal{D}$ . The assembler structure on  $\mathcal{C} \setminus \mathcal{D}$  is determined by taking as covering families in  $\mathcal{C} \setminus \mathcal{D}$  those collections  $\{f_i : X_i \rightarrow X\}_{i \in I}$  with  $X_i, X$  objects in  $\mathcal{C} \setminus \mathcal{D}$  that can be completed to a covering family in  $\mathcal{C}$ , namely such that there exists  $\{f_j : X_j \rightarrow X\}_{j \in J}$  with  $X_j$  in  $\mathcal{D}$  such that  $\{f_i : X_i \rightarrow X\}_{i \in I} \cup \{f_j : X_j \rightarrow X\}_{j \in J}$  is a covering family in  $\mathcal{C}$ . There is a morphism of assemblers  $\mathcal{C} \rightarrow \mathcal{C} \setminus \mathcal{D}$  that maps objects of  $\mathcal{D}$  to  $\emptyset$  and objects of  $\mathcal{C} \setminus \mathcal{D}$  to themselves and morphisms with source in  $\mathcal{C} \setminus \mathcal{D}$  to themselves and morphisms with source in  $\mathcal{D}$  to the unique morphism to the same target with source  $\emptyset$ . The data  $\mathcal{C}, \mathcal{D}, \mathcal{C} \setminus \mathcal{D}$  are an *abstract scissor congruence*.

The construction of spectra from assembler categories uses the general construction of spectra from categorical data is provided by the Segal construction [Se74] of spectra from  $\Gamma$ -spaces, that we recalled in section 4.2 above.

The main construction of [Za17a] associates to an assembler  $\mathcal{C}$  a homotopy-theoretic spectrum, whose homotopy groups provide a family of associated topological invariants satisfying versions of scissor congruence relations. The main steps of the construction can be summarized as follows (see [Za17a]):

(1) One associates to an assembler  $\mathcal{C}$  a category  $\mathcal{W}(\mathcal{C})$  with objects  $\{A_i\}_{i \in I}$ , collections of non-initial objects  $A_i$  of  $\mathcal{C}$  indexed by a finite set  $I$ , and morphisms

$f : \{A_i\}_{i \in I} \rightarrow \{B_j\}_{j \in J}$  given by a map of finite sets  $f : I \rightarrow J$  and morphisms  $f_i : A_i \rightarrow B_{f(i)}$  such that  $\{f_i : A_i \rightarrow B_j : i \in f^{-1}(j)\}$  is a disjoint covering family for all  $j \in J$ .

(2) For a finite pointed set  $(X, x_0)$  and an assembler  $\mathcal{C}$ , one considers the assembler  $X \wedge \mathcal{C} := \bigvee_{x \in X \setminus \{x_0\}} \mathcal{C}$ . The assignment  $X \mapsto \mathcal{N}\mathcal{W}(X \wedge \mathcal{C})$ , where  $\mathcal{N}$  is the nerve, is a  $\Gamma$ -space in the sense of [Se74] recalled above, hence it defines a spectrum  $K(\mathcal{C})$  by

$$X_n = \mathcal{N}\mathcal{W}(S^n \wedge \mathcal{C})$$

with structure maps  $S^1 \wedge X_n \rightarrow X_{n+1}$  determined by the maps

$$X \wedge \mathcal{N}\mathcal{W}(\mathcal{C}) \rightarrow \mathcal{N}\mathcal{W}(X \wedge \mathcal{C}).$$

(3) The group  $K_0(\mathcal{C}) := \pi_0 K(\mathcal{C})$  is the free abelian group generated by objects of  $\mathcal{C}$  modulo the scissor-congruence relations  $[A] = \sum_{i \in I} [A_i]$  for each finite disjoint covering family  $\{A_i \rightarrow A\}_{i \in I}$ .

(4) Given a morphism  $\varphi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  of assemblers, there is an assembler  $\mathcal{C}_2/\varphi$  and a morphism  $\iota : \mathcal{C}_2 \rightarrow \mathcal{C}_2/\varphi$  such that the diagram

$$K(\iota) \circ K(\varphi) : K(\mathcal{C}_1) \rightarrow K(\mathcal{C}_2) \rightarrow K(\mathcal{C}_2/\varphi)$$

is a cofiber sequence.

**4.5. Assembler for the equivariant Grothendieck ring.** As we have seen in the previous section, the equivariant version of the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{K}})$  is generated by isomorphism classes of varieties with a “good”  $\hat{\mathbf{Z}}$ -action, where as before good means that each orbit is contained in an affine open subvariety of  $X$  and that the action factors through some finite level  $\mathbf{Z}/N\mathbf{Z}$ . The scissor congruence relations in  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{K}})$  are of the form  $[X] = [Y] + [X \setminus Y]$  where  $Y \hookrightarrow X$  and  $X \setminus Y \hookrightarrow X$  are  $\hat{\mathbf{Z}}$ -equivariant embeddings. The product is given by the Cartesian product endowed with the induced diagonal  $\hat{\mathbf{Z}}$ -action.

**4.5.1. Lemma.** *The category  $\mathcal{C}^{\hat{\mathbf{Z}}}$  with objects that are varieties  $X$  with a good  $\hat{\mathbf{Z}}$ -action and morphisms that are equivariant locally closed embeddings, endowed with the Grothendieck topology generated by the covering families  $\{Y \hookrightarrow X, X \setminus Y \hookrightarrow X\}$  of  $\hat{\mathbf{Z}}$ -equivariant embeddings, is an assembler category. The spectrum  $K^{\hat{\mathbf{Z}}}(\mathcal{V})$*

determined by the assembler  $\mathcal{C}^{\hat{\mathbf{Z}}}$  has  $\pi_0$  given by the equivariant Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$ .

