Supplemental Material

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In this Supplemental Material, we provide the detailed calculations for the results in the main text.

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SI. DIMER PROPAGATOR MATRIX

We consider finite total momentum Q for each two-body pairing state. As shown in Fig. S1, the inverse of the dimer propagator matrix is given by the Dyson equation

$$\begin{aligned} D^{-1}(E_0,Q) &= \begin{pmatrix} (ig_S)^{-1} & 0 \\ 0 & (ig_P)^{-1} \end{pmatrix} - \begin{pmatrix} \Pi_{SS}(E_0,Q) & \Pi_{SP}(E_0,Q) \\ \Pi_{PS}(E_0,Q) & \Pi_{PP}(E_0,Q) \end{pmatrix} \\ &= \begin{pmatrix} (ig_S)^{-1} - \Pi_{SS}(E_0,Q) & -\Pi_{SP}(E_0,Q) \\ -\Pi_{PS}(E_0,Q) & (ig_P)^{-1} - \Pi_{PP}(E_0,Q) \end{pmatrix}, \end{aligned} \tag{S1}$$

where the polarization bubble is given by (the derivations are given in the Appendix)

$$\Pi_{\alpha\beta}(E_0,Q) = -\int \frac{dpdp_0}{(2\pi)^2} \frac{p^{l_\alpha+l_\beta}}{2} \mathrm{Tr}\left[G^T(p_0,Q/2+p)\sigma_\alpha G(E_0-p_0,Q/2-p)\sigma_\beta^\dagger\right], \tag{S2}$$

$E_0 = Q^2/(4m) + k^2/m$ is the total energy, $\alpha,\beta \in \{S,P\}$, $l_S = 0$, $l_P = 1$, and Tr denotes the trace over the spin degrees of freedom.

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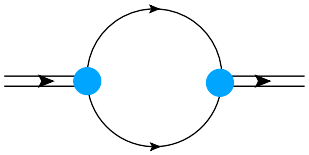

FIG. S1. Feynman diagrams for the matrix elements of the dimer-atom interaction operator. The single line denotes the bare atom propagator matrix G, the double lines denote the matrix elements of the dimer propagator martrix $D_{\alpha\beta}$ with $\alpha,\beta\in\{S,P\}$, and the blue dot represents the interaction vertex: $-i\sigma_\alpha$ or $-i\sigma_\beta$.

For convenience, one can also write the dimer propagator matrix as

$$D(E_0,Q)=\frac{1}{\det[D^{-1}(E_0,Q)]}\begin{pmatrix}(ig_P)^{-1}-\Pi_{PP}(E_0,Q) & \Pi_{SP}(E_0,Q)\\ \Pi_{PS}(E_0,Q) & (ig_S)^{-1}-\Pi_{SS}(E_0,Q)\end{pmatrix}=\begin{pmatrix}D_{SS}(E_0,Q) & D_{SP}(E_0,Q)\\ D_{PS}(E_0,Q) & D_{PP}(E_0,Q)\end{pmatrix}, \tag{S3}$$

where $\det[D^{-1}(E_0,Q)]$ is the determinant of $D^{-1}(E_0,Q)$.

In the absence of the Raman coupling, the s-wave polarization bubble is given by

$$\Pi_{SS}(E_0,Q)=-\int\frac{dpdp_0}{(2\pi)^2}\frac{i}{p_0-(Q/2+p)^2/(2m)+i0^+}\frac{i}{E_0-p_0-(Q/2-p)^2/(2m)+i0^+}=-\frac{m}{2k}, \tag{S4}$$

where we use $E_0=Q^2/(4m)+k^2/m$.

The 1D scattering amplitude can be written as [1, 2] $f_{1D}(k)=-1/(1+i\cot\delta_k)\simeq -1/(1+ika_s)$, where δ_k is the scattering phase shift. With the s-wave T matrix $T_s(k)=ikf_{1D}(k)/m_r$ and $D_{SS}(k)=iT_s(k)$, one can get

$$a_s=-\frac{2}{mg_S}, \tag{S5}$$

where a_s is the effective 1D s-wave scattering length.

In the absence of the Raman coupling, the p-wave polarization bubble is given by

$$\begin{aligned}\Pi_{PP}(E_0,Q)&=-\frac{1}{2}\int\frac{dpdp_0}{(2\pi)^2}\frac{ip^2}{p_0-(Q/2+p)^2/(2m)+i0^+}\frac{i}{E_0-p_0-(Q/2-p)^2/(2m)+i0^+}\\&=-\frac{1}{2}\int\frac{dp}{2\pi}\frac{ip^2}{E_0-Q^2/(4m)-p^2/m+i0^+}=\frac{im}{2}\int_{-\Lambda}^{\Lambda}\frac{dp}{2\pi}\frac{(p^2-k^2)+k^2}{[p-(k+i0^+)][p+(k+i0^+)]}\\&=\frac{im}{2}\left(\frac{\Lambda}{\pi}+i\frac{k}{2}\right),\end{aligned} \tag{S6}$$

where Λ is the ultraviolet momentum cutoff and we use $E_0=Q^2/(4m)+k^2/m$.

