Supplementary Material

Hydrodynamic stabilization of self-organized criticality in a driven Rydberg gas

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Here we provide additional details on the following subjects: *(i)* the operator splitting scheme implemented for numerical simulation of the Langevin equations, *(ii)* the derivation of the coarse-grained equations of motion, and *(iii)* the dynamics in the absence of atomic motion.

NUMERICAL INTEGRATION SCHEME

We simulate the Langevin equations on a discrete spatio-temporal lattice by means of an operator-splitting update scheme. Consider starting from a general stochastic differential equation (SDE) over the density field $\rho_{\vec{x},t}$ like

$$\partial_t \rho_{\vec{x},t} = D\nabla^2 \rho_{\vec{x},t} + a + b\rho_{\vec{x},t} + c\rho^2_{\vec{x},t} + \sigma^2 \sqrt{\rho_{\vec{x},t} + d}\eta, \tag{1}$$

where the Markovian noise kernel η has unit variance and zero mean. On a lattice the Laplacian is discretized so that it gets absorbed into the coefficients a and b. Then under appropriate change of variables $\rho \to \rho' \equiv \rho + d$ we may eliminate the constant offset in the noise term. The temporal update is then decomposed into two steps: a stochastic step and a deterministic step. For the former we drop the quadratic term, yielding an SDE of the form

$$\partial_t \rho_{\vec{x},t} = \alpha + \beta\rho_{\vec{x},t} + \sigma\sqrt{\rho}\eta. \tag{2}$$

This class of SDEs with a multiplicative noise kernel admits an exact solution to the corresponding Fokker-Planck equation [1]. Denoting the current value as ρ_0, the distribution of values ρ after a time step δt is given by

$$P(\rho) = \lambda e^{-\lambda(\rho_0 e^{\beta\delta t} + \rho)} \left(\frac{\rho}{\rho_0 e^{\beta\delta t}}\right)^{\mu/2} I_\mu\left(2\lambda\sqrt{\rho_0 \rho e^{\beta\delta t}}\right). \tag{3}$$

Here we have denoted $\lambda = \frac{2\beta}{\sigma^2(e^{\beta\delta t}-1)}$ and $\mu = \frac{2\alpha}{\sigma^2} - 1$. This distribution may be efficiently sampled by rewriting it as a mixed Gamma distribution

$$\rho = \Gamma[\mu + 1 + \text{Poisson}[\lambda\rho_0 e^{\beta\delta t}]]/\lambda. \tag{4}$$

The stochastic evolution of the state at time t reduces to randomly sampling from the above distribution at each time step. For $d \neq 0$ in Eq. (1) this requires that we enforce non-negativity of ρ in terms of the transformed variables by taking all $\rho' < d$ to d. The remaining deterministic

quadratic term may be treated by a standard Euler scheme. Similarly, we treat the evolution of the total density $n_{\vec{x},t}$ by an Euler scheme.

EQUATIONS OF MOTION

The equations of motion for the density $n_{\vec{x},t}, \rho_{\vec{x},t}$ follow from acting the derivative on their definition and applying the chain rule

$$\begin{aligned}\partial_t n_{\vec{x},t} &= \partial_t \sum_l \langle \sigma_l^{rr} + \sigma_l^{gg} \rangle \theta(r_{\text{fac}}^2 - |\vec{r}_l - \vec{x}|^2) \\ &= \sum_l \theta(r_{\text{fac}}^2 - |\vec{r}_l - \vec{x}|^2) \langle \partial_t \sigma_l^{rr} + \partial_t \sigma_l^{gg} \rangle + \sum_l \langle \sigma_l^{rr} + \sigma_l^{gg} \rangle \nabla_{\vec{r}_l} \theta(r_{\text{fac}}^2 - |\vec{r}_l - \vec{x}|^2) \partial_t \vec{r}_l .\end{aligned} \tag{5}$$