**Proof.** The first part of the statement follows as in Example 1 of sec. 1 of [Za17c]. The empty set is the initial object. Finite disjoint covering families are  $\hat{\mathbf{Z}}$ -equivariant maps  $f_i : X_i \hookrightarrow X$  where  $X_i = Y_i \setminus Y_{i-1}$  for a chain of  $\hat{\mathbf{Z}}$ -equivariant embeddings  $\emptyset = Y_0 \hookrightarrow Y_1 \hookrightarrow \dots \hookrightarrow Y_n = X$ . The property that any two finite disjoint covering families have a common refinement follows since the category has pullbacks, [Zak1]. Morphisms are compositions of closed and open  $\hat{\mathbf{Z}}$ -equivariant embeddings, hence they are all monomorphisms. For the second part, by Theorem 2.3 of [Za17a], if  $K(\mathcal{C})$  is the spectrum determined by an assembler  $\mathcal{C}$ , then  $\pi_0 K(\mathcal{C})$  is generated, as an abelian group, by the objects of  $\mathcal{C}$  with the scissor–congruence relations determined by disjoint covering families. In this case this means that  $K(\mathcal{C}^{\hat{\mathbf{Z}}})$  is generated by the pairs  $(X, \alpha)$  of a variety  $X$  with a good  $\hat{\mathbf{Z}}$ -action  $\alpha$  with relations  $[X] = [Y] + [X \setminus Y]$  for the covering families given by  $\hat{\mathbf{Z}}$ -equivariant embeddings  $\{Y \hookrightarrow X, X \setminus Y \hookrightarrow X\}$ . The ring structure is coming from the symmetric monoidal structure on the category of assemblers, which induces an  $E_\infty$ -ring structure on the spectrum  $K(\mathcal{C})$ . In this case it induces the product on  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  given by the Cartesian product of varieties with the diagonal  $\hat{\mathbf{Z}}$ -action (see also Theorem 1.4 of [Ca15]). The ring structure is induced by an  $E_\infty$ -ring spectrum structure on  $K(\mathcal{C})$  which is in turn induced by a symmetric monoidal structure on the category of assembler, cf. [Za17a]. ■

Following Theorem 4.25 of [Ca15], the ring structure on  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  can also be seen, as in the case of the ordinary Grothendieck ring  $K_0(\mathcal{V})$ , as induced on  $\pi_0$  by an  $E_\infty$ -ring spectrum structure obtained from the fact that the cartesian product of varieties determines a biexact symmetric monoidal structure on  $\mathcal{V}$ , seen as an SW-category (a subtractive Waldhausen category).

**4.6. Lifting the Bost–Connes algebra to spectra.** We will now show how to lift the maps  $\sigma_n$  and  $\tilde{\rho}_n$  of the Bost–Connes system from the level of the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  to the level of the spectrum  $K^{\hat{\mathbf{Z}}}(\mathcal{V})$ .

**4.6.1. Proposition.** *The maps  $\sigma_n(X, \alpha) = (X, \alpha \circ \sigma_n)$  and  $\tilde{\rho}_n(X, \alpha) = (X \times Z_n, \Phi_n(\alpha))$  on  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  determine endofunctors of the assembler category  $\mathcal{C}^{\hat{\mathbf{Z}}}$ . The endofunctors  $\sigma_n$  are compatible with the monoidal structure induced by the Cartesian product of varieties with diagonal  $\hat{\mathbf{Z}}$ -action.*

**Proof.** The endofunctors  $\sigma_n$  of  $\mathcal{C}^{\hat{\mathbf{Z}}}$  map an object  $(X, \alpha)$  to  $(X, \alpha \circ \sigma_n)$  and a

pair of  $\hat{\mathbf{Z}}$ -equivariant embeddings

$$(Y, \alpha|_Y) \hookrightarrow (X, \alpha) \leftarrow (X \setminus Y, \alpha|_{X \setminus Y})$$

to the pair of embedding

$$(Y, \alpha|_Y \circ \sigma_n) \hookrightarrow (X, \alpha \circ \sigma_n) \leftarrow (X \setminus Y, \alpha|_{X \setminus Y} \circ \sigma_n).$$

This determines the functor  $\sigma_n$  on both objects and morphisms of  $\mathcal{C}^{\hat{\mathbf{Z}}}$ . The compatibility with the monoidal structure comes from the compatibility with Cartesian products  $\sigma_n(X, \alpha) \times \sigma_n(X', \alpha') = (X \times X', (\alpha \times \alpha') \circ \Delta \circ \sigma_n) = \sigma_n((X, \alpha) \times (X', \alpha'))$ . The group homomorphisms  $\tilde{\rho}_n$  of  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  are also induced by endofunctors of  $\mathcal{C}^{\hat{\mathbf{Z}}}$ , which map objects by  $\tilde{\rho}_n(X, \alpha) = (X \times Z_n, \Phi_n(\alpha))$  and pairs of  $\hat{\mathbf{Z}}$ -equivariant embeddings

$$(Y, \alpha|_Y) \hookrightarrow (X, \alpha) \leftarrow (X \setminus Y, \alpha|_{X \setminus Y})$$

to pair of embedding

$$(Y \times Z_n, \Phi_n(\alpha|_Y)) \hookrightarrow (X \times Z_n, \Phi_n(\alpha)) \leftarrow ((X \setminus Y) \times Z_n, \Phi_n(\alpha|_{X \setminus Y}))$$

where  $\Phi_n(\alpha|_Y) = \Phi_n(\alpha)|_Y$  and  $\Phi_n(\alpha|_{X \setminus Y}) = \Phi_n(\alpha)|_{X \setminus Y}$ . The functors  $\tilde{\rho}_n$ , however, are not compatible with the monoidal structure, and this reflects the fact that they only induce group homomorphisms on  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  rather than ring homomorphisms. ■

One can obtain a similar argument working with subtractive Waldhausen categories as in [Ca15] in place of assemblers as in [Za17a].

**4.7. The Kontsevich–Tschinkel Burnside ring.** In a similar way, instead of working with the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$ , we can consider the refinement of the Grothendieck ring constructed in [KoTsch17]. We discuss here briefly how to adapt the previous construction to this case.

In [KoTsch17] a refinement of the Grothendieck ring of varieties is introduced, which is based on birational equivalence. More precisely, for  $\mathbf{K}$  a field of characteristic zero, the Burnside semiring  $\text{Burn}_+(\mathbf{K})$  is defined as the set of equivalence classes of smooth  $\mathbf{K}$ -varieties under the  $\mathbf{K}$ -birational equivalence relation, with addition and multiplication are given by disjoint union and product over  $\mathbf{K}$  (Definition 2 of [KoTsch17]). The Burnside ring  $\text{Burn}(\mathbf{K})$  is the Grothendieck ring generated by

the semiring  $\text{Burn}_+(\mathbf{K})$ . Equivalently, the Burnside ring  $\text{Burn}(\mathbf{K})$  is generated by isomorphism classes  $[X]$  of smooth varieties over  $\mathbf{K}$  with the equivalence relation  $[X] = [U]$  for  $U \hookrightarrow X$  an open embedding with dense image.