Therefore, the p-wave T matrix is given by

$$\frac{T_{pp}(E_0,Q)}{qq'}=\frac{D_{PP}(E_0,Q)}{i}=\frac{1}{\frac{1}{g_P}-i\Pi_{pp}(E_0,Q)}=\frac{1}{\frac{1}{g_P}+\frac{m\Lambda}{2\pi}+i\frac{mk}{4}}. \tag{S7}$$

Comparing Eq. (S7) with [3]

$$\frac{T_{pp}(E_0,Q)}{qq'}=\frac{1}{\frac{m}{4a_p}+i\frac{mk}{4}}, \tag{S8}$$

one can get

$$\frac{m}{4a_p}=\frac{1}{g_P}+\frac{m\Lambda}{2\pi}, \tag{S9}$$

where a_p is the effective 1D p-wave scattering length.

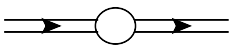

FIG. S2. Feynman diagrams for the matrix elements of the dimer local operator $\varphi_\alpha^\dagger(R)\varphi_\beta(R)$ and its derivatives $\varphi_\alpha^\dagger(R)\left[i\partial_t+\partial_R^2/(4m)\right]^u(-i\partial_R)^v\varphi_\beta(R)$ with $u,v=0,1,2,3,\cdots$. The open dot represents the operators.

In the presence of the Raman fields, the bubble with zero total momentum ($Q=0$) can be calculated as

$$\Pi_{SS}(E_0,0)=\frac{-m^3\Omega^2}{2(mE_0k_0^2-k_0^4+m^2\Omega^2)\sqrt{mE_0+i0^+-k_0^2}}-\frac{mk_0^2}{4}\left[\frac{\sqrt{mE_0+k_0^2+2\sqrt{mE_0k_0^2+m^2\Omega^2}}}{m^2\Omega^2+k_0^2(mE_0+\sqrt{mE_0k_0^2+m^2\Omega^2})}\right.$$
$$\left.+\frac{\sqrt{mE_0+i0^++k_0^2-2\sqrt{mE_0k_0^2+m^2\Omega^2}}}{m^2\Omega^2+k_0^2(mE_0-\sqrt{mE_0k_0^2+m^2\Omega^2})}\right]\approx\frac{-m}{2\sqrt{mE_0}},\tag{S10}$$

$$\Pi_{PP}(E_0,0)=\frac{im}{2}\left(\frac{\Lambda}{\pi}+i\frac{\sqrt{mE_0+i0^+-k_0^2}}{2}\right)-\frac{m^3\Omega^2}{16}\left(\frac{\sqrt{mE_0+k_0^2+2\sqrt{mE_0k_0^2+m^2\Omega^2}}}{mE_0k_0^2+m^2\Omega^2+k_0^2\sqrt{mE_0k_0^2+m^2\Omega^2}}\right.$$
$$\left.+\frac{\sqrt{mE_0+i0^++k_0^2-2\sqrt{mE_0k_0^2+m^2\Omega^2}}}{mE_0k_0^2+m^2\Omega^2-k_0^2\sqrt{mE_0k_0^2+m^2\Omega^2}}\right)\approx\frac{im}{2}\left(\frac{\Lambda}{\pi}+i\frac{\sqrt{mE_0+i0^+-k_0^2}}{2}\right),\tag{S11}$$

$$\Pi_{SP}(E_0,0)=\Pi_{PS}(E_0,0)$$
$$=\frac{m^2k_0\Omega\sqrt{mE_0+i0^+-k_0^2}}{4(mE_0k_0^2-k_0^4+m^2\Omega^2)}-\frac{m^2k_0\Omega}{8}\left(\frac{\sqrt{mE_0+k_0^2+2\sqrt{mE_0k_0^2+m^2\Omega^2}}}{mE_0k_0^2+m^2\Omega^2+k_0^2\sqrt{mE_0k_0^2+m^2\Omega^2}}\right.$$
$$\left.+\frac{\sqrt{mE_0+i0^++k_0^2-2\sqrt{mE_0k_0^2+m^2\Omega^2}}}{mE_0k_0^2+m^2\Omega^2-k_0^2\sqrt{mE_0k_0^2+m^2\Omega^2}}\right)\approx\frac{\sqrt{m}k_0\Omega}{8E_0^{-3/2}},\tag{S12}$$

where k_0 and Ω are treated perturbatively up to the k_0^2 and Ω order. Note that Eq. (S9) can be used to cancel the divergence of the p-wave bubble (S11), and the s- and p-wave mixing bubble (S12) needs both finite k_0 and Ω, as expected from having a non-trivial SOC.

