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Non-Gaussian Fluctuations

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Abstract

Natural primordial mass density fluctuations are those for which the probability distribution, for the mass density fluctuations averaged over the horizon volume, is independent of time. This criterion determines the two-point correlation of the mass density fluctuations to have a Zeldovich power spectrum but allows for many types of higher correlations. If the connected higher correlations vanish the primordial fluctuations are Gaussian. In this case the probability distribution develops into a non-Gaussian one due to the non-linear time evolution. The nature of this non-Gaussian distribution and its effects on the large scale distribution of galaxies or clusters of galaxies and their large scale streaming velocities is explored. Next the possibility of natural primordial non-Gaussian fluctuations is examined. These can give rise to a very different large scale distribution of galaxies (or clusters of galaxies) than the Gaussian primordial fluctuations.

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I. Introduction

The large scale structure of the universe probably arose from small fluctuations in the energy density that grew due to their gravitational instability. The main purpose of these lectures is to introduce some techniques that I believe will prove useful if the primordial fluctuations in the energy density are not Gaussian[1]. As preparation for the study of non-Gaussian fluctuations many of the features of Gaussian primordial mass density fluctuations will be explored. Throughout these lectures I assume that the Universe is at critical density ($\Omega = 1$) and that the cosmological constant vanishes. I denote the Hubble constant by H and the Robertson Walker scale factor by a. The horizon length is defined to be equal to H^{-1}. Note that this is not necessarily the size of a region that has been in casual contact. In an inflationary cosmology[2], for example, the size of a region that has been in casual contact is very much greater than the horizon length.

Since the energy density fluctuations were once small linear perturbation theory should provide an adequate description of their evolution at early times. In linear perturbation theory it is convenient to Fourier transform the energy density fluctuations.

$$\frac{\delta\rho(\vec{x},t)}{\langle\rho(t)\rangle} = \int d\vec{k}\frac{\delta\rho(\vec{k},t)}{\langle\rho\rangle}e^{i\vec{k}\cdot\vec{x}}. \tag{1}$$

In eq. (1) $\vec{x}$ is the comoving coordinate and k is the comoving wavenumber. In linear perturbation theory modes of different wavevectors $\vec{k}$ evolve independently. Note that the physical wavelength associated with a mode of wavenumber k is $a(t)/k$. Since the horizon length is increasing linearly with cosmic time t while the Robertson Walker scale factor grows like $t^{1/2}$ during the radiation dominated era, and $t^{2/3}$ during the matter dominated era, fluctuations with physical wavelengths that are less than the horizon length today had wavelengths greater than the horizon length at early times. It is this feature that makes it hard to come up with reasonable ways for generating the primordial fluctuations.

Modes with physical wavelength less than the horizon length do not grow during the radiation dominated era and grow like $t^{2/3}$ during the matter dominated era (in linear perturbation theory). The evolution of modes with wavelength greater than the horizon length is gauge dependent[3]. I shall work in a gauge where they do not grow in the radiation or matter dominated eras.

2. The Principle Of Scale Invariance

There are many different possibilities for the form of the primordial fluctuations in the energy density. In order to make progress it is necessary to introduce some principle that narrows down the number of possibilities. For fluctuations which cross the horizon in the matter dominated era we can write

$$\frac{\delta\rho(\vec{k},t)}{\langle\rho\rangle} = a(\vec{k})(t/t_{\rm h.c.})^{2/3}, \tag{2}$$

where $a(\vec{k})$ is a random variable and $t_{\rm h.c.}$ is the time that the fluctuations with comoving wavenumber k entered the horizon. This is

$$k/a(t_{\rm h.c.}) = H(t_{\rm h.c.}). \tag{3}$$

Using

$$a = (t/t_0)^{2/3},\ H = 2/3t, \tag{4}$$

the time of horizon crossing can be expressed in terms of the present time t_0 and the comoving wavenumber k (Note that I have normalized the scale factor so that today

the scale factor is unity and physical and comoving wavenumbers coincide).

$$t_{\rm h.c.}^{-1/3} = \frac{3}{2}kt_0^{2/3}. \tag{5}$$

Putting eq. (5) into eq. (2) gives

$$\frac{\delta\rho(\vec{k},t)}{\langle\rho\rangle} = \frac{9}{4}a(\vec{k})k^2t^{2/3}t_0^{4/3} \equiv \epsilon(\vec{k})t^{2/3}t_0^{4/3}. \tag{6}$$

For fluctuations which cross the horizon in the radiation dominated era this gets modified by a computable function of k that goes to unity a k goes to zero.

In the matter dominated era fluctuations in the energy density become fluctuations in the mass density. Since the Fourier transform of the fluctuations in the mass density $\frac{\delta\rho(\vec{k},t)}{\langle\rho\rangle}$ have dimensions of (length)^3, (I shall use the particle physics convention that $c = \hbar = 1$) eq. (6) implies that ϵ has dimensions of (length). It is the probability distribution for ϵ that determines the nature of the primordial fluctuations. Equivalently one can specify the primordial fluctuations by the moments of the probability distribution (assuming of course that all the moments exist).

$$\langle\epsilon(\vec{k}_1)\dots\epsilon(\vec{k}_n)\rangle = \int[d\epsilon]\epsilon(\vec{k}_1)\dots\epsilon(\vec{k}_n)P[\epsilon]. \tag{7}$$

In equation (7) the measure of integration $[d\epsilon]$ means that the value of ϵ at each wavevector $\vec{k}$ is integrated over. This is called a functional integral[4].

If the length scales associated with the process which generated the primordial fluctuations are small compared to astrophysically relevant length scales, then they can be neglected. Under this circumstance dimensional analysis gives[5]

$$\langle\epsilon(\lambda\vec{k}_1)\dots\epsilon(\lambda\vec{k}_n)\rangle = \lambda^{-n}\langle\epsilon(\vec{k}_1)\dots\epsilon(\vec{k}_n)\rangle. \tag{8}$$

Eq. (8) is called the principle of scale invariance. I shall adhere religiously to this principle during these lectures. Equation (8) implies that the mass density fluctuations averaged over the horizon volume have a probability distribution that is independent of time.

Scale invariance plus the homogeneity and isotropy of space determines the two-point correlation of ϵ up to normalization to have the Zeldovich[6] form.

$$\langle \epsilon(\vec{k}_1)\epsilon(\vec{k}_2)\rangle \propto k_1 \delta^3(\vec{k}_1 + \vec{k}_2). \tag{9}$$

In eq. (9) the δ function of wavevectors is required by the homogeneity of space. It ensures that the correlation $\langle \epsilon(\vec{x})\epsilon(\vec{y})\rangle$ depends only on the separation between the points $\vec{x}$ and $\vec{y}$. Scale invariance gives that the coefficient of the δ function must go like a single power of a wavevector. For example, the magnitude of $\vec{k}_1$ or its component along some particular axis. The only rotationally invariant choice is the magnitude of $\vec{k}_1$.

Scale invariance does not determine the constant of proportionality in eq. (9). However, positivity of the probability distribution restricts this constant to be positive (recall that since the mass density fluctuations are real $\epsilon^*(\vec{k}) = \epsilon(-\vec{k})$ so $\langle \epsilon(\vec{k})\epsilon(-\vec{k})\rangle$ is the average value of a positive quantity).

3. Gaussian Scale Invariant Fluctuations

The simplest probability distribution consistent with the principle of scale invariance is a Gaussian one.

$$P[\epsilon] = \frac{1}{Z} \exp -\frac{1}{2} \int d\vec{k} \epsilon(\vec{k})\epsilon(-\vec{k}) f(\vec{k}). \tag{10}$$

Since $\epsilon(\vec{k})$ has dimensions of length, scale invariance demands that $f(k) \propto 1/k$ (where the constant of proportionality is dimensionless). The function f determines the two

point function

$$\langle \epsilon(\vec{k}_1)\epsilon(\vec{k}_2)\rangle = (1/f(k_1))\delta^3(\vec{k}_1 + \vec{k}_2). \tag{11}$$

The probability distribution (10) is not completely correct since it assigns a non-zero probability to configurations $\epsilon(\vec{k})$ that correspond to a negative mass. Fortunately the error made is incredibly small because these configurations are highly exponentially suppressed.