To construct an assembler and an associated spectrum that recovers the Burnside ring  $\text{Burn}(\mathbf{K})$  as its zeroth homotopy group, we proceed again as in [Za17a].

**4.7.1. Lemma.** *Let  $\mathcal{C}_{\text{Burn}}$  be the category with non-initial objects given by the smooth  $\mathbf{K}$ -varieties  $X$  and morphisms given by the open embeddings  $U \hookrightarrow X$  with dense image. Consider the Grothendieck topology which is generated by the open dense embeddings  $U \hookrightarrow X$ . The category  $\mathcal{C}_{\text{Burn}}$  is an assembler and the associated spectrum  $K(\mathcal{C}_{\text{Burn}})$  has  $\pi_0 K(\mathcal{C}_{\text{Burn}}) = \text{Burn}(\mathbf{K})$ .*

**Proof.** The initial object is the empty scheme. If  $X$  is irreducible, a disjoint covering family consists of a single dense open set  $U \hookrightarrow X$  and the common refinement of two disjoint covering families  $U_1 \hookrightarrow X$  and  $U_2 \hookrightarrow X$  is the dense open set  $U_1 \cap U_2 \hookrightarrow X$ . Morphisms are monomorphisms given by compositions of open dense embeddings. This shows that the category  $\mathcal{C}_{\text{Burn}}$  is an assembler. As an abelian group,  $\pi_0 K(\mathcal{C}_{\text{Burn}})$  is generated by the objects of  $\mathcal{C}_{\text{Burn}}$  with relations  $[X] = \sum_i [X_i]$  for  $\{f_i : X_i \rightarrow X\}$  a finite disjoint covering family. In this case this means identifying  $[X] = [U]$  for any dense open embedding  $U \hookrightarrow X$ , which is the equivalence relation of  $\text{Burn}(\mathbf{K})$ . ■

It is shown in [KoTsch17] that the Burnside ring  $\text{Burn}(\mathbf{K})$  with the grading given by the transcendence degree, maps surjectively to the associated graded object  $\text{gr } K_0(\mathcal{V}_{\mathbf{K}})$  with respect to the filtration of  $K_0(\mathcal{V}_{\mathbf{K}})$  by dimension

$$\text{Burn}(\mathbf{K}) \rightarrow \text{gr } K_0(\mathcal{V}_{\mathbf{K}}). \quad (4.2)$$

As we did in the case of the Grothendieck ring, we can also consider an equivariant version of the Kontsevich–Tschinkel Burnside ring  $\text{Burn}(\mathbf{K})$  with respect to the group  $\hat{\mathbf{Z}}$ , see sec. 5 of [KoTsch17]. The corresponding assembler and spectrum are obtained as a modification of the case discussed above. The following statement can be proved by arguments as in Lemma 4.7.1 and Lemma 4.5.1.

**4.7.2. Lemma.** *Let  $\text{Burn}^{\hat{\mathbf{Z}}}(\mathbf{K})$  be generated by equivalence classes of smooth  $\mathbf{K}$ -varieties with a good  $\hat{\mathbf{Z}}$ -action with respect to the equivalence relation  $[X] = [U]$  for  $U \hookrightarrow X$  a  $\hat{\mathbf{Z}}$ -equivariant dense open embedding. The category  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$  with objects the*

smooth  $\mathbf{K}$ -varieties  $X$  with a good  $\hat{\mathbf{Z}}$ -action and morphisms the  $\hat{\mathbf{Z}}$ -equivariant dense open embeddings  $U \hookrightarrow X$  is an assembler with  $\pi_0 K(\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}) = \text{Burn}^{\hat{\mathbf{Z}}}(\mathbf{K})$ .

We refer to  $K(\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}})$  as the  $\hat{\mathbf{Z}}$ -equivariant Burnside spectrum.

The notion of an epimorphic assembler with a sink was introduced in Section 4 of [Za17a]. It denotes an assembler  $\mathcal{C}$  with a sink object  $S$  such that  $\text{Hom}(X, S) \neq \emptyset$  for all other objects  $X \in \mathcal{C}$ , and with the properties that all morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$  with non-initial  $X$  are epimorphisms with the set  $\{f : X \rightarrow Y\}$  a covering family, and for  $X, Y \neq \emptyset$  no two morphisms  $X \rightarrow Z$  and  $Y \rightarrow Z$  are disjoint. There is a group  $G_{\mathcal{C}}$  associated to epimorphic assembler with a sink, whose elements are the equivalence classes of pairs of morphisms  $f_1, f_2 : X \rightarrow S$  from a non-initial object to the sink, where the equivalence  $[f_1, f_2 : X \rightarrow S] = [g_1, g_2 : Y \rightarrow S]$  is determined by the existence of an object  $Z$  and maps  $h_X : Z \rightarrow X$  and  $h_Y : Z \rightarrow Y$  such that the following diagram commutes

$$\begin{array}{ccccc}
 & & Y & & \\
 & g_1 \nearrow & \uparrow h_Y & \searrow g_2 & \\
 S & & Z & & S \\
 & f_1 \nwarrow & \downarrow h_X & \nearrow f_2 & \\
 & & X & & 
 \end{array}$$

Composition of morphisms is given by any (equivalent) completion to a commutative diagram of the form

$$\begin{array}{ccccc}
 & & W & & \\
 & h_1 \swarrow & & \searrow h_2 & \\
 & X & & Y & \\
 f_1 \swarrow & & & & \searrow g_2 \\
 S & & & & S \\
 & f_2 \searrow & & \nearrow g_1 & \\
 & & S & & 
 \end{array}$$

It is shown in Theorem 4.8 of [Za17a] that any choice of a morphism  $f_X : X \rightarrow S$  from each object of  $\mathcal{C}$  to the sink object  $S$  determines a morphism of assemblers

$\mathcal{C} \rightarrow \mathbf{S}_G$ , where  $\mathbf{S}_G$  is the assembler with objects  $\emptyset$  and  $\star$ , a non-invertible morphism  $\emptyset \rightarrow \star$  and invertible morphisms  $\text{Aut}(\star) = G$ , which has spectrum  $K(\mathbf{S}_G) = \Sigma_+^\infty BG$  (see Example 3.2 of [Za17a]). This morphism of assemblers  $\mathcal{C} \rightarrow \mathbf{S}_G$  induces an equivalence on  $K$ -theory.