The first part on the RHS of Eq. (5) describes the evolution of the internal degrees of freedom of the particles in the unit cell. It has been discussed in detail in Ref. [2] and it captures the loss of particles into the dark state. The second part considers the evolution of the motional degrees of freedom. Evaluating the equations of motion $\partial_t \langle \hat{O} \rangle = \text{Tr}(\hat{O} \partial_t \rho)$ with the master equation from the main text yields

$$\partial_t n_{\vec{x},t} = -\gamma_{\downarrow 0} \rho_{\vec{x},t} - \nabla_{\vec{x}} \underbrace{\sum_l \langle \sigma_l^{rr} + \sigma_l^{gg} \rangle \theta(r_{\text{fac}}^2 - |\vec{r}_l - \vec{x}|^2) \vec{v}_l}_{\vec{j}_{\vec{x}}} . \tag{6}$$

Here, we have replaced $\vec{v}_l = \partial_t \vec{r}_l$ and we have exploited $\nabla_{\vec{r}_l} f(\vec{r}_l - \vec{x}) = -\nabla_{\vec{x}} f(\vec{r}_l - \vec{x})$ in order to pull the gradient out of the sum. The average over all the particles' velocities is the coarse grained current $\vec{j}$ in the unit cell. This yields the equation of motion for $n_{\vec{x},t}$ in the main text. The current can be evaluated using the classical equations of motion in the relaxation time approximation. The Hamilton equation of motion for the velocity of thermal particles in the presence of a potential V reads as

$$M \partial_t \vec{v}_l = -\nabla_{\vec{r}_l} V(\vec{r}_l) - \frac{1}{\eta} \vec{v}_l + \vec{\xi}_{t,\vec{r}_l} . \tag{7}$$

Here η is the mobility and $\vec{\xi}_{t,\vec{r}_l}$ describes random forces (i.e. collisions with other atoms, photon absorption, etc.) which may change the momentum. The product $M\eta$ defines the typical timescale at which the velocity departs from a characteristic path set by the potential, which may either happen due to collisions or in a (coarse grained) description, due to a random and/or anharmonic potential. While the latter is of particular interest for a trapped gas of non-interacting particles, for which the atomic motion becomes chaotic [3, 4], the microscopic origin of η is not relevant for the following steps. We emphasize, however, that this is an effective description and in the

collisionless regime, the density and velocity distribution of the atom cloud could be computed exactly from the Boltzmann equation without a collision integral.

On timescales of the self-organization, we can consider the gas to be in the overdamped regime $\vec{v}_l = -\eta\nabla_{\vec{r}_l}V(\vec{r}_l) + \eta\vec{\xi}_{t,\vec{r}_l}$. One can insert this result into the definition of the current

$$\vec{j}_{\vec{x}} = \sum_l \langle\sigma_l^{rr} + \sigma_l^{gg}\rangle\theta(r_{\text{fac}}^2 - |\vec{r}_l - \vec{x}|^2)(-\eta\nabla_{\vec{r}_l}V(\vec{r}_l) + \eta\vec{\xi}_{t,\vec{r}_l}) = -\eta n_{\vec{x},t}\nabla_{\vec{x}}V(\vec{x}) - D_T\nabla n_{\vec{x},t}. \tag{8}$$

Here, we first approximated $V(\vec{r}_l)$ by $V(\vec{x})$ for all $\vec{r}_l$ in the unit cell in order to pull the average potential force out of the sum. This is justified as long as the potential varies on much larger scales than $r_{\text{fac}} \ll w$. Second, we made the intuitive approximation that the average over the random forces felt by an atom in the unit cell, which is a purely statistical effect trying to distribute particles equally in phase space, pushes the atom towards the region of the lowest instantaneous density. This direction is given by $-\nabla n_{\vec{x},t}$. This is the common relaxation time approximation, leading to Brownian diffusion of the density. The proportionality constant $D_T = \eta k_B T$ follows from the Einstein relation.

DYNAMICS WITHOUT ATOMIC MOTION

Particle motion will always be present in the experiment, but we can examine numerically what happens when motion is absent, which is shown in Fig. 1. As discussed in the main text, without particle motion there is no refilling mechanism to restore the central density and allow for subsequent avalanches. Therefore only the initial extensive avalanche may be triggered, which then pulls the density profile below the critical density n_c. After one single initial avalanche there appear no subsequent avalanches and no shrinking of the atomic cloud. The density profile remains subcritical everywhere and exhibits a central dip which never recovers.