SII. OTHER UNIVERSAL RELATIONS

A. Adiabatic relations

The traditional s- and p-wave adiabatic relations are given by [2, 3]

$$\frac{C_{SS}}{2m}\equiv\frac{\partial E}{\partial a_s}=-\int dR\left\langle\frac{\partial\mathcal{L}(R)}{\partial a_s}\right\rangle=\frac{m}{2}\int dR\left\langle\varphi_S^\dagger(R)\varphi_S(R)\right\rangle,\tag{S13}$$

$$\frac{C_{PP}}{4m}\equiv-\frac{\partial E}{\partial a_p^{-1}}=\int dR\left\langle\frac{\partial\mathcal{L}(R)}{\partial a_p^{-1}}\right\rangle=\frac{m}{4}\int dR\left\langle\varphi_P^\dagger(R)\varphi_P(R)\right\rangle,\tag{S14}$$

where E is the total energy of the many-body system, $\mathcal{L}(R)$ is the density of the Lagrangian (**??**), C_{SS} (C_{PP}) is the 1D $s(p)$-wave contact.

When SOC is present, there are two new parameters k_0 and Ω. One can define two new contacts C_λ and C_Ω as

$$C_\lambda\equiv-\int dR\left\langle\frac{\partial\mathcal{L}(R)}{\partial k_0}\right\rangle,\tag{S15}$$

$$C_\Omega\equiv-\int dR\left\langle\frac{\partial\mathcal{L}(R)}{\partial\Omega}\right\rangle.\tag{S16}$$

Here, C_λ and C_Ω refer to only single-atom operators which give nonzero matrix elements in the single-atom sector. The momentum distribution under single-particle states is just a delta function, so that C_λ and C_Ω will not contribute to the large-momentum tail, which is different from C_{SS} and C_{PP} [4]. However, both k_0 and Ω have nonzero energy scale, so that they would appear in the pressure relation and viral theorem.

B. Pressure relation

For a uniform gas, the pressure relation can be derived following the expression of the Helmholtz free energy density $\mathcal{F} = F/L$ which can be expressed in terms of [2, 5, 6]

$$\mathcal{F}(T, n_\uparrow, n_\downarrow, a_s, a_p, k_0, \Omega) = \frac{k_F^3}{2m} f\left(\frac{2mT}{k_F^2}, \frac{n_\uparrow}{k_F}, \frac{n_\downarrow}{k_F}, a_s k_F, a_p k_F, \frac{k_0}{k_F}, \frac{2m\Omega}{k_F^2}\right), \tag{S17}$$

where L is the length along the x direction, f is a dimensionless function, T is the temperature, $n = n_\uparrow + n_\downarrow = k_F/\pi$ is the Fermi particle number density, and k_F is the Fermi wave vector.

Equation (S17) implies the scaling behavior of the Helmholtz free energy density as follows:

$$\tilde{\lambda}^3 \mathcal{F}(T, n_\uparrow, n_\downarrow, a_s, a_p, k_0, \Omega) = \mathcal{F}\left(\tilde{\lambda}^2 T, \tilde{\lambda} n_\uparrow, \tilde{\lambda} n_\downarrow, \tilde{\lambda}^{-1} a_s, \tilde{\lambda}^{-1} a_p, \tilde{\lambda} k_0, \tilde{\lambda}^2 \Omega\right), \tag{S18}$$

where $\tilde{\lambda}$ is a dimensionless and arbitrary parameter.

Taking the derivative of Eq. (S18) with respect to $\tilde{\lambda}$ at $\tilde{\lambda} = 1$, we have

$$3\mathcal{F} = \left(2T\frac{\partial}{\partial T} + n_\uparrow \frac{\partial}{\partial n_\uparrow} + n_\downarrow \frac{\partial}{\partial n_\downarrow} - a_s \frac{\partial}{\partial a_s} - a_p \frac{\partial}{\partial a_p} + k_0 \frac{\partial}{\partial k_0} + 2\Omega \frac{\partial}{\partial \Omega}\right)\mathcal{F}. \tag{S19}$$