**4.7.3. Lemma.** *The assembler  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$  is a coproduct of epimorphic assemblers with sinks*

$$\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}} = \bigvee_{[X, \alpha]} \mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)$$

where

$$K(\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)) \simeq \Sigma_+^\infty B\text{Aut}^{\hat{\mathbf{Z}}}(\mathbf{K}(X, \alpha)).$$

Here  $\text{Aut}^{\hat{\mathbf{Z}}}(\mathbf{K}(X, \alpha))$  is the group of  $\hat{\mathbf{Z}}$ -equivariant birational automorphisms of  $X$  with a good  $\hat{\mathbf{Z}}$ -action  $\alpha$ .

**Proof.** For an irreducible smooth projective variety  $X$  with a good  $\hat{\mathbf{Z}}$ -action  $\alpha : \hat{\mathbf{Z}} \times X \rightarrow X$ , consider the assembler  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)$  whose objects  $(U, \alpha_U) \hookrightarrow (X, \alpha)$  are the  $\hat{\mathbf{Z}}$ -equivariant dense open embeddings, with  $\alpha, \alpha_U$  the compatible good actions of  $\hat{\mathbf{Z}}$  on  $X$  and  $U$ , respectively. Arguing as in Theorem 5.3 of [Za17a] for the non-equivariant case, we see that  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X)$  satisfies the conditions of an epimorphic assembler with sink. The associated group  $G_{(X, \alpha)}^{\hat{\mathbf{Z}}}$  consists of equivalence classes of pairs  $f_1, f_2 : (U, \alpha_U) \rightarrow (X, \alpha)$ , and the  $f_i$  are equivariant with respect to these actions. The group  $G_{(X, \alpha)}^{\hat{\mathbf{Z}}}$  is therefore given by the group  $\text{Aut}^{\hat{\mathbf{Z}}}(\mathbf{K}(X, \alpha))$  of  $\hat{\mathbf{Z}}$ -equivariant birational automorphisms of the variety with good  $\hat{\mathbf{Z}}$  action  $(X, \alpha)$ . We then have  $K(\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)) \simeq \Sigma_+^\infty B\text{Aut}^{\hat{\mathbf{Z}}}(\mathbf{K}(X, \alpha))$ . Moreover, we can identify the assembler  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$  with the coproduct over equivalence classes  $[X, \alpha]$  of the assemblers  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)$  as above, since the morphisms of  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$  between non-initial objects come from morphisms of the  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)$  and the objects of  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$  consist of an initial object  $\emptyset$  and the non-initial objects of the  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)$  for a choice of representatives of the classes  $[X, \alpha]$ . ■

The relation between the Kontsevich–Tschinkel Burnside ring  $\text{Burn}^{\hat{\mathbf{Z}}}(\mathbf{K})$  and the equivariant Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{K}})$  can then be formulated at the level of assemblers and spectra in a form similar to Theorem 5.2 of [Za17a], using the same argument, adapted to the equivariant case. Let  $\mathcal{C}_{\mathbf{K}}^{\hat{\mathbf{Z}}, (\ell)}$  denote the full sub-assembler

of the assembler  $\mathcal{C}_{\mathbf{K}}^{\hat{\mathbf{Z}}}$  of Lemma 4.5.1 above, consisting of varieties of dimension at most  $n$  with good  $\hat{\mathbf{Z}}$ -action.

**4.7.4. Proposition.** *Let  $B_n^{\hat{\mathbf{Z}}}$  denote the set of birational isomorphism classes of varieties of dimension  $n$  with good  $\hat{\mathbf{Z}}$ -action, through  $\hat{\mathbf{Z}}$ -equivariant birational isomorphisms. The coproduct assembler*

$$\mathcal{C}_{\text{Burn},n}^{\hat{\mathbf{Z}}} := \bigvee_{[X,\alpha] \in B_n^{\hat{\mathbf{Z}}}} \mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}(X, \alpha)$$

satisfies

$$K(\mathcal{C}_{\text{Burn},n}^{\hat{\mathbf{Z}}}) \simeq \bigvee_{[X,\alpha] \in B_n^{\hat{\mathbf{Z}}}} \Sigma_+^{\infty} B\text{Aut}^{\hat{\mathbf{Z}}}(\mathbf{K}(X, \alpha)) \simeq \text{hocofib}(K(\mathcal{C}_{\mathbf{K}}^{\hat{\mathbf{Z}},(n-1)}) \rightarrow K(\mathcal{C}_{\mathbf{K}}^{\hat{\mathbf{Z}},(n)})).$$

**4.8. Burnside spectrum and Bost–Connes endomorphisms.** The same procedure we used to lift the Bost–Connes maps  $\sigma_n$  and  $\tilde{\rho}_n$  to the Grothendieck ring  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  and the spectrum  $K^{\hat{\mathbf{Z}}}(\mathcal{V})$  can be adapted to lift the same maps to the Kontsevich–Tschinkel Burnside ring  $\text{Burn}^{\hat{\mathbf{Z}}}(\mathbf{K})$  and the spectrum  $K(\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}})$ .

**4.8.1. Proposition.** *The maps  $\sigma_n$  and  $\tilde{\rho}_n$  of the integral Bost–Connes algebra lift to endofunctors of the assembler category  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$ , with the  $\sigma_n$  compatible with the monoidal structure induced by the Cartesian product. These endofunctors induce the corresponding maps  $\sigma_n$  and  $\tilde{\rho}_n$  on the Kontsevich–Tschinkel Burnside ring  $\text{Burn}^{\hat{\mathbf{Z}}}(\mathbf{K})$ .*