One can also characterize a Gaussian probability distribution by the behavior of all its moments. For Gaussian primordial mass density fluctuations all the connected correlations of $\epsilon(\vec{k})$ vanish, except the two-point correlation function. For example, the four-point function can be written in the form (using the property that the average value of ϵ is zero).

$$\langle \epsilon(\vec{k}_1)\epsilon(\vec{k}_2)\epsilon(\vec{k}_3)\epsilon(\vec{k}_4)\rangle = \langle \epsilon(\vec{k}_1)\epsilon(\vec{k}_2)\rangle\langle \epsilon(\vec{k}_3)\epsilon(\vec{k}_4)\rangle + \langle \epsilon(\vec{k}_1)\epsilon(\vec{k}_3)\rangle\langle \epsilon(\vec{k}_2)\epsilon(\vec{k}_4)\rangle$$

$$+\langle \epsilon(\vec{k}_1)\epsilon(\vec{k}_4)\rangle\langle \epsilon(\vec{k}_2)\epsilon(\vec{k}_3)\rangle + \langle \epsilon(\vec{k}_1)\epsilon(\vec{k}_2)\epsilon(\vec{k}_3)\epsilon(\vec{k}_4)\rangle_c. \tag{12}$$

In eq. (12) the subscript c denotes the connected part of the correlation. For Gaussian fluctuations this part of the four-point correlation vanishes.

The main motivation for Gaussian fluctuations is the small value of the primordial mass density fluctuations. The mass density fluctuations averaged over the horizon volume are about equal to 10^{-5}. This suggests that it was a weakly interacting field which gave rise to the mass density fluctuations; in the standard new inflationary model[7] for how the mass density fluctuations are generated[8] this is indeed the case. The small value of primordial fluctuations restricts the inflaton field to be very weakly interacting and hence for the resulting fluctuations to be approximately Gaussian. Of course there is no observational evidence to support the standard inflationary model. In fact since the final cosmological constant must be fine tuned to zero, it seems silly to imagine that the details of this model are correct, (although the general idea may be).

4. Non-Linear Time Evolution

Even if the primordial mass density fluctuations are Gaussian the non-linear time evolution[9] will ensure that at late times the mass density fluctuations are highly non-Gaussian. It is important to understand the nature of the connected correlations of the mass density induced by the non-linear time evolution in order to distinguish their effects from those of the primordial non-Gaussian fluctuations. In this section the connected correlations of the mass density and the peculiar velocity are studied at wavenumbers corresponding to distances that are large compared with those that have undergone very non-linear evolution, but small compared to the horizon length. The mass is treated as a pressureless Newtonian fluid. The time evolution of the mass density fluctuation field $\delta(\vec{x},t) = \frac{\rho(\vec{x},t)-\langle\rho\rangle}{\langle\rho\rangle}$ and the peculiar velocity field $\vec{v}(\vec{x},t)$ is governed by the equation of continuity, Eulers equation and the Newtonian expression for the gravitational field $\vec{g}(\vec{x},t)$.

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\vec{\nabla}\cdot(1+\delta)\vec{v} = 0 \qquad (13a)$$

$$\frac{\partial\vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{g} \qquad (13b)$$

$$\vec{\nabla}\cdot\vec{g} = -4\pi G\langle\rho\rangle a\delta, \quad \vec{\nabla}\times\vec{g} = 0. \qquad (13c)$$

A perturbative expansion for the mass density fluctuation field and the peculiar

velocity field is developed by linearizing these equations about the background

$$\delta = 0 \quad \text{and} \quad \vec{v} = 0, \qquad (14)$$

and keeping only the fastest growing mode, then linearizing about this solution and again keeping only the fastest growing mode, etc. The resulting perturbation expansions for δ and $\vec{v}$ have the form

$$\delta(\vec{x},t) = \sum_{n=1}^{\infty}\epsilon_n(\vec{x}_1)t^{2n/3}, \qquad (15)$$

$$\vec{v}(\vec{x},t) = a\sum_{n=1}^{\infty}\vec{v}_n(\vec{x})t^{\frac{2n}{3}-1}, \vec{\nabla}\times\vec{v}_n = 0. \qquad (16)$$

At early times (small t) the series is dominated by the first term so ϵ_1 and $\vec{v}_1$ characterize the primordial mass density fluctuations. The equation of continuity determines $\vec{v}_1$ in terms of ϵ_1. The equations of motion (13) determine ϵ_n and $\vec{v}_n$ in terms of the primordial fluctuations. In wavenumber space this relationship has the form:

$$\epsilon_n(\vec{k}) = \int\frac{d\vec{q}_1}{(2\pi)^3}\cdots\int\frac{d\vec{q}_n}{(2\pi)^3}(2\pi)^3\delta^3(\vec{q}_1+\ldots+\vec{q}_n-\vec{k})P_n^{(s)}(\vec{q}_n\ldots,\vec{q}_n) \qquad (17)$$

$$\cdot\epsilon_1(\vec{q}_1)\ldots\epsilon_1(\vec{q}_n),$$

$$\vec{v}_n(\vec{k}) = \frac{i\vec{k}}{k^2}\int\frac{d\vec{q}_1}{(2\pi)^3}\cdots\int\frac{d\vec{q}_n}{(2\pi)^3}\delta^3(\vec{q}_1+\ldots+\vec{q}_n-\vec{k})$$

$$\cdot Q_n^{(s)}(\vec{q}_1,\ldots,\vec{q}_n)\epsilon_1(\vec{q}_1)\ldots\epsilon_1(\vec{q}_n). \qquad (18)$$

Here $P_n^{(s)}$ and $Q_n^{(s)}$ are symmetric homogeneous functions of the wavevectors $\vec{q}_1,\ldots\vec{q}_n$ of degree zero. Recursion relations relate $P_n^{(s)}$ and $Q_n^{(s)}$ to $P_1^{(s)} = 1$ and $Q_1^{(s)} = 2/3$.

For example

$$P_2^{(s)}(\vec{p},\vec{q}) = \frac{5}{7} + \frac{\vec{p}\cdot\vec{q}}{2pq}\left(\frac{p}{q}+\frac{q}{p}\right) + \frac{2}{7}\left(\frac{\vec{p}\cdot\vec{q}}{pq}\right)^2 \tag{19a}$$

$$Q_2^{(s)}(\vec{p},\vec{q}) = \frac{2}{7} + \frac{\vec{p}\cdot\vec{q}}{3pq}\left(\frac{p}{q}+\frac{q}{p}\right) + \frac{8}{21}\left(\frac{\vec{p}\cdot\vec{q}}{pq}\right)^2. \tag{19b}$$

For $n > 2$ $P_n^{(s)}$ and $Q_n^{(s)}$ are quite complicated. The Zeldovich approximation[10] is sometimes used to describe the non-linear time evolution. Since it is exact in one dimension it should be useful when the dominant non-linear effect is the collapse of pancakes. In appendix A it is shown that the Zeldovich approximation is equivalent to the choice[11]

$$P_n^{(s)}(\vec{q}_1,\dots\vec{q}_n) = \frac{1}{n!}\frac{(\vec{k}\cdot\vec{q}_1)}{q_1^2}\cdots\frac{(\vec{k}\cdot\vec{q}_n)}{q_n^2} \tag{20}$$

$$Q_n^{(s)}(\vec{q}_1,\dots\vec{q}_n) = \frac{2}{3}P_n^{(s)}(\vec{q}_1,\dots\vec{q}_n), \tag{21}$$

where $\vec{k} = \vec{q}_1 + \dots + \vec{q}_n$. From the recursion relations it can be deduced that the $P_n^{(s)}(\vec{q}_1,\dots\vec{q}_n)$ and $Q_n^{(s)}(\vec{q}_1,\dots\vec{q}_n)$ have the following special properties.