Replacing the free energy density $\mathcal{F}$ in the left side of Eq. (S19) by $n_\uparrow\mu_\uparrow + n_\downarrow\mu_\downarrow - \mathcal{P}$ and substituting $S = -\partial F/\partial T$ and $\mu_\sigma = \partial F/\partial n_\sigma$ into Eq. (S19), one gets

$$3(n_\uparrow\mu_\uparrow + n_\downarrow\mu_\downarrow - \mathcal{P}) = -2TS + n_\uparrow\mu_\uparrow + n_\downarrow\mu_\downarrow - a_s\frac{\partial \mathcal{F}}{\partial a_s} - a_p\frac{\partial \mathcal{F}}{\partial a_p} + k_0\frac{\partial \mathcal{F}}{\partial k_0} + 2\Omega\frac{\partial \mathcal{F}}{\partial \Omega}, \tag{S20}$$

where $\mathcal{P}$ is the pressure density, S is the entropy, and μ_σ is the chemical potentail with spin σ.

Using the adiabatic relations (S13), (S14), (S15) and (S16), we can get the pressure relation as

$$\mathcal{P} = 2\mathcal{E} + \frac{a_s C_{SS}}{2mL} + \frac{C_{PP}}{4ma_pL} - \frac{k_0 C_\lambda}{L} - \frac{2\Omega C_\Omega}{L}, \tag{S21}$$

where $\mathcal{E} = E/L$ is the energy density and we use $E = F + TS$ and

$$-a_p\frac{\partial E}{\partial a_p} = -a_p\frac{\partial E}{\partial a_p^{-1}}\frac{\partial a_p^{-1}}{\partial a_p} = -\frac{C_{PP}}{4ma_p}. \tag{S22}$$

C. Virial theorem

For an atomic gas in a harmonic potential $V_T = m\omega^2 x^2/2$ with the trapping frequency ω, the free energy can be expressed in terms of [2, 5, 6]

$$F(T, \omega, a_s, a_p, k_0, \Omega, N_\uparrow, N_\downarrow) = \omega \tilde{f}(T/\omega, \omega/\omega, a_s/a_{\rm ho}, a_p/a_{\rm ho}, k_0 a_{\rm ho}, \Omega/\omega, N_\uparrow, N_\downarrow), \tag{S23}$$

where $N = N_\uparrow + N_\downarrow$ is the particle number, $a_{\rm ho} = \sqrt{2/(m\omega)}$ is the harmonic oscillator length and the dimensionless function $\tilde{f}$ is dependent on the dimensionless ratios T/ω, $a_s/a_{\rm ho}$, $a_p/a_{\rm ho}$, $k_0 a_{\rm ho}$, Ω/ω, and particle numbers $N_\uparrow$ and $N_\downarrow$.

With Eq. (S23), we can get the scaling law

$$\tilde{\lambda} F(T, \omega, a_s, a_p, k_0, \Omega, N_\uparrow, N_\downarrow) = F(\tilde{\lambda}T, \tilde{\lambda}\omega, \tilde{\lambda}^{-1/2} a_s, \tilde{\lambda}^{-1/2} a_p, \tilde{\lambda}^{1/2} k_0, \tilde{\lambda}\Omega, N_\uparrow, N_\downarrow), \tag{S24}$$

where $\tilde{\lambda}$ is a dimensionless and arbitrary parameter.

The derivative of Eq. (S24) with respect to $\tilde{\lambda}$ at $\tilde{\lambda}=1$ gives

$$F=\left(T\frac{\partial}{\partial T}+\omega\frac{\partial}{\partial\omega}-\frac{1}{2}a_s\frac{\partial}{\partial a_s}-\frac{1}{2}a_p\frac{\partial}{\partial a_p}+\frac{1}{2}k_0\frac{\partial}{\partial k_0}+\Omega\frac{\partial}{\partial\Omega}\right)F. \tag{S25}$$

Substituting $E=F+TS$ and $S=-\partial F/\partial T$ into Eq. (S25), one gets

$$E=\left(\omega\frac{\partial}{\partial\omega}-\frac{1}{2}a_s\frac{\partial}{\partial a_s}-\frac{1}{2}a_p\frac{\partial}{\partial a_p}+\frac{1}{2}k_0\frac{\partial}{\partial k_0}+\Omega\frac{\partial}{\partial\Omega}\right)E, \tag{S26}$$

which, together with the Hellmann-Feynman theorem and the adiabatic relations (S13), (S14), (S15) and (S16), gives

$$E=2\langle V_T\rangle-\frac{a_sC_{SS}}{4m}-\frac{C_{PP}}{8ma_p}+\frac{k_0C_\lambda}{2}+\Omega C_\Omega. \tag{S27}$$