**Proof.** We argue as in Proposition 4.6.1. The endofunctors  $\sigma_n$  of  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$  map an object  $(X, \alpha)$  to  $(X, \alpha \circ \sigma_n)$  and a  $\hat{\mathbf{Z}}$ -equivariant dense open embedding

$$(U, \alpha|_U) \hookrightarrow (X, \alpha)$$

to the  $\hat{\mathbf{Z}}$ -equivariant dense open embedding

$$(U, \alpha|_U \circ \sigma_n) \hookrightarrow (X, \alpha \circ \sigma_n).$$

This determines the functor  $\sigma_n$  on both objects and morphisms of  $\mathcal{C}_{\text{Burn}}^{\hat{\mathbf{Z}}}$ . As in Proposition 4.6.1 one sees the  $\sigma_n$  are compatible with Cartesian products and induce

ring homomorphisms of  $\text{Burn}^{\hat{\mathbf{Z}}}(\mathbf{K})$ . The  $\tilde{\rho}_n$  map objects by  $\tilde{\rho}_n(X, \alpha) = (X \times Z_n, \Phi_n(\alpha))$  and  $\hat{\mathbf{Z}}$ -equivariant dense open embeddings  $(U, \alpha|_U) \hookrightarrow (X, \alpha)$  by

$$(U \times Z_n, \Phi_n(\alpha|_U)) \hookrightarrow (X \times Z_n, \Phi_n(\alpha))$$

with  $\Phi_n(\alpha|_U) = \Phi_n(\alpha)|_U$ . The  $\tilde{\rho}_n$  are not compatible with the monoidal structure and only induce group homomorphism on  $\text{Burn}^{\hat{\mathbf{Z}}}(\mathbf{K})$ . ■

## 5. Expectation values, motivic measures, and zeta functions

**5.1. The Bost–Connes expectation values.** In the case of the original Bost–Connes system, one considers representations  $\pi$  of the Bost–Connes algebra (either the integral  $\mathcal{A}_{\mathbf{Z}}$  or the rational  $\mathcal{A}_{\mathbf{Q}} = \mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$ ) on a Hilbert space  $\mathcal{H} = \ell^2(\mathbf{N})$  and associates to the algebra and the representation a dynamical system, namely the one–parameter group of automorphism  $\sigma : \mathbf{R} \rightarrow \text{Aut}(\mathcal{A})$  of the  $C^*$ -algebra generated by  $\mathcal{A}_{\mathbf{Q}}$ , seen as an algebra of bounded operators on  $\mathcal{H}$ . The time evolution satisfies the covariance condition

$$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH},$$

where  $H$  is an (unbounded) linear operator on  $\mathcal{H}$ , the Hamiltonian of the system. In the Bost–Connes case the time evolution is determined by  $\sigma_t(\mu_n) = n^{it} \mu_n$  and  $\sigma_t(x) = x$  for  $x \in \mathbf{Q}[\mathbf{Q}/\mathbf{Z}]$ . The Hamiltonian acts on the standard orthonormal basis of  $\ell^2(\mathbf{N})$  as  $H\epsilon_n = \log(n) \epsilon_n$  and the partition function  $Z(\beta) = \text{Tr}(e^{-\beta H})$  is the Riemann zeta function, [BC]. For any element  $a \in \mathcal{A}_{\mathbf{Q}}$  the expectation value with respect to the Bost–Connes dynamics is then given by

$$\langle a \rangle_{\beta} = \zeta(\beta)^{-1} \text{Tr}(\pi(a) e^{-\beta H}) = \zeta(\beta)^{-1} \sum_{n \in \mathbf{N}} \langle \epsilon_n, \pi(a) e^{-\beta H} \epsilon_n \rangle. \quad (5.1)$$

We can similarly construct Bost–Connes expectation values associated to the non-commutative ring  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V})$  defined in sec. 3.4.

**5.2. The equivariant Euler characteristic.** As we discussed above, the equivariant Euler characteristic  $\chi : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow K_0^{\hat{\mathbf{Z}}}(\mathbf{Q})$  induces a ring homomorphism  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow \mathcal{A}_{\mathbf{Z}}$  where  $\mathcal{A}_{\mathbf{Z}}$  is the integral Bost–Connes algebra. After tensoring with  $\mathbf{Q}$ , one obtains a morphism of crossed product algebras  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \otimes \mathbf{Q} = K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})_{\mathbf{Q}} \rtimes \mathbf{N} \rightarrow \mathcal{A}_{\mathbf{Q}} = \mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$ , with  $K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})_{\mathbf{Q}} = K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \otimes \mathbf{Q}$ .

**5.2.1. Proposition.** *Let  $\pi$  be a representation of the Bost–Connes algebra  $\mathcal{A}_{\mathbf{Q}}$  on the Hilbert space  $\mathcal{H} = \ell^2(\mathbf{N})$  with  $\pi(\mu_n)\epsilon_m = \epsilon_{nm}$  and  $\pi(e(r))\epsilon_n = \zeta_r^n \epsilon_n$  for  $r \rightarrow \zeta_r$  an embedding of  $\mathbf{Q}/\mathbf{Z}$  as the group of roots of unity in  $\mathbf{C}^*$ . Then  $\pi$  determines a one-parameter family of group homomorphism  $\varphi_\beta : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \rightarrow \mathbf{C}$ , with  $\beta \in \mathbf{R}_+^*$ , such that for all  $[X, \alpha] \in K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})$  the product  $\zeta(\beta) \cdot \langle [X, \alpha] \rangle_\beta$ , with  $\zeta(\beta)$  the Riemann zeta function, is a  $\mathbf{Z}$ -combination of values at roots of unity of the polylogarithm function  $\text{Li}_\beta(x)$ .*

**Proof.** For the generators  $a = e(r)$  of  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  the expectation value (5.1) is a polylogarithm function evaluated at a root of unity normalized by the Riemann zeta function,

$$\langle e(r) \rangle_\beta = \zeta(\beta)^{-1} \sum_{n \geq 1} \zeta_r^n n^{-\beta} = \frac{\text{Li}_\beta(\zeta_r)}{\zeta(\beta)},$$

where  $\pi(e(r))\epsilon_n = \zeta_r^n \epsilon_n$  with  $r \mapsto \zeta_r$  an embedding of  $\mathbf{Q}/\mathbf{Z}$  as the roots of unity in  $\mathbf{C}^*$ . Given a representation  $\pi$  of the Bost–Connes algebra, we compose the equivariant Euler characteristic  $\mathbf{K}_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}}) \otimes \mathbf{Q} = K_0^{\hat{\mathbf{Z}}}(\mathcal{V}_{\mathbf{Q}})_{\mathbf{Q}} \rtimes \mathbf{N} \rightarrow \mathcal{A}_{\mathbf{Q}} = \mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$  with the Bost–Connes expectation value  $\varphi(X, \alpha) = \langle \chi(X, \alpha) \rangle_\beta$ . ■