(a) As $\vec{k} = \vec{q}_1 + \dots + \vec{q}_n$ goes to zero (but the individual $\vec{q}_i$ do not) $P_n^{(s)} \propto k^2$. This property is essentially a consequence of momentum conservation. In the Zeldovich approximation this property is modified to $P_n^{(s)} \propto k^n$. It appears that the Zeldovich approximation makes successive orders in the perturbative expansion for δ less important at large scales (i.e. small k) than they actually are.

(b) As some of the arguments of $P_n^{(s)}$ (or $Q_n^{(s)}$) get large but the vector sum of all the arguments of $P_n^{(s)}$ (or $Q_n^{(s)}$) stays fixed, $P_n^{(s)}$ (or $Q_n^{(s)}$) vanishes like a power of

the large arguments. For example, for $p >> q_j$,

$$P_n^{(s)}(\vec{q}_1,\dots\vec{q}_{n-2},\vec{p},-\vec{p}) \propto 1/p^2 \quad Q_n^{(s)}(\vec{q}_1,\dots\vec{q}_{n-2},\vec{p},-\vec{p}) \propto 1/p^2. \tag{22}$$

This is essentially a decoupling property. It limits the importance of primordial fluctuations of high wavenumber to the mass density fluctuations at small wavenumbers.

(c) If one of the arguments of $P_n^{(s)}(\vec{q}_1,\dots\vec{q}_n)$ or $Q_n^{(s)}(\vec{q}_1,\dots\vec{q}_n)$ goes to zero then there is an infrared divergence of the form

$$\vec{q}_i\,/\,q_i^2. \tag{23}$$

This property is of kinematical origin. These infrared singularities would be absent if the fluctuations had been expanded in terms of $\vec{v}_1$ instead of ϵ_1. There are no infrared divergences as partial sums of several wavevectors go to zero.

The effects of nonlinearities in the time evolution can be taken into account once the probability distribution for ϵ_1 is specified. In this section it is assumed that ϵ_1 is a Gaussian random variable with a Zeldovich spectrum.

$$\langle\epsilon_1(\vec{k}_1)\epsilon_1(\vec{k}_2)\rangle t^{4/3} = (2\pi)^3 A(k_1)k_1\delta^3(\vec{k}_1+\vec{k}_2), \tag{24}$$

where A goes to a constant as k_1 goes to zero. With cold dark matter $A(k) \propto (\ln k/k^2)^2$ for large k. A arises because fluctuations which cross the horizon in the radiation dominated era only grow logarithmically until the time of matter domination.

To compute correlations of δ and $\vec{v}$ one expands them in a "power series" in ϵ_1 and then evaluates the correlations of ϵ_1 by factorizing them into products of two-point functions (see for example eq. (12)). As a simple example consider the leading perturbative contribution to the three-point correlation of $\langle\delta(\vec{k}_1,t)\delta(\vec{k}_2,t)\delta(\vec{k}_3,t)\rangle$. It arises when one of the δ's is expanded to second order in ϵ_1

$$\langle\delta(\vec{k}_1,t)\delta(\vec{k}_2,t)\delta(\vec{k}_3,t)\rangle = t^{8/3}\left\{\langle\epsilon_2(\vec{k}_1)\epsilon_1(\vec{k}_2)\epsilon_1(\vec{k}_2)\rangle + \text{perms}\right\}$$

$$= t^{8/3}\int\frac{d\vec{q}_1}{(2\pi)^3}\int\frac{d\vec{q}_2}{(2\pi)^3}P_2^{(s)}(\vec{q}_1,\vec{q}_2)(2\pi)^3\delta^3(\vec{q}_1+\vec{q}_2-\vec{k}_1)$$

$$\cdot\langle\epsilon_1(\vec{q}_1)\epsilon_1(\vec{q}_2)\epsilon_1(\vec{k}_2)\epsilon_1(\vec{k}_3)\rangle + \text{perms.} \tag{25}$$

Evaluating the average value of the product of four in eq. (25) using eq. (24) gives

$$\langle\delta(\vec{k}_1,t)\delta(\vec{k}_2,t)\delta(\vec{k}_3,t)\rangle$$

$$= 2\int\frac{d\vec{q}_1}{(2\pi)^3}\int\frac{d^3q_2}{(2\pi)^3}P_2^{(s)}(\vec{q}_1,\vec{q}_2)(2\pi)^3\delta^3(\vec{q}_1+\vec{q}_2-\vec{k}_1)A(k_1)k_1$$

$$\cdot(2\pi)^3\delta^3(\vec{k}_1+\vec{q}_1)A(k_2)k_2(2\pi)^3\delta^3(\vec{k}_2+\vec{q}_2) + \text{perms}$$

$$= (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)2\Big[P_2^{(s)}(\vec{k}_1,\vec{k}_2)A(k_1)A(k_2)k_1k_2$$

$$+P_2^{(s)}(\vec{k}_1,\vec{k}_3)A(k_1)A(k_3)k_1k_3 + P_2^{(s)}(\vec{k}_2,\vec{k}_3)A(k_2)A(k_3)k_2k_3\Big]. \tag{26}$$

There is a simple diagrammatic way to visualize this computation. This is shown in Fig. (1). Each δ is denoted by a solid line and the primordial fluctuation ϵ_1 is denoted by a dotted line. One of the δ's split into two ϵ_1's via second order perturbation theory while the other two δ's just went into a single dotted line (linear perturbation theory). Then the four ϵ_1's (dotted lines) are sewn together two at a time as is appropriate for Gaussian primordial fluctuations. This diagrammatic approach

can be generalized to all orders in the perturbative expansion and is analogous to the Feynman diagrams used to compute correlations of quantum fields. Contributions to a connected n-point correlation of δ, come from connected diagrams with n external solid lines and an arbitrary number of internal dotted lines. Each internal line is labeled by a wavevector that is integrated over. The vertices which join the lines and the factors associated with them and the internal lines are shown in Fig. (2). There are also combinatorial factors associated with the number of ways of putting the ϵ_1's into two-point correlations (24). Some of the perturbative contributions to the two-point function are shown in Fig. (3).

According to the diagrammatic rules the diagram at the top of the second column in Fig. (3), for example, gives the following contribution to the two-point function.

$$(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)2\int\frac{d\vec{q}}{(2\pi)^3}\left(P_2^{(s)}(\vec{k}_1-\vec{q},\vec{q})\right)^2A(q)A(\vec{k}-\vec{q})q|\vec{k}-\vec{q}|. \tag{27}$$

Note that there is an integral over the internal wavevector $\vec{q}$ in eq. (27). It is straightforward to see that in general the number of integrations over internal wavenumbers in the contribution of a diagram is equal to the number of loops in the diagram. Note also that with $A(q)$ falling as $1/q^4$ for large q this integral is convergent.

To evaluate the mean square value, of the mass density fluctuations averaged over a ball of large radius R (with a fuzzy edge) $\delta(R)$, the diagrammatic contributions to the two-point correlation of δ must be examined at small external wavenumbers k. Fig. (3) gives the leading k behavior for small external wavenumber of each of the diagrams. Most of these factors follow straightforwardly from the properties of the $P_n^{(s)}$ mentioned earlier. For example the first diagram in the second column of Fig.

(3) has a factor of k^4. This comes from the part of this diagram that has large loop wavenumbers compared with k. At loop wavenumbers comparable with k the diagram has a factor of k^3 associated with the measure of integration and one factor of k for each internal dotted line. There are no factors of k associated with the two vertices since they are homogeneous functions of degree zero and all their arguments are of order k. Thus the contribution from loop wavenumbers comparable with k is of order k^5. At large loop wavevectors there are no factors of k associated with the measure of integrations or the internal dotted lines. There are however, two factors of k^2 associated with the vertices. According to the decoupling property each vertex falls off like the square of the wavenumber associated with the loop integration. Since they are homogeneous functions this necessitates a compensating factor of k^2 in each vertex. (The presence of this factor could have also been deduced from property (a).)