Closer examination of the Rydberg density (Fig. 2) shows that after the initial avalanche, all non-zero Rydberg density may be attributed to rare off-resonant excitations which do not trigger avalanches. Rather, they are excited and then rapidly decay without spreading. On very long timescales set by the rate of off-resonant excitations τ, these rare excitations will slowly deplete the trap of all atomic density. This may be contrasted with the case shown in the main text where loss and atomic motion conspire to pin the central density to the critical density while the flanks of the cloud shrink.

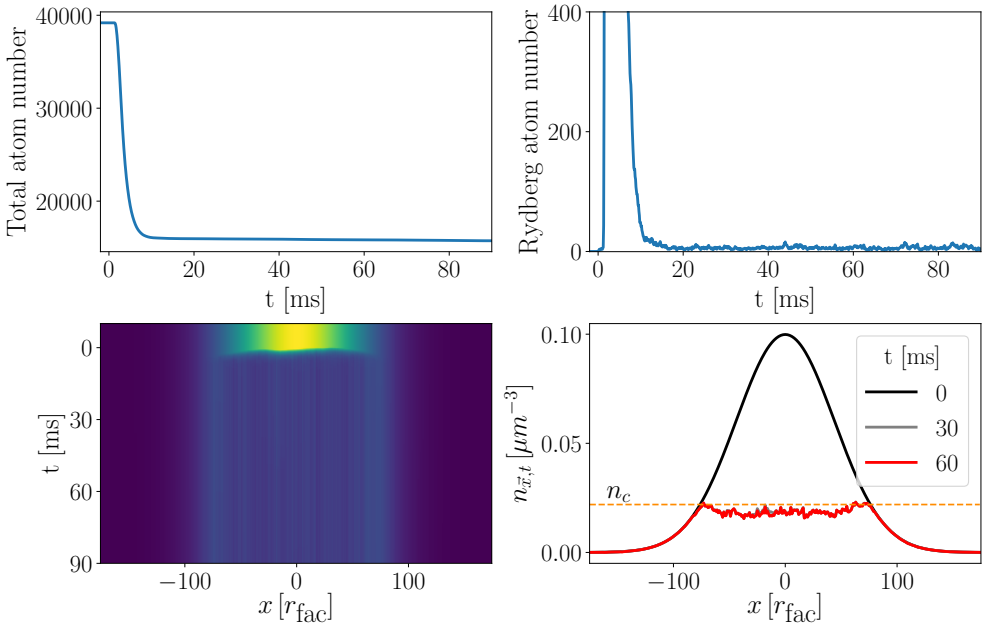

Figure 1. Numerical results for the same parameters taken in the main text but with no particle motion (i.e. $\eta = D_n = 0$). (Top left) The total particle number drops rapidly until the cloud dips below the critical density, after which loss occurs on very long timescales. (Top right) The total number of atoms in the excited Rydberg state remains vanishingly small after the initial avalanche. (Bottom) Evolution of the density profile in space and time shows an unchanging profile with subcritical density after the initial avalanche.

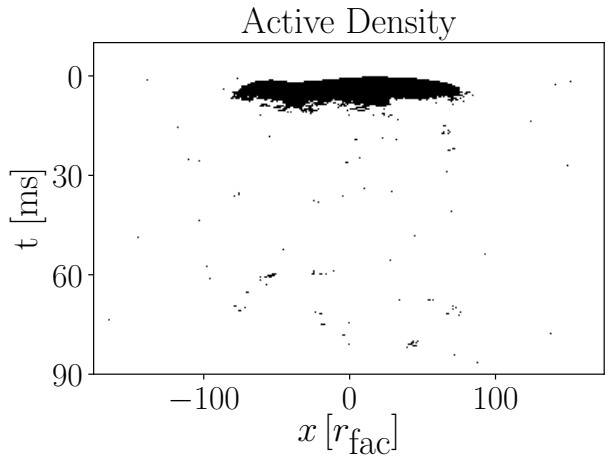

Figure 2. Evolution of the Rydberg density in space and time. The data here are plotted on a binary scale to highlight where and when the rare off-resonant excitations occur.

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