**5.3. Expectation values of motivic measures.** Other examples can be constructed using other motivic measures. For example, one can consider the mixed Hodge motivic measure  $h : K_0(\mathcal{V}_{\mathbf{C}}) \rightarrow K_0(HS)$  with  $h(X) = \sum_r (-1)^r [H_c^r(X, \mathbf{Q})] \in K_0(HS)$ . This is a refinement of the Hodge–Deligne polynomial motivic measure  $P : K_0(\mathcal{V}_{\mathbf{C}}) \rightarrow \mathbf{Z}[u, v, u^{-1}, v^{-1}]$  with  $P(X, u, v) = \sum_{p,q} \dim H^{p,q} u^p v^q$ . In the case of complex varieties with a good  $\hat{\mathbf{Z}}$ -action that factors through a finite quotient  $\mathbf{Z}/n\mathbf{Z}$ , the graded pieces  $H^{p,q}$  are  $\hat{\mathbf{Z}}$ -modules. The equivariant Hodge–Deligne polynomial is then defined as the polynomial in  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}][u, v]$  given by  $P^{\hat{\mathbf{Z}}}(X, \alpha, u, v) = \sum_{p,q} E^{p,q}(X, \alpha) u^p v^q$  with  $E^{p,q}(X, \alpha) = \sum_k (-1)^k H^{p,q}(H_c^k(X, \mathbf{C}))$ , with the  $\hat{\mathbf{Z}}$ -module structure determined by the action  $\alpha$ . The equivariant weight polynomial is given by  $W^{\hat{\mathbf{Z}}}(X, w) = P^{\hat{\mathbf{Z}}}(X, w, w)$  while evaluation at  $w = 1$  recovers the equivariant Euler characteristic. The associated expectation values are then of the form

$$\varphi_\beta(X, \alpha) = \sum_{p,q} \langle E^{p,q}(X, \alpha) \rangle_\beta u^p v^q$$

**5.4. Zeta functions and assemblers.** Passing from the level of Grothendieck rings to assemblers, spectra, and  $K$ -theory, as in [Za17a]–[Za17c], also provides possible methods for lifting the zeta functions at the level of  $K$ -theory. One approach,

currently being developed [Za18], directly uses assemblers and the construction of a map of assemblers between the assemblers underlying the Grothendieck ring (and its equivariant version as discussed above) and an assembler of almost-finite  $G$ -sets, by mapping a variety  $X$  to the almost-finite set  $X(\bar{K})$ . Another approach to the lifting of zeta functions was developed in [CaWoZa17], using étale cohomology and SW-categories. We will return to discussing zeta functions and the lifts of the Bost–Connes system to assemblers and spectra in the second part of this paper (in preparation).

Following the approach being developed in [Za18], one can show that the equivariant Euler characteristic

$$\chi^{\hat{\mathbf{Z}}} : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}) \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$$

discussed above in sections 3.1–3.2 lifts to a map of assembler by considering, as in section 3.2, the morphism

$$\chi^G : K_0^G(\mathcal{V}) \rightarrow A(G) \rightarrow R(G)$$

with  $A(G)$  the Burnside ring, for a finite group  $G$ , with the equivariant Euler characteristics defined as in [Gu-Za17] as mapping  $\chi^G(X) = \sum_k [X_k]$  with  $[X_k]$  the classes in  $A(G)$  of the  $k$ -skeleta. In the case of  $\hat{\mathbf{Z}}$  one considers the completion  $\hat{A}(\hat{\mathbf{Z}}) = \varprojlim A(\mathbf{Z}/n\mathbf{Z})$  as discussed in section 3.2, where the complete Burnstein ring  $\hat{A}(\hat{\mathbf{Z}})$  is seen as the Grothendieck ring of almost-finite  $\hat{\mathbf{Z}}$ -sets [DrSi88].

According to [Za18], there is a construction of an assembler of almost-finite  $G$ -sets, which we denote by  $\mathcal{AF}^G$ . The equivariant Euler characteristic

$$\chi^G : K_0^G(\mathcal{V}) \rightarrow A(G)$$

then lifts to a morphism of assemblers

$$\chi^G : \mathcal{C}^G \rightarrow \mathcal{AF}^G$$

since the assignment of  $X$  to the union of the  $X_k$  as  $G$ -sets and  $G$ -equivariant embeddings  $Y \hookrightarrow X$  and  $X \setminus Y \hookrightarrow X$  to the corresponding maps of the skeleta as  $G$ -sets maps disjointness morphisms in the assembler  $\mathcal{C}^G$  to disjoint morphisms in the assembler  $\mathcal{AF}^G$ . In particular, the equivariant Euler characteristic

$$\chi^{\hat{\mathbf{Z}}} : K_0^{\hat{\mathbf{Z}}}(\mathcal{V}) \rightarrow \hat{A}(\hat{\mathbf{Z}})$$

can be lifted to a morphism of assemblers

$$\chi^{\hat{\mathbf{Z}}} : \mathcal{C}^{\hat{\mathbf{Z}}} \rightarrow \mathcal{AF}^{\hat{\mathbf{Z}}}.$$

## 6. Dynamical $\mathbf{F}_1$ -structures and the Bost–Connes algebra

**6.1. The spectrum as Euler characteristic.** The point of view we adopt here is similar to [EbGu–Za17]. We consider the Grothendieck ring  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  of pairs  $(X, f)$  of a complex quasi-projective variety  $X$  with an automorphism  $f : X \rightarrow X$ , such that  $f_*$  in homology is quasi-unipotent. The addition is given by disjoint union and the product by the Cartesian product.

The quasi-unipotent condition ensures that the spectrum of the induced action  $f_* : H_*(X, \mathbf{Z}) \rightarrow H_*(X, \mathbf{Z})$  is contained in the set of roots of unity. We can then consider the spectrum of  $f_*$  as an Euler characteristic.