Note that there appear to be an infinite number of diagrams that behave like k^4 (this is different from what happens in the Zeldovich approximation where successive orders of perturbation theory produce additional factors of k). Clearly the contribution of order k^4 cannot be computed. However, only the first diagram (linear perturbation theory) in Fig. (3) goes like k. Thus for large R

$$\langle \delta(R)^2 \rangle \propto 1/R^4, \tag{28}$$

with the constant of proportionality calculable from linear perturbation theory.

To get information on the behavior of the higher moments of $\delta(R)$, connected correlations with more external lines must be considered. Consider the average value, for large R, of $\delta(R)^3$. This is determined by the behavior of the three-point correlation function at small wavenumbers k (here k stands for any of the three wavenumbers that are needed to specify this function). Fig. (4) shows some low order diagrams and

the power of k associated with their contribution. Only the first diagram produces a factor of k^2. This behavior persists to all orders in the perturbative expansion. Thus for large R

$$\langle \delta(R)^3 \rangle \propto 1/R^8, \tag{29}$$

with the constant of proportionality given by a computation of the first diagram in Fig. (4) (see eq. (26)).

Diagrams without loops are called trees. In general the tree graphs dominate the large R behavior of $\langle \delta(R)^n \rangle_c$. Since the trees produce a factor of k for each internal dotted line

$$\langle \delta(R)^n \rangle_c \propto \frac{1}{R^{4(n-1)}}. \tag{30}$$

For n even the primordial fluctuations contribute to $\langle \delta(R)^n \rangle$ through it's disconnected part. They give

$$\langle \delta(R)^n \rangle_{\text{primordial}} \propto 1/R^{2n}. \tag{31}$$

This dominates over the connected contribution (30) at large R. So for large R, $\delta(R)$ becomes a Gaussian random variable with a variance determined by the primordial fluctuations. The leading corrections to this Gaussian behavior are computable through the tree graphs.

A similar analysis can be done for the peculiar velocity field. Perturbative contributions to connected correlations of $\vec{v}$ come from connected diagrams with n external wavy lines and an arbitrary number of internal dotted lines. The factors associated with the vertices are shown in Fig. (5). The behavior of moments of the velocity field averaged over a large ball (with a fuzzy edge) $\vec{v}(R)$, is determined by the behavior of the correlations of $\vec{v}$ at small external wavenumbers $\vec{k}$. In general

$$\langle \vec{v}(R)^{2n} \rangle_c \propto 1/R^{6n-4} \tag{32}$$

with the constant of proportionality determined by the tree graphs.

The perturbative expansions for δ and $\vec{v}$ will not be valid for late times t; the pressureless mass density fluid develops a pressure at late times due to orbit crossing. Clearly the present time t_0 is late enough for orbit crossing to have occurred. Nonetheless, the conclusions of this section probably remain valid since they are completely insensitive to the form of the primordial power spectrum at high wavenumbers.

5. Objects

In the previous section properties of the large scale distribution of mass were explored. Unfortunately it is not possible to observe directly the distribution of mass. What is really needed are predictions for the large scale distribution of the various types of luminous objects observed (e.g., galaxies or rich clusters of galaxies). Since the details of galaxy formation are not well understood one is forced to resort to models for the relationship between the number density n of the objects observed and the underlying mass density fluctuations. The simplest possible assumption one can make is that the objects trace the mass. That is, the number density of the objects is proportional to the mass density and the peculiar velocity of the objects is equal to the peculiar velocity of the mass at the location of the objects. One way the objects could trace the mass is if all the mass ends up in the objects; another way is if the objects are a fair sample of the mass distribution. It is clear from observations that not all of the objects observed can trace the mass since different types of objects have different spatial distributions. For example, the two-point correlation for galaxies is unity at

about $5h^{-1}$Mpc. While the two-point correlation for rich clusters of galaxies[12] is unity at about $25h^{-1}$ Mpc.

In order to make progress, I will assume that the number density of the various classes of luminous objects can be written as a local function of the mass density fluctuations, (and derivatives of the mass density fluctuations), filtered on the comoving scale that collapsed to the object observed.

$$n(\vec{x}) = \sum_{n=0}^{\infty} C_n \delta_f^n(\vec{x}) + \sum_{n=0}^{\infty} C_n' \vec{\nabla}\delta_f(\vec{x}) \cdot \vec{\nabla}\delta_f(x)\delta_f^n(\vec{x}) + \dots \tag{33}$$

where

$$\delta_f(\vec{x}) = \int d\vec{y} W(\vec{x}-\vec{y})\delta(\vec{y}), \tag{34}$$

and W is the appropriate filter. I shall also assume that the peculiar velocity of the objects is equal to that of the mass density fluid at the location of the objects.

An example of a number density of the type in eq. (33) is

$$n_>(\vec{x}) = C \exp T\delta_f(\vec{x}). \tag{34}$$

For large T (i.e. $T\delta_f(0) >> 1$) this corresponds to the objects forming preferentially where the mass density fluctuations are unusually large[13]. Eq. (34) has the advantage that it is easy to display how the two point correlation of such objects, $\xi_>$, defined by

$$1 + \xi_>(|\vec{x}-\vec{y}|) = \frac{\langle n_>(\vec{x}) n_>(\vec{y}) \rangle}{\langle n_> \rangle^2} \tag{35}$$

depends on the correlations of the underlying mass density fluctuations. For a source

J it can easily be shown that the generating functional

$$Z[J] = \int [d\,\delta] P[\delta] e^{\int d\vec{x} J(x)\delta(x)} \tag{36}$$

can be expressed in terms of the connected correlations of δ in the following fashion

$$Z[J] = \exp\left\{\sum_{n=2}^{\infty} \frac{1}{n!} \int d\,\vec{x}_1 \ldots d\vec{x}_n \langle\delta(\vec{x}_1)\ldots\delta(\vec{x}_n)\rangle_c J(\vec{x}_1)\ldots J(\vec{x}_n)\right\}. \tag{37}$$

To verify this, just expand the exponentials in eqs. (36) and (37) in a power series and equate powers of the source J. The two-point function for the objects is essentially the generating functional evaluated at a particular source.

$$1 + \xi_>(|\vec{x} - \vec{y}|) = \frac{Z[J(\vec{z}) = W(\vec{x} - \vec{z}) + W(\vec{y} - \vec{z})]}{Z[J(\vec{z}) = W(\vec{z})]^2}. \tag{38}$$

Using eq. (37) this becomes

$$1 + \xi_>(|\vec{x}_1 - \vec{x}_2|) = \exp\Big\{\sum_{n=2}^{\infty} \frac{T^n}{n!} \int d\,\vec{y}_1 \ldots d\,\vec{y}_n \sum_{m=1}^{n-1} \binom{n}{m} \langle\delta(\vec{y}_1)\ldots\delta(\vec{y}_n)\rangle_c$$

$$\cdot [W(\vec{x}_1 - \vec{y}_1)\ldots W(\vec{x}_1 - \vec{y}_m)][W(\vec{x}_2 - \vec{y}_{m+1})\ldots W(\vec{x}_2 - \vec{y}_n)]\Big\}. \tag{39}$$

The two-point correlation of the objects depends on all the connected correlations of the mass density fluctuations. If the mass density fluctuations are approximated as Gaussian so that only the connected two-point correlation of δ is non-zero the above becomes[13,14]

$$\xi_>(|\vec{x}_1 - \vec{x}_2|) = \exp T^2 \xi_f(|\vec{x}_1 - \vec{x}_2|) - 1. \tag{40}$$

where

$$\xi_f(r) = \int d\,\vec{k}\, e^{i\vec{k}\cdot\vec{r}} k A(k) \tilde{W}(k)^2. \tag{41}$$

For T large the two-point correlation of the objects is enhanced compared with that of the mass and the enhancement increases with T. This seems to be what is needed

to explain the enhanced rich cluster correlations mentioned earlier. There is, however, one problem with this. Since the two-point correlation of the filtered mass distribution ξ_f has a Fourier transform that vanishes at $k = 0$, the two-point correlation must integrate to zero. That means that the filtered two-point correlation of the mass must cross zero somewhere (the zero crossing of $\xi_>$ coincides with that of ξ_f, see eq. (40)). The location of the zero crossing depends (with cold dark matter) on the time of matter domination and the filtering scale. For rich clusters it is at $17h^{-2}$ Mpc. Using a more realistic model where the objects form at peaks (local maxima) of the mass filtered mass density fluctuations[15] only makes matters worse. The peak condition introduces an anticorrelation (peaks do not occur right next to each other) which reduces the size of the enhancement and moves the zero crossing in[16]. If the mass density fluctuations are approximately Gaussian it does not seem likely that this model can accommodate significant (positive) rich-cluster two-point correlations at a distance of $30h^{-1}$ Mpc.