SIII. CONTACTS IN TWO-BODY BOUND STATES

The inverse of the two-body T matrix can be written as

$$T^{-1}(E_0,Q)\sim iD^{-1}(E_0,Q)=\begin{pmatrix} g_S^{-1}-i\Pi_{SS}(E_0,Q) & \delta_{SP}-i\Pi_{SP}(E_0,Q)\\ \delta_{PS}-i\Pi_{PS}(E_0,Q) & g_P^{-1}-i\Pi_{PP}(E_0,Q)\end{pmatrix}, \tag{S28}$$

where $\delta_{SP}=\delta_{PS}$ is a phenomenological parameter which is used to describe the bare coupling between s- and p-wave interactions on the two-body level. It corresponds to add a term $\sum_Q\left(\delta_{SP}\varphi^\dagger_{Q,S}\varphi_{Q,P}+\delta_{PS}\varphi^\dagger_{Q,P}\varphi_{Q,S}\right)$ in the Lagrangian. We introduce this additional coupling to derive the contact using the adiabatic relation.

The binding energy E_b can be calculated by the pole of the T-matrix as $\det(T^{-1}(E_b,0))=0$, where $E_b=-\kappa^2/m$ with momentum $k=i\kappa$ [3]. We consider the case where k_0 and Ω can be treated perturbatively. To the k_0^2 and Ω order, we find

$$\begin{pmatrix} g_S^{-1}-i\Pi_{SS}(E_b,0) & \delta_{SP}-i\Pi_{SP}(E_b,0)\\ \delta_{PS}-i\Pi_{PS}(E_b,0) & g_P^{-1}-i\Pi_{PP}(E_b,0)\end{pmatrix}=\begin{pmatrix} -\frac{ma_s}{2}+\frac{m}{2\sqrt{-mE_b}} & \delta_{SP}+\frac{\sqrt{m}k_0\Omega}{8(-E_b)^{3/2}}\\ \delta_{SP}+\frac{\sqrt{m}k_0\Omega}{8(-E_b)^{3/2}} & \frac{m-a_pm\sqrt{-mE_b+k_0^2}}{4a_p}\end{pmatrix}, \tag{S29}$$

We assume $a_s>0$ and $a_p>0$. Without the coupling Ω and δ_{sp}, the s- and p-wave dimers decouple. The binding energies are

$$E_b^{(s)}=-1/(ma_s^2),\qquad E_b^{(p)}=k_0^2/m-1/(ma_p^2). \tag{S30}$$

When $E_b^{(s)}$ and $E_b^{(p)}$ are almost degenrate, we expect strong mixing between the $s-$wave and the $p-$wave dimer. As a result, we expect a small off-diagonal term is important only when

$$1/(a_s^0)^2=1/(a_p^0)^2-k_0^2. \tag{S31}$$

we thus approximate the off-diagonal terms as $\delta_{SP}+\frac{k_0\Omega m^2(a_s^0)^3}{8}$. We further expand the diagonal terms near their own pole as

$$T^{-1}(E_b,0)\sim\begin{pmatrix} \frac{m^2a_s^3}{4}\left(E_b+\frac{1}{ma_s^2}\right) & \delta_{SP}+\frac{k_0\Omega m^2(a_s^0)^3}{8}\\ \delta_{SP}+\frac{k_0\Omega m^2(a_s^0)^3}{8} & \frac{m^2a_p}{8}\left(E_b+\frac{1}{ma_p^2}-\frac{k_0^2}{m}\right)\end{pmatrix}. \tag{S32}$$

Therefore, the pole of the T-matrix as $\det(T^{-1}(E_b,0))=0$ gives:

$$\begin{aligned}E_b^{(\pm)}=&-\frac{1}{2a_p^2a_s^2m^2}\left\{\left[a_s^2+a_p^2(1-a_s^2k_0^2)\right]m\right.\\&\left.\pm\sqrt{a_s^4m^2+a_p^4m^2(1+a_s^2k_0^2)^2-2a_p^2a_s^2m^2(1+a_s^2k_0^2)+2a_p^3a_s[8\delta_{SP}+(a_s^0)^3k_0\Omega m^2]^2}\right\}.\end{aligned} \tag{S33}$$