**6.1.1. Lemma.** *The spectrum of the induced map on homology determines a ring homomorphism*

$$\sigma : K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]. \quad (6.1)$$

**Proof.** To a pair  $(X, f)$  we associate the spectrum of the map  $f_* : H_*(X, \mathbf{Z}) \rightarrow H_*(X, \mathbf{Z})$ , seen as a subset  $\Sigma(f_*) \subset \mathbf{Q}/\mathbf{Z}$  of roots of unity counted with integer multiplicities. Thus, we have  $\sigma(X, f) = \sum_{\lambda \in \Sigma(f_*)} m_{\lambda} \lambda$ . The spectrum of a tensor product is given by the set of products of eigenvalues of the two matrices, hence the compatibility with the ring structure of  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ . ■

Under suitable assumptions on the induced map on  $H^*(X, \mathbf{C})$  and its Hodge decomposition, one can also consider other kinds of motivic measures associated to the spectrum  $\Sigma$ , for example generalizations of the Hodge–Deligne polynomial, see [EbGu–Za17].

**6.2. Lifting the Bost–Connes algebra to dynamical  $\mathbf{F}_1$ -structures.** Let  $Z_n$  be a zero-dimensional variety with  $\#Z_n(\mathbf{C}) = n$ . Then, for a given  $(X, f) \in K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ , the *Verschiebung* pair  $(X \times Z_n, \Phi_n(f))$  consists of the variety  $X \times Z_n$  with the automorphism  $\Phi_n(f)(x, a_i) = (x, a_{i+1})$  for  $i = 1, \dots, n-1$  and  $\Phi_n(f)(x, a_n) = (f(x), a_1)$ .

**6.2.1. Lemma.** *The induced map in homology  $\Phi_n(f)_* : H_*(X \times Z_n, \mathbf{Z}) \rightarrow H_*(X \times Z_n, \mathbf{Z})$  is the Verschiebung map.*

**Proof.** We have  $H_k(Z_n, \mathbf{Z}) = \mathbf{Z}^n$  for  $k = 0$  and zero otherwise, hence we can identify  $H_*(X \times Z_n, \mathbf{Z}) \simeq H_*(X, \mathbf{Z})^{\oplus n}$ . Then the action  $\Phi_n(f)(x, a_i) = (x, a_{i+1})$  for  $i = 1, \dots, n-1$  and  $\Phi_n(f)(x, a_n) = (f(x), a_1)$  induces the action  $\Phi_n(f)_* = V(f_*)$  in homology. ■

**6.3. Proposition.** *The maps  $\sigma_n(X, f) = (X, f^n)$  and  $\tilde{\rho}_n(X, f) = (X \times Z_n, \Phi_n(f))$  lift the maps  $\sigma_n$  and  $\tilde{\rho}_n$  of the integral Bost–Connes algebra to the Grothendieck ring  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ , compatibly with the spectrum Euler characteristic  $\sigma : K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ .*

**Proof.** The argument is analogous to the case of  $\hat{\mathbf{Z}}$ -actions analyzed in the previous section. Because of the relations between Frobenius  $F_n(f) = f^n$  and Verschiebung  $V_n(f)$  we have

$$\sigma_n \circ \tilde{\rho}_n(X, f) = \sigma_n(X \times Z_n, \Phi_n(f)) = (X \times Z_n, \Phi_n(f)^n) = (X \times Z_n, f \times 1) = (X, f)^{\oplus n}$$

$$\tilde{\rho}_n \circ \sigma_n(X, f) = \tilde{\rho}_n(X, f^n) = (X \times Z_n, \Phi_n(f^n)) = (X, f) \times (Z_n, \gamma),$$

with  $\gamma = \Phi_n(1) : a_i \mapsto a_{i+1}$  and  $a_n \mapsto a_1$ , where as before we used the relation  $V_n(F_n(a)b) = aV_n(b)$ , which gives  $\Phi_n(f^n) = f\Phi_n(1)$ . Under the spectrum Euler characteristic map  $\sigma : K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  we then see that we have commutative diagrams

$$\begin{array}{ccc} K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) & \xrightarrow{\sigma} & \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \\ \downarrow \sigma_n & & \downarrow \sigma_n \\ K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) & \xrightarrow{\sigma} & \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \end{array}$$

and where  $\sigma_n$  are ring homomorphism and  $\tilde{\rho}_n$  are group homomorphisms. ■

Thus, we can consider a non-commutative version of the Grothendieck ring  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ .

**6.4. Definition.** *Let  $\mathbf{K}_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  be the non-commutative ring generated by  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  together with generators  $\tilde{\mu}_n$  and  $\mu_n^*$  satisfying the relations (3.1) for all  $n, m \in \mathbf{N}$ , and (3.2) for all  $x = (X, f) \in K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  and all  $n \in \mathbf{N}$ .*

Consider then the algebra  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})_{\mathbf{Q}} = K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{Q}$ . As in the  $\hat{\mathbf{Z}}$ -equivariant case analyzed in the previous section, the maps  $\sigma_n$  and  $\tilde{\rho}_n$  induce endomorphisms  $\sigma_n$  and  $\rho_n$  of  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})_{\mathbf{Q}}$ , which determine a non-commutative semigroup crossed product algebra. The spectrum Euler characteristic (6.1) extends to an algebra homomorphism to the rational Bost–Connes algebra.

**6.5. Proposition.** *The algebra  $\mathbf{K}_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{Q}$  is isomorphic to a semigroup crossed product algebra  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})_{\mathbf{Q}} \rtimes \mathbf{N}$  with the semigroup action given by  $x \mapsto n^{-1}\tilde{\rho}_n(x)$ . The spectrum Euler characteristic (6.1) extends to an algebra homomorphism  $\sigma : \mathbf{K}_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \mathcal{A}_{\mathbf{Q}}$ , where  $\mathcal{A}_{\mathbf{Q}} = \mathbf{Q}[\mathbf{Q}/\mathbf{Z}] \rtimes \mathbf{N}$  is the rational Bost–Connes algebra.*

**Proof.** The algebra  $\mathbf{K}_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{Q}$  is generated by the elements of  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})_{\mathbf{Q}}$  and additional generators  $\mu_n = n^{-1}\tilde{\mu}_n$  and  $\mu_n^*$ , which satisfy the relations

$$\mu_n^* \mu_n = 1, \quad \mu_{nm} = \mu_n \mu_m, \quad \mu_{nm}^* = \mu_n^* \mu_m^*, \quad \forall n, m \in \mathbf{N}, \quad \mu_n \mu_m^* = \mu_m^* \mu_n \text{ if } (n, m) = 1$$

$$\mu_n x \mu_n^* = \rho_n(x) \quad \text{with} \quad \rho_n(x) = \frac{1}{n} \tilde{\rho}_n(x),$$

with  $\sigma_n \rho_n(x) = x$ , for all  $x = (X, f) \in K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ . The semigroup action in the crossed product algebra  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})_{\mathbf{Q}} \rtimes \mathbf{N}$  is given by  $x \mapsto \rho_n(x) = \mu_n x \mu_n^*$ , hence one obtains an identification of these two algebras. The morphism  $\sigma : \mathbf{K}_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \mathcal{A}_{\mathbf{Q}}$  is the map given by the spectrum Euler characteristic on elements of  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ , extended to  $\mathbf{Q}$ -coefficients, and it maps  $\chi(\mu_n) = \mu_n$  and  $\chi(\mu_n^*) = \mu_n^*$ . By Proposition 6.3, it determines a homomorphism of crossed-product algebras. ■

**6.6. Assembler and Bost–Connes endofunctors.** First we consider an assembler category  $\mathcal{C}_{\mathbf{C}}^{\mathbf{Z}}$  associated to the ring  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  and the associated  $\Gamma$ -space and spectrum  $K^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  with  $\pi_0 K^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) = K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ , then we show that the maps  $\sigma_n$  and  $\tilde{\rho}_n$  define endofunctors of the assembler  $\mathcal{C}_{\mathbf{C}}^{\mathbf{Z}}$ , hence they induce maps of spectra and induced map of the homotopy groups that recover the Bost–Connes map on  $\pi_0$ .

**6.6.1. Proposition** *Let  $\mathcal{C}_{\mathbf{C}}^{\mathbf{Z}}$  be the following category. Its objects the pairs  $(X, f)$  of a complex quasi-projective variety  $X$  with an automorphism  $f : X \rightarrow X$ , such that the induced map  $f_*$  in homology is quasi-unipotent. Its morphisms  $\varphi : (Y, h) \hookrightarrow (X, f)$  are given by embeddings  $Y \hookrightarrow X$  of components preserved by the map  $f$  and  $h = f|_Y$ . This is an assembler category, and the associated spectrum  $K^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) := K(\mathcal{C}_{\mathbf{C}}^{\mathbf{Z}})$  has  $\pi_0 K(\mathcal{C}_{\mathbf{C}}^{\mathbf{Z}}) = K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ . The maps  $\sigma_n$  and  $\tilde{\rho}_n$  on  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  lift to endofunctors of the assembler  $\mathcal{C}_{\mathbf{C}}^{\mathbf{Z}}$ , in which  $\sigma_n$  also compatible with the monoidal structure.*

**Proof.** The argument is similar to the  $\hat{\mathbf{Z}}$ -equivariant case we discussed before. In the category  $\mathcal{C}_{\mathbf{C}}^{\mathbf{Z}}$  the Grothendieck topology is generated by the covering families  $\{(X_1, f|_{X_1}) \hookrightarrow (X, f), (X_2, f|_{X_2}) \hookrightarrow (X, f)\}$  with  $X = X_1 \sqcup X_2$  and the  $X_i$  are preserved by the map  $f : X \rightarrow X$ . The empty  $X$  is the initial object. The

finite disjoint covering families are given by embeddings  $\varphi_i : (X_i, f|_{X_i}) \hookrightarrow (X, f)$ , where the  $X_i$  are unions of components preserved by the map,  $f|_{X_i} = f \circ \varphi_i$ . Any two finite disjoint families have a common refinement since the category has pullbacks, [Zak1] and morphisms are compositions of embeddings hence monomorphisms. The abelian group structure on  $\pi_0 K(\mathcal{C}_{\mathbf{Z}}^{\mathbf{Z}})$  is determined by the relation  $(X, f) = (X_1, f|_{X_1}) + (X_2, f|_{X_2})$  for each decomposition  $X = X_1 \sqcup X_2$  that is preserved by the map  $f : X \rightarrow X$ . The product is determined by the symmetric monoidal structure induced by the Cartesian product. Thus, we obtain the ring  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$ . The endofunctors  $\sigma_n$  map objects by  $\sigma_n(X, f) = (X, f^n)$  and maps pairs of embeddings with  $X = X_1 \sqcup X_2$

$$\{(X_1, f|_{X_1}) \hookrightarrow (X, f) \leftarrow (X_2, f|_{X_2})\}$$

to pairs of embeddings

$$\{(X_1, f^n|_{X_1}) \hookrightarrow (X, f^n) \leftarrow (X_2, f^n|_{X_2})\}$$

These functors are compatible with Cartesian products, hence with the monoidal structure. The endofunctors  $\tilde{\rho}_n$  act on objects as  $\tilde{\rho}_n(X, f) = (X \times Z_n, \Phi_n(f))$  and map a pair of embeddings as above to the pair

$$\{(X_1 \times Z_n, \Phi_n(f)|_{X_1}) \hookrightarrow (X \times Z_n, \Phi_n(f)) \leftarrow (X_2 \times Z_n, \Phi_n(f)|_{X_2})\},$$

where  $\Phi_n(f)|_{X_i} = \Phi_n(f|_{X_i})$ . The functors  $\tilde{\rho}_n$  are not compatible with the monoidal structure hence they induce group homomorphisms of  $\pi_0 K(\mathcal{C}_{\mathbf{Z}}^{\mathbf{Z}})$ . ■

Note that, unlike the  $\hat{\mathbf{Z}}$ -equivariant cases considered in the previous sections, the spectrum  $K(\mathcal{C}_{\mathbf{Z}}^{\mathbf{Z}})$  is not so interesting topologically, since in the assembler we are only using decompositions into connected components. The reason for wanting only this type of scissor-congruence relations in  $K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}})$  is the spectrum Euler characteristic  $\sigma : K_0^{\mathbf{Z}}(\mathcal{V}_{\mathbf{C}}) \rightarrow \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ , which should map a splitting  $X = X_1 \sqcup X_2$  compatible with  $f : X \rightarrow X$  to a corresponding splitting  $H_*(X, \mathbf{Z}) = H_*(X_1, \mathbf{Z}) \oplus H_*(X_2, \mathbf{Z})$  with quasi-unipotent maps  $f_*|_{X_i}$ , so that the spectrum as an element of  $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$  satisfies  $\sigma(X, f) = \sigma(X_1, f_1) + \sigma(X_2, f_2)$ .

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