Even if the primordial fluctuations are Gaussian there are connected correlations of δ induced by the non-linear time evolution. It is easy to characterize how the connected correlations can effect the behavior of $\xi_>(r)$ at large r[17]. An n-point correlation of δ depends on the locations of n-points $\vec{x}_1, \ldots, \vec{x}_n$. Suppose m of those points are kept close together but separated a large distance r from the remaining $n - m$ points (which are also close to each other). If a connected correlation falls like r^{-p} in this limit then from eq. (39) it is evident that typically this will give rise to a two-point correlation $\xi_>(r)$ that also falls as r^{-p} for large r. This condition can also be given in wavenumber space. The Fourier transform of a connected n-point correlation of the mass density is a function of n wavevectors $\vec{k}_1, \ldots, \vec{k}_n$. If it diverges like k_T^{-p} as a partial sum $\vec{k}_T$ of those wavevectors (e.g., $\vec{k}_T = \vec{k}_i$, $\vec{k}_T = \vec{k}_i + \vec{k}_j$ with $i \neq j$, etc.,) goes to zero then at small k the power spectrum $P_>(k)$ typically also diverges like k^{-p}. Fig. (6a) shows a contribution to the connected four-point correlation of δ that behaves like k_T^0 as the partial sum $\vec{k}_T = \vec{k}_1 + \vec{k}_2$ goes to zero (but the individual wavevectors k_1 and k_2 do not). This gives a contribution to $P_>(k)$ that behaves like k^o as k gets small. Clearly there are an infinite number of diagrams that behave in this fashion. Fig. (6b) shows a tree graph that contributes to the six-point correlation of δ. Since each of the vertices diverges linearly as $\vec{k}_T/k_T^2$, where $\vec{k}_T = \vec{k}_1 + \vec{k}_2 + \vec{k}_3$, this diagram seems like it would give a contribution to $P_>(k)$ that diverges like k^{-1} as k goes to zero. This is not the case, however. It is easy to see that rotational invariance causes the most divergent piece to cancel when the integrals over $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ are done. This behavior persists to all orders in the perturbative expansion. In general $P_>(k)$ goes to a constant as k goes to zero. Unless the threshold T is so large that linear perturbation theory is valid on the comoving scale that collapsed to the objects, the behavior of $P_>(k)$, at small k, is not computable since an infinite number of diagrams (like Fig.(6a)) contribute significantly. These diagrams are sensitive to physics on short distance scales (i.e., distances of order the comoving scale that collapsed to form the objects). Clearly numerical simulations of the non-linear evolution must correctly handle physics at these short distances if they are to draw believable conclusions about the large scale distribution of objects that arise wherever the mass density fluctuations are unusually large. Similar conclusions hold for any biasing (of the type in eq. (33)) in the definition of the objects.

It is possible to observe not only the positions of objects but also their peculiar velocities[18]. Imagine that the peculiar velocities of some type of object are averaged over a large ball of radius R (with fuzzy edges). The resulting average velocity is $\vec{v}(R)$ where,

$$\vec{v}(R) \simeq \frac{1}{\bar{n}} \int d\vec{r}\, W_R(\vec{r}) n(\vec{r}) \vec{v}(\vec{r}). \tag{42}$$

In eq. (42) n is the number density of the objects observed, $\bar{n}$ its average value and W_R is a smooth function that integrates to unity and becomes small outside the ball. Note that if n is set to unity in eq. (42) then the results of Section 4 imply that, for large R, $\vec{v}(R)$ becomes a Gaussian random variable with a variance determined by linear perturbation theory.

In general the mean square value of $v(R)$ is given by

$$\langle \vec{v}(R)^2 \rangle = \int d\vec{r}\, W_R(\vec{r}) \int d\vec{r}\,'\, W_R(\vec{r}\,') \cdot \left(\frac{1}{\bar{n}^2}\right) \langle n(\vec{r})\vec{v}(\vec{r}) n(\vec{r}\,')\vec{v}(\vec{r}\,') \rangle. \tag{43}$$

Let

$$n(\vec{r}) = \bar{n}(1 + \lambda(\vec{r})), \tag{44}$$

then the relevant correlation in the expression for $\vec{v}(R)$ breaks into four pieces

$$(1/\bar{n}^2)\langle n(\vec{r})\vec{v}(\vec{r}) n(\vec{r}\,')\vec{v}(\vec{r}\,') \rangle = \langle \vec{v}(\vec{r}) v(\vec{r}\,') \rangle + \langle \lambda(\vec{r}) \vec{v}(\vec{r}) \vec{v}(\vec{r}\,') \rangle$$

$$+ \langle \vec{v}(\vec{r}) \lambda(\vec{r}\,') \vec{v}(\vec{r}\,') \rangle + \langle \lambda(\vec{r}) \vec{v}(\vec{r}) \lambda(\vec{r}\,') \vec{v}(\vec{r}\,') \rangle. \tag{45}$$

I have already argued that (if $\vec{v}$ is the peculiar velocity of the mass) the first of the four terms in eq. (45) is dominated by the primordial fluctuations (after the integrations over $\vec{r}$ and $\vec{r}\,'$, are performed). It is easy to see that the other two terms are negligible if λ is uncorrelated with $\vec{v}$ or if the only connected correlations that λ has with $\vec{v}$ is a two-point correlation.

If the objects being observed trace the mass

$$\lambda(\vec{r}) \propto \delta(\vec{r}). \tag{46}$$

It appears naively that in this case the non-linear time evolution will cause the last three terms in eq. (45) to contribute as significantly as the first (even after integrating $\vec{r}$ and $\vec{r}\,'$, over the large balls). Writing

$$\langle \delta(\vec{r})\vec{v}(\vec{r}) \cdot \vec{v}(\vec{r}\,') \rangle = \int \frac{d\vec{q}}{(2\pi)^3} \int \frac{d\vec{k}}{(2\pi)^3} \int \frac{d\vec{k}\,'}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} e^{i\vec{k}\,'\cdot\vec{r}\,'} \cdot \langle \delta(\vec{q})\vec{v}(\vec{k}-\vec{q})\vec{v}(\vec{k}\,') \rangle, \tag{47}$$

the diagrams which diverge $1/k$ as k goes to zero and contribute to $\langle \delta(\vec{q})\vec{v}(\vec{k}-\vec{q})\vec{v}(\vec{k}\,') \rangle$ in the lowest non-trivial order of gravitational perturbation theory are shown in Fig. (7). (Note homogeneity implies $\vec{k}\,' = -\vec{k}$.) Each of these graphs diverges like $1/k$ as k goes to zero, however, when they are summed the most divergent piece cancels. This cancellation is a consequence of the equation of continuity.

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\vec{\nabla}(1+\delta)\vec{v} = 0. \tag{48}$$

It is straightforward to show that at small k, $[(1+\delta)\vec{v}](k)$ is curl free (i.e. $(1+\delta)\vec{v} \times \vec{k} = 0$). This implies that

$$[(1+\delta)v](\vec{k}) = \frac{ia\vec{k}}{k^2} \frac{\partial \delta(\vec{k})}{\partial t}. \tag{49}$$

It has already been shown that at small k, δ is dominated by linear perturbation theory and eq. (49) implies that the same is true of $(1+\delta)\vec{v}$.