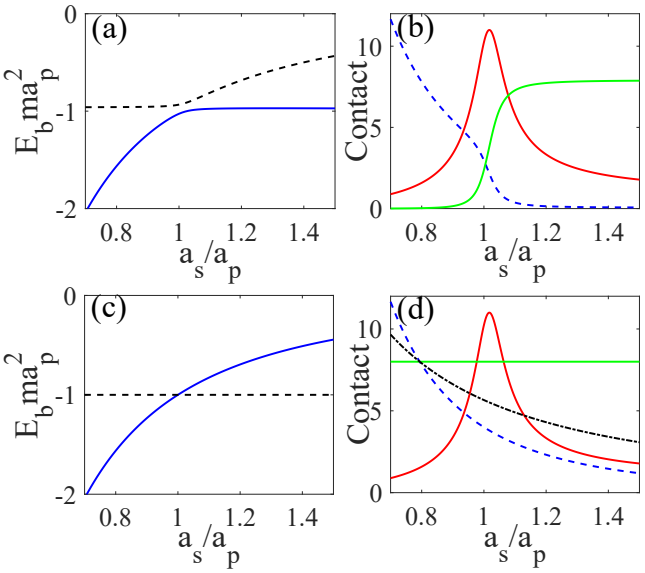

FIG. S3. (a) Dimensionless two-body banding energy versus a_s/a_p with SOC. The black dashed curve denotes $E_b^{(+)}ma_p^2$ and the blue solid curve denotes $E_b^{(-)}ma_p^2$. (b) Dimensionless two-body contacts versus a_s/a_p with SOC. (c) Dimensionless two-body banding energy versus a_s/a_p without SOC. The black dashed curve denotes $E_b^{(p)}ma_p^2$ and the blue solid curve denotes $E_b^{(s)}ma_p^2$. (d) Dimensionless two-body contacts versus a_s/a_p without SOC. As a comparsion, we also plot the C_{SP} with finite SOC (the same curve as (b)). The red solid curve denotes $C_{SP}a_p^2$, the blue dashed curve denotes $C_{SS}a_p^3$, the green solid curve denotes $C_{PP}a_p$, and the black DotDashed curve denotes $\sqrt{C_{SS}C_{PP}}a_p^2$. Here, we choose the SOC parameters as $k_0a_p = 0.2$ and $m\Omega a_p^2 = 0.3$.

The explicit formula for all contacts are given in the supplementary material. A plot for $E_b^{(\pm)}$ and contacts for $E_b^{(-)}$ are shown in Fig. S3 (a) and (c). Away from the degenerate point, $E_b^{(\pm)}$ approaches $E_b^{(s)}$ or $E_b^{(p)}$. Comparing Fig. S3 (a) with (c), it is found that the SOC parameters can open a gap between the two banding energies $E_b^{(+)}$ and $E_b^{(-)}$.

Further, with the adiabatic relations (S13), (S14), we can get the two-body contacts as

$$C_{SS} \equiv 2m\frac{\partial E_b^{(-)}}{\partial a_s} = \frac{2}{a_s^3} + \frac{-2a_s^2 + 2a_p^2(1+a_s^2k_0^2) + 3a_pa_s(a_s^0)^6k_0^2m^2\Omega^2}{a_s^2\sqrt{a_s^2\{[a_s^2 - a_p^2(1+a_s^2k_0^2)]^2 + 2a_p^3a_s(a_s^0)^6k_0^2m^2\Omega^2\}}}, \tag{S34}$$

$$C_{PP} \equiv -4m\frac{\partial E_b^{(-)}}{\partial a_p^{-1}} = \frac{4}{a_p} + \frac{2\{a_s[2a_s^2 - 2a_p^2(1+a_s^2k_0^2)] + a_p^3(a_s^0)^6k_0^2m^2\Omega^2\}}{a_pa_s\sqrt{[a_s^2 - a_p^2(1+a_s^2k_0^2)]^2 + 2a_p^3a_s(a_s^0)^6k_0^2m^2\Omega^2}}, \tag{S35}$$

$$C_{SP} = C_{PS} \equiv -2m^2\frac{\partial E_b^{(-)}}{\partial \delta_{SP}}\bigg|_{\delta_{SP}=0} = \frac{16a_p(a_s^0)^3k_0m\Omega}{a_s\sqrt{[a_s^2 - a_p^2(1+a_s^2k_0^2)]^2 + 2a_p^3a_s(a_s^0)^6k_0^2m^2\Omega^2}}, \tag{S36}$$

where we use $\delta_{SP} = 0$, $a_s^0 = a_s$, and $E_b^{(-)}$ to calculate the contacts as shown in Fig. S3(b). Consequently, for the diagonal components of the contact matrix, we have $C_{SS} \approx 0$ for $a_s/a_p \gg 1$ and $C_{PP} \approx 0$ for $a_s/a_p \ll 1$. Near the degenerate point $a_s/a_p \sim 1$, we see a peak for C_{SP}, indicating a large mixing between s- and $p-$wave dimers as expected.

Without SOC, one can have

$$C_{SS} \equiv 2m\frac{\partial E_b^{(s)}}{\partial a_s} = \frac{4}{a_s^3}, \tag{S37}$$

$$C_{PP} \equiv -4m\frac{\partial E_b^{(p)}}{\partial a_p^{-1}} = \frac{8}{a_p}, \tag{S38}$$

$$C_{SP} = C_{PS} = 0. \tag{S39}$$

Moreover, we also calculate the amplitude of the hybridized new contacts compared to the s- and p-wave ones without SOC as shown in Fig. S3 (d) to give the possibility of the measurement.

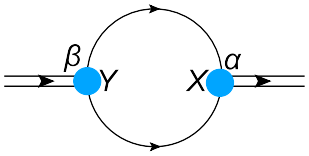

FIG. S4. Feynman diagram for the polarization bubble Π_s. Here, $X=(x_1,t_1)$ and $Y=(x_2,t_2)$.

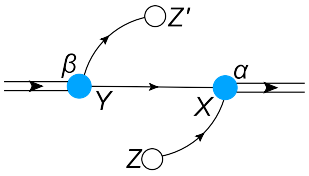

FIG. S5. Feynman diagram for the matrix of the operator $\Psi^{\dagger T}(Z)\Psi^{T}(Z')$. Here, $X=(x_1,t_1)$, $Y=(x_2,t_2)$, $Z=(x_3,t_3)$, and $Z'=(x_4,t_4)$.

SIV. DERIVATION OF SOME FORMULAS

A. Derivations of Eq. (S2)

The polarization bubble can be calculated by the diagram in Fig. S4 as

$$
\begin{aligned}
\Pi_{\alpha\beta}(E_0,Q) &= \int dXdY\ \mathrm{Tr}\left\langle \mathcal{T}\left[\frac{1}{2}\Psi^{T}(X)\sigma_\alpha\Psi(X)\right]\left[\frac{1}{2}\Psi^{\dagger}(Y)\sigma_\beta^{\dagger}\Psi^{\dagger T}(Y)\right]\right\rangle \\
&= \frac{1}{4}\int dXdY\ \mathrm{Tr}\left[\langle \mathcal{T}\overbrace{\Psi^{T}(X)\sigma_\alpha\Psi(X)\Psi^{\dagger}(Y)\sigma_\beta^{\dagger}\Psi^{\dagger T}(Y)}\rangle + \langle \mathcal{T}\overbrace{\Psi^{T}(X)\sigma_\alpha\Psi(X)\Psi^{\dagger}(Y)\sigma_\beta^{\dagger}\Psi^{\dagger T}(Y)}\rangle\right] \\
&= \frac{2}{4}\int dXdY\ \mathrm{Tr}\langle \mathcal{T}\overbrace{\Psi^{T}(X)\sigma_\alpha\Psi(X)\Psi^{\dagger}(Y)\sigma_\beta^{\dagger}\Psi^{\dagger T}(Y)}\rangle\ (\textit{two equivalent contractions}), \\
&= \frac{1}{2}\int dXdY\ \mathrm{Tr}\left\langle \mathcal{T}\left[\Psi^{T}(X)\right]_{1a}(\sigma_\alpha)_{ab}\left[\Psi(X)\right]_{b1}\left[\Psi^{\dagger}(Y)\right]_{1m}\left(\sigma_\beta^{\dagger}\right)_{mn}\left[\Psi^{\dagger T}(Y)\right]_{n1}\right\rangle \\
&= \frac{1}{2}\int dXdY\ \mathrm{Tr}\left\langle \mathcal{T}\left[\Psi^{T}(X)\right]_{1a}\left[\Psi^{\dagger T}(Y)\right]_{n1}(\sigma_\alpha)_{ab}\left[\Psi(X)\right]_{b1}\left[\Psi^{\dagger}(Y)\right]_{1m}\left(\sigma_\beta^{\dagger}\right)_{mn}\right\rangle \\
&= \frac{1}{2}\int dXdY\ \mathrm{Tr}\left\langle \mathcal{T}\left[\Psi(X)\right]_{a1}\left[\Psi^{\dagger}(Y)\right]_{1n}\right\rangle(\sigma_\alpha)_{ab}\left[G(X-Y)\right]_{bm}\left(\sigma_\beta^{\dagger}\right)_{mn} \\
&= \frac{1}{2}\int dXdY\ \mathrm{Tr}\left[G(X-Y)\right]_{an}(\sigma_\alpha)_{ab}\left[G(X-Y)\right]_{bm}\left(\sigma_\beta^{\dagger}\right)_{mn} \\
&= \frac{1}{2}\int dXdY\ \mathrm{Tr}\left[G^{T}(X-Y)\right]_{na}(\sigma_\alpha)_{ab}\left[G(X-Y)\right]_{bm}\left(\sigma_\beta^{\dagger}\right)_{mn} \\
&= \int dXdY\ \frac{1}{2}\mathrm{Tr}\left[G^{T}(X-Y)\sigma_\alpha G(X-Y)\sigma_\beta^{\dagger}\right],
\end{aligned}
\tag{S40}
$$