Although for objects which trace the mass, $\vec{v}(\vec{R})$ is dominated by linear perturbation theory that is not true is general. As a simple example imagine a biased model

in which

$$\lambda(\vec{r}) = \delta_f(\vec{r}) + b\delta_f^2(\vec{r}) - b\langle \delta_f^2(\vec{r}) \rangle \tag{50}$$

but with the peculiar velocities of the objects the same as that of the mass density fluid at the location of the objects. Now the second term in eq. (45) will be dominated by

$$b\langle \delta_f^2(\vec{r})\vec{v}(\vec{r})\vec{v}(\vec{r}\,') \rangle_c = b \int \frac{d\vec{q}}{(2\pi)^3} \int \frac{d\vec{p}}{(2\pi)^3} \int \frac{d\vec{k}\,'}{(2\pi)^3} \int \frac{d\vec{k}}{(2\pi)^3} \tilde{W}(\vec{p})\tilde{W}(\vec{q})$$

$$\cdot e^{i\vec{k}\cdot\vec{r}} e^{i\vec{k}\,'\cdot\vec{r}\,'} \langle \tilde{\delta}(\vec{p})\delta(\vec{q})\vec{v}(\vec{k}-\vec{p}-\vec{q})\vec{v}(\vec{k}\,') \rangle_c. \tag{51}$$

In the lowest non-trivial order of perturbation theory the graphs which give a contribution to (51) that goes like $1/k$ at small k are shown in Fig. (8). Explicit computation of these graphs reveals that the $1/k$ piece does not cancel. In addition they are sensitive to physics a short distances since large wavevectors $\vec{q}$ and $\vec{p}$ are important. At higher orders in gravitational perturbation theory there are also contributions of order $1/k$ (e.g. Fig. (9)). Not every diagram has a $1/k$ piece however. The diagram in Fig. (10), for example, goes like k^4 as k goes to zero.

Clearly $\vec{v}(R)$ is sensitive to the definition of the objects and to non-linear dynamics even for arbitrarily large R: different objects yield different $\vec{v}(R)$. The relative orientations of $\vec{v}(\vec{R})$'s corresponding to different objects is studied by examining the two-point correlation.

$$\langle \vec{v}_1(R) \cdot \vec{v}_2(R) \rangle = \frac{1}{\bar{n}_1\bar{n}_2} \int d\vec{r} \int d\vec{r}\,' \, W_R(\vec{r})W_R(\vec{r}\,') \langle n_1(\vec{r})\vec{v}(\vec{r}) \cdot n_2(\vec{r}\,')\vec{v}(\vec{r}\,') \rangle, \tag{52}$$

where the two types of objects have number densities n_1 and n_2 but are assumed to have the same peculiar velocity as the underlying mass density fluid.

The diagrams which dominate the large R behavior of this quantity have the form shown in Fig. (11). The shaded region denotes an arbitrary number of dotted lines. The dotted line joining the two shaded circles carries the small wavevector k. This implies that for large R

$$\langle \vec{v}_1(R) \cdot \vec{v}_2(R) \rangle = \frac{C_1 C_2}{R^2} \tag{53}$$

with C_1 and C_2 constants independent of R that depend on the definitions of objects one and two respectively. Therefore

$$\langle (C_2 \vec{v}_1(R) - C_1 \vec{v}_2(R))^2 \rangle = 0 \tag{54}$$

which means that

$$C_2 \vec{v}_1(R) = C_1 \vec{v}_2(R). \tag{55}$$

All the large scale streaming velocities $\vec{v}(R)$ are proportional to that of the mass. The constant of proportionality, however, gets contributions from non-linear gravitational dynamics. For objects that arise where mass density fluctuations are very large one expects the discrepancy between the large scale streaming velocity of the objects and that of the mass to be small if linear perturbation theory is valid on the comoving scale that collapsed to form the objects.

6. Primordial Non-Gaussian Fluctuations

Previously, in a few sections, I have explored the consequences of natural primordial Gaussian fluctuations. In this section the possibility of primordial non-Gaussian fluctuations will be examined. It will be shown that natural primordial non-Gaussian fluctuations can give rise to very different expectations for the large scale distribution of the observed luminous objects (e.g., galaxies or rich clusters of galaxies).

The strongest motivation for Gaussian primordial mass density fluctuations is the small value of the fluctuations (when averaged over the horizon volume). In the new inflationary cosmology this restricts the field that is doing the inflating to be very weakly coupled so that fluctuations in that field (which are equivalent to energy density fluctuations) are approximately Gaussian. However, there can be other contributions to the energy density fluctuations. Suppose that there is another field a that has a negligible mass in the inflationary era but in a later era develops a mass for dynamical reasons. At late times the field a contributes to the mass density but does not dominate it. Even if the field a contributes a small amount to the mass density quantum fluctuations in the field a, generated in the inflationary era, could dominate the fluctuations in the mass density if they are not small. Thus the field a is not restricted to be weakly coupled in the inflationary era and the fluctuations in the mass density that it gives rise to can be highly non-Gaussian.

For scale invariant fluctuations (as shown in eq. (8)),

$$\langle \epsilon(\lambda \vec{k}_1) \ldots \epsilon(\lambda \vec{k}_n) \rangle = \lambda^{-n} \langle \langle \epsilon(\vec{k}_1) \ldots \epsilon(\vec{k}_n) \rangle. \tag{8}$$

This is determined the two-point function, up to normalization (as shown in eq. (9)),

$$\langle \epsilon(\vec{k}_1) \epsilon(\vec{k}_2) \rangle \propto k_1 \delta^3(\vec{k}_1 + \vec{k}_2) \tag{9}$$

However the higher correlations are not uniquely determined. Writing

$$\langle \epsilon(\vec{k}_1) \ldots \epsilon(\vec{k}_n) \rangle_c = \xi_n(\vec{k}_1, \ldots \vec{k}_n)(2\pi)^3 \delta^3(\vec{k}_1 + \ldots + \vec{k}_n), \tag{56}$$

the function $\xi_n(\vec{k}_1, \ldots \vec{k}_n)$ is restricted by eq.(8) to be homogeneous of degree $3 - n$ (of course it must also be a symmetric rotationally invariant function of the wavevectors

$\vec{k}_1, \ldots, \vec{k}_n$). It is easy to generalize the diagrammatic rules introduced in section (4) to the case when the primordial fluctuations are non-Gaussian. One must add new vertices that take into account the fact that the ϵ's can now be sewn together with the correlations in eq. (56). The new vertices and the factors associated with them are shown in Fig. (12). All the dotted lines that are joined to these new vertices must originate from a solid (or wavy) line. The diagrammatic rules forbid there to be dotted lines joining the new vertices to each other. Unfortunately, it is difficult to make any general statements about the effect of non-linear gravitational evolution on primordial non-Gaussian fluctuations, without a particular model for these fluctuations.

Not every set of correlations (56) that are consistent with the principle of scale invariance (8) and the homogeneity and isotropy of space is permitted. One must be careful to make sure that the $\xi_n(\vec{k}_1, \ldots \vec{k}_n)$ are consistent with the positivity of the probability distribution for the primordial mass density fluctuations. For example, positivity of the probability distribution forbids any of the $\xi_n(\vec{k}_1, \ldots \vec{k}_n)$ to diverge like k_i^{-p}, $p > 1$, as a single wavevector $\vec{k}_i$ gets small. (Strictly speaking I assume this divergence does not vanish when integrated against a function of k_i, therefore my remarks don't apply to a divergence of the form $\vec{k}_i/k_i^{p+1}$.) To see this, note that positivity of the probability distribution for the primordial fluctuations means that

$$\langle f(\epsilon) g(\epsilon)\rangle^2 \leq \langle\{f(\epsilon)\}^2\rangle\langle\{g(\epsilon)\}^2\rangle \tag{57}$$

for any two functions f and g. Suppose now that a connected m-point function diverges as k_i^{-p} as one of its arguments gets small. Choose

$$f(\epsilon) = \int d\vec{k} e^{-k^2R^2}\epsilon(\vec{k}) \tag{58}$$

and

$$g(\epsilon) = \left\{\int d\vec{q} e^{-q^2r^2}\epsilon(\vec{q})\right\}^{m-1} \tag{59}$$

Putting this into eq. (57) it is easy to see that for R large (but r fixed) the divergence of the connected m-point function as one of its arguments gets small, causes the left-hand side to scale as R^{2p-6}. The R behavior of the right-hand side however, is just determined by the primordial two-point function. It scales like R^{-4} for large R. Therefore the inequality in eq. (57) will not be satisfied for very large R if $p > 1$.

Primordial non-Gaussian could have a significant effect on the power spectrum for the two-point correlation of biased objects[17,19]. Recall that in section (5) it was shown that if one of the correlations $\xi_n(\vec{k}_1, \ldots \vec{k}_n)$ diverges as a partial sum of its wavevectors $\vec{k}_T = \vec{k}_1 + \ldots + \vec{k}_i, i < n$, gets small like k_T^{-p}, then the power spectrum for the objects (typically) diverges like k^{-p} for small k. The case $p = 1$ is very interesting since in coordinate space this corresponds to a two-point function that falls like r^{-2} for large r. This could explain the significant correlations of rich clusters of galaxies at large distances. In ref. (17) it was shown that it is possible to construct a natural positive probability distribution that gives rise to this behavior. This model was quite awkward and it required a fine tuning of counter terms to preserve the naturalness of the probability distribution (in the sense of eq. (8)). It is not clear how general this difficulty is. It is worth noting however, that models for the probability distribution of the primordial fluctuations that do not require a fine tuning of counter terms, can easily be constructed to give a primordial power spectrum for objects that goes to a constant at small wavenumbers. This may be enough to make the zero crossing of the two-point correlation for rich clusters of Galaxies occur at distances significantly larger than $17h^{-2}$ Mpc in a model where they occur at peaks of the filtered mass density field that are unusually high.

Appendix[11]

The Zeldovich approximation[10] takes the Lagrangian coordinates of the mass to be

$$\vec{r}(\vec{s},t) = a(t)\vec{s} + b(t)\vec{\alpha}(\vec{s}), \tag{A.1}$$

where the mass is evenly distributed in $\vec{s}$-space. The comoving coordinates are obtained by dividing by the Robertson Walker scale factor $a(t)$. In eq (A.1) $b(t) = t^{2/3}a(t)$. Writing the peculiar velocity as

$$\vec{v} = a(t)\frac{d}{dt}(\vec{r}/a(t)) = \frac{2}{3}\left(\frac{a(t)}{t}\right)\vec{p}. \tag{A.2}$$

Eq.(A.1) implies that in the Zeldovich approximation $\vec{p} = \vec{\alpha}(\vec{s})$. To show that eqs. (20) and (21) are the Eulerian equivalent of the Zeldovich approximation (A.1), it is convenient to introduce a phase space distribution; $f(\vec{x},\vec{p}:t)d\vec{x}d\vec{p}$ is the number of particles (which I assume to have mass m) in the six-dimensional phase space volume $d\vec{x}d\vec{p}$ at the time t. In the Zeldovich approximation

$$f(\vec{x},\vec{p}:t) = \frac{\langle\rho\rangle}{m}\delta^3(\vec{p}-\vec{\alpha}(\vec{s})) \tag{A.3}$$

with

$$\vec{s} = \vec{x} - (b(t)/a(t))\,\vec{p}. \tag{A.4}$$

Before there is orbit crossing one can expand the Dirac δ function yielding

$$f(\vec{x},\vec{p}:t) = \frac{\langle\rho\rangle}{m}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\alpha_{i_1}\ldots\alpha_{i_n}\frac{\partial}{\partial p_{i_1}}\cdots\frac{\partial}{\partial p_{i_n}}\delta^3(\vec{p}). \tag{A.5}$$

Putting this into the equation for the mass density

$$\rho(\vec{x},t) = m \int f(\vec{x},\vec{p}:t) d\vec{p}, \tag{A.6}$$

integrating by parts to remove the derivatives from the delta functions and using eq. (A.4) gives

$$\frac{\rho(\vec{x},t)}{\langle\rho\rangle} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{b(t)}{a(t)}\right)^n \frac{\partial}{\partial x_{i_1}}, \dots \frac{\partial}{\partial x_{i_n}} (\alpha_{i_1}(\vec{x}) \dots \alpha_{i_n}(\vec{x})). \tag{A.7}$$

The $n = 1$ term in the sum corresponds to linear perturbation theory and it implies that $\epsilon_1(\vec{x}) = -\vec{\nabla} \cdot \vec{\alpha}$. Eqs. (20) and (21) follow (when $\vec{\alpha}$ is curl free) from Fourier transforming eq. (A.7).

For a one dimensional perturbation

$$\begin{aligned} x &= a(t)s_1 + b(t)\alpha_1(s_1) \\ y &= a(t)s_2 \\ z &= a(t)s_3 \end{aligned} \tag{A.8}$$

where $\vec{r} = (x, y, z)$. In this case the Zeldovich approximation in Eulerian coordinates is given by eqs. (20) and (21) where the wavevectors are replaced by scalars and vector dot products by ordinary multiplication, (one only Fourier transforms the coordinate x since the density perturbation is independent of the y and z coordinates). The Zeldovich approximation is actually the exact solution in the case of one-dimensional perturbations. The equations of motion that the exact solution must satisfy are

$$\nabla^2\phi = -4\pi G\rho, \quad \frac{\partial^2 \vec{r}}{\partial t^2} = \vec{\nabla}\phi \tag{A.9}$$

where the gradient is derivatives with respect to x, y, and z. Using the fact that

$\ddot{b}/b = -2\ddot{a}/a$ and eq. (A.8) it is straightforward to show that

$$\vec{\nabla} \cdot \frac{\partial^2 \vec{r}}{\partial t^2} = \left(\frac{3\ddot{a}}{a}\right) \frac{1}{1 + (b/a)(d\alpha_1/ds_1)}. \tag{A.10}$$

Since the particles are evenly distributed in $\vec{s}$-space the mass density is proportional to the inverse of the Jacobian for the transformation from x, y, z coordinates to s_1, s_2, s_3 coordinates. Therefore

$$\rho = \langle\rho\rangle \frac{1}{1 + (b/a)(d\alpha_1/ds_1)}. \tag{A.11}$$

Since $4\pi G\langle\rho\rangle = -3\ddot{a}/a$ eq. (A.10) implies that the Zeldovich approximation solves the equations of motion (A.9).

References

1. The suggestion that primordial non-Gaussian fluctuations might have an interesting effect on the large scale distribution of objects was originally given in: Peebles, P.J.E., *Ap.J.* **274**, 1 (1983).
2. Guth, A. *Phys. Rev.* **D23**, 347, (1981).
3. For an exposition on a gauge invariant formulation for density perturbations see: Bardeen, J.M., *Phys. Rev.* **D22**, 1882, (1980).
4. For a review of functional integration and a discussion of its application to probability theory see: Feynman, R.P. and Hibbs, A.R., *Quantum Mechanics and Path Integrals*, McGraw Hill, New York (1965).
5. Otto, S., Preskill, J., Politzer, H.D. and Wise, M.B., *Ap.J.* **304**, 62, (1986).
6. Harrison, E., *Phys. Rev.*, **D1**, 2726 (1970); Peebles, P.J.E. and Yu J., *Ap. J.* **162**, 815 (1970); Zeldovich, Ya.B., *M.N.R.A.S.*, **160**, 1P, (1972).
7. Linde, A.D. *Phys. Lett.*, **108B**, 289 (1982); Albrecht, A. and Steinhardt, P.J., *Phys. Rev. Lett.* **48**, 1220 (1982).
8. Guth, A. and Pi, S.Y.,*Phys. Rev. Lett.* **49**, 1110 (1982); Bardeen, J.M., Steinhardt, P.J. and Turner, M., *Phys. Rev.*, **D28**, 679 (1983); Starobinaskii, A., *Phys. Lett.*, **117B**, 175 (1982); Hawking, S., Phys. Lett., **115B**, 295, (1982).
9. The formalism discussed in this section was developed in; Goroff, M., Grinstein, B., Rey, S-Y. and Wise, M.B., *Ap. J.* **311**, 6, (1986). See also Fry, J., *Ap. J.* **279**, 499 (1984).
10. Zeldovich, Ya. B. *Astron and Astrophys*, **5**, 84 (1970).
11. Grinstein, B., Kamien, R., and Wise, M.B. (1986), unpublished.
12. Hauser, M. and Peebles, P.J.E., *Ap. J.*, **185**, 757 (1973); Kopolov, A. and Klypin, A., *Soviet. Astr. Lett.*, **9**, 41 (1983); Bachall, N. and Soniera, R., *Ap. J.*, **276**, 20 (1983).
13. Kaiser, N., *Ap. J. Lett.*, **284**, L9 (1984).
14. Politzer, H.D. and Wise, M.B., *Ap. J. Lett.*, **285**, L1, (1984).
15. Bardeen, J.M., Bond, J.R., Kaiser, N. and Szalay, A.S., *Ap. J.*, **304**, 15 (1986).
16. Otto, S., Politzer, H.D., Wise, M.B., *Phys. Rev. Lett.*, **56**, 1878 (1986).
17. Grinstein, B. and Wise, M.B., *Ap. J.*, **310**, 19 (1986).
18. This discussion follows from the paper: Grinstein, B., Politzer, H.D., Rey, S-J. and Wise, M.B., to appear in *Ap.J.*, **314** (1987).
19. Matarrese, S., Lucchin, F., Bonometto, S.A., *Ap.J. Lett.*, **310**, L21 (1986).