where $\int dXdY = \frac{1}{2}\int d(X+Y)d(X-Y)$, $\Psi=(\psi_\uparrow,\psi_\downarrow)^T$, $X=(x_1,t_1)$, $Y=(x_2,t_2)$, and we use the definition $G(X-Y)\equiv\langle\mathcal{T}\Psi(X)\Psi^{\dagger}(Y)\rangle$.

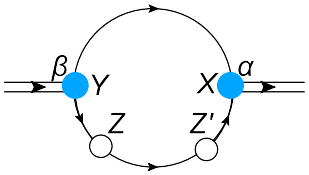

FIG. S6. (Color online) Feynman diagram for the matrix element of $\mathcal{O}_{\sigma 3}(R+x,t)\mathcal{O}^{\dagger}_{\sigma' 3}(R,0)$.

B. Derivations of Eq. (14)

As shown in Fig. S5, the expectation of the operator $\Psi^{\dagger T}(Z)\Psi^{T}(Z')$ is calculated as follows:

$$
\begin{aligned}
&\left\langle O_{\alpha_o}\left|\Psi^{\dagger T}(Z)\Psi^{T}(Z')\right|I_{\alpha_i}\right\rangle = \begin{pmatrix} \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\uparrow}(Z)\psi_{\uparrow}(Z')\right|I_{\alpha_i}\right\rangle & \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\uparrow}(Z)\psi_{\downarrow}(Z')\right|I_{\alpha_i}\right\rangle \\ \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\downarrow}(Z)\psi_{\uparrow}(Z')\right|I_{\alpha_i}\right\rangle & \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\downarrow}(Z)\psi_{\downarrow}(Z')\right|I_{\alpha_i}\right\rangle \end{pmatrix} \\
&= (-i)^2 D_{\alpha_o\alpha}(E_0,Q)D_{\beta\alpha_i}(E_0,Q)\int dXdY\ \left\langle \mathcal{T}\left[\frac{1}{2}\Psi^{T}(X)\sigma_{\alpha}\Psi(X)\right]\Psi^{\dagger T}(Z)\Psi^{T}(Z')\left[\frac{1}{2}\Psi^{\dagger}(Y)\sigma^{\dagger}_{\beta}\Psi^{\dagger T}(Y)\right]\right\rangle \\
&= -\frac{D_{\alpha_o\alpha}(E_0,Q)D_{\beta\alpha_i}(E_0,Q)}{4}\int dXdY \\
&\quad\Big[\langle\mathcal{T}\overbrace{\Psi^{T}(X)\sigma_{\alpha}\underbrace{\Psi(X)\Psi^{\dagger T}(Z)}\Psi^{T}(Z')\underbrace{\Psi^{\dagger}(Y)\sigma^{\dagger}_{\beta}}\Psi^{\dagger T}(Y)}\rangle + \langle\mathcal{T}\overbrace{\Psi^{T}(X)\sigma_{\alpha}\underbrace{\Psi(X)\Psi^{\dagger T}(Z)}\Psi^{T}(Z')\Psi^{\dagger}(Y)\sigma^{\dagger}_{\beta}}\Psi^{\dagger T}(Y)\rangle \\
&\quad+\langle\mathcal{T}\overbrace{\Psi^{T}(X)\sigma_{\alpha}\underbrace{\Psi(X)\Psi^{\dagger T}(Z)}\Psi^{T}(Z')\Psi^{\dagger}(Y)\sigma^{\dagger}_{\beta}}\Psi^{\dagger T}(Y)\rangle + \langle\mathcal{T}\overbrace{\Psi^{T}(X)\sigma_{\alpha}\underbrace{\Psi(X)\Psi^{\dagger T}(Z)}\Psi^{T}(Z')\Psi^{\dagger}(Y)\sigma^{\dagger}_{\beta}}\Psi^{\dagger T}(Y)\rangle\Big] \\
&\sim \int dXdY\ \langle\mathcal{T}\overbrace{\Psi^{T}(X)\sigma_{\alpha}\underbrace{\Psi(X)\Psi^{\dagger T}(Z)}\Psi^{T}(Z')\Psi^{\dagger}(Y)\sigma^{\dagger}_{\beta}}\Psi^