20. Zeldovich, Ya. B., 'Fluid Dynamics Transactions'; Volume **8**, *Polish Scientific Publishers*, Warsaw Poland (1976).

Figure Captions

Fig.(1) : Diagrammatic view of the computation in eq. (25).

Fig.(2) : Factors associated with vertices and dotted lines.

Fig.(3) : Some diagrams contributing to the two-point correlation of δ.

Fig.(4) : Some diagrams contributing to the three-point correlation of δ.

Fig.(5) : Vertices needed for correlations of $\vec{v}$.

Fig.(6) : Two diagrams contributing to correlations of δ.

Fig.(7) : Contributions to $\langle\delta(\vec{q})\vec{v}(\vec{k}-\vec{q})v(\vec{k}')\rangle$ that appear to diverge like $1/k$.

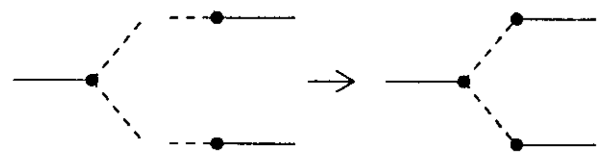
Fig.(8) : Contributions to $\langle\delta(\vec{p})\delta(\vec{q})\vec{v}(\vec{k}-\vec{p}-\vec{q})\vec{v}(\vec{k}')\rangle$ that diverge like $1/k$.

Fig.(9) : Higher order contribution to $\langle\delta(\vec{p})\delta(\vec{q})\vec{v}(\vec{k}-\vec{p}-\vec{q})\vec{v}(\vec{k}')\rangle$ that diverges like k^{-1} (for small k).

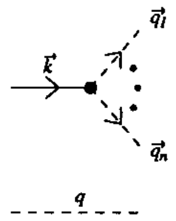
Fig.(10) : Contribution to $\langle\delta(\vec{p})\delta(\vec{q})\vec{v}(\vec{k}-\vec{p}-\vec{q})\vec{v}(\vec{k}')\rangle$ that doesn't diverge as k goes to zero.

Fig.(11) : Generic diagram contributing to the two-point correlation of $(n\vec{v})(\vec{k})$ that diverges like $1/k$ for small k.

Fig.(12) : New vertices for primordial non-Gaussian fluctuations.



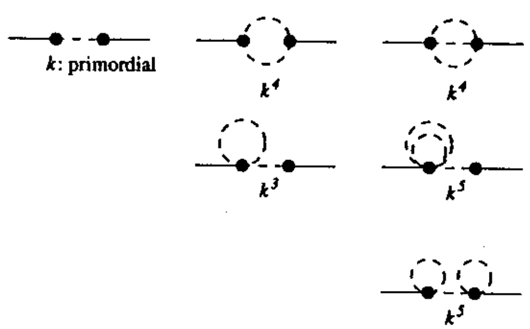
(1)



$$\frac{1}{(2\pi)^{3n}}(2\pi)^3\delta^3(\vec{q}_1+\cdots+\vec{q}_n-\vec{k})P_n^{(s)}(\vec{q}_1,\cdots,\vec{q}_n)$$

$$(2\pi)^3 A(q)q$$

(2)



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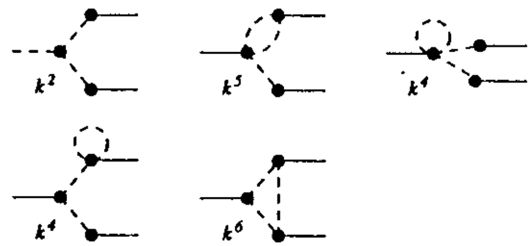
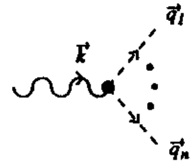
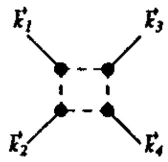


(4)

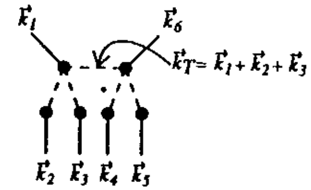


$$\frac{i\vec{k}}{k^2}\frac{a(t)}{t}\frac{(2\pi)^3}{(2\pi)^{3n}}\delta^3(\vec{q}_1+\cdots+\vec{q}_n-\vec{k})Q_n^{(s)}(\vec{q}_1,\cdots,\vec{q}_n)$$

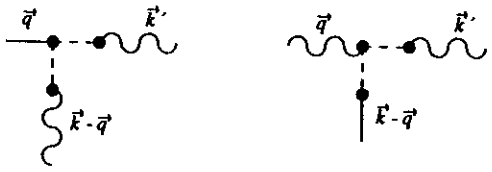
(5)



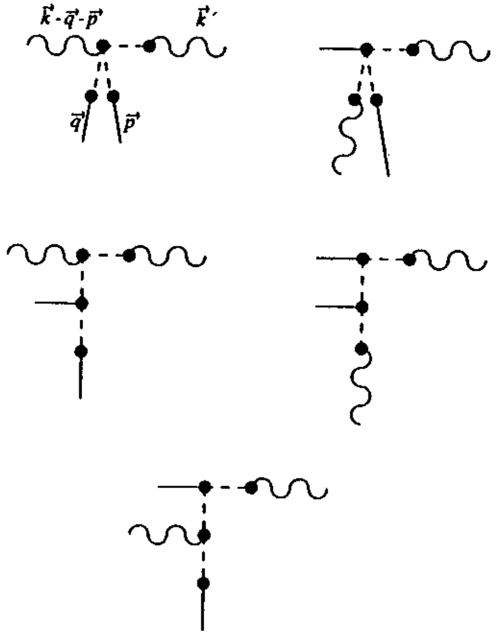
(6a)



(6b)



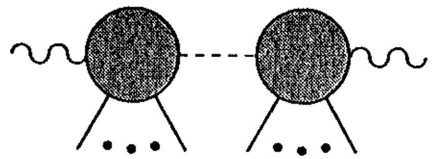
(7)



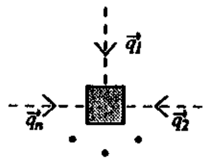
(8)

(9)

(10)



(11)



$$\frac{(2\pi)^{3(1-n)}\delta^3(\vec{q}_1+\cdots+\vec{q}_n)}{n!\,q_1\sqrt{A(q_1)}\cdots q_n\sqrt{A(q_n)}}\xi_n(\vec{q}_1,\cdots\vec{q}_n)$$

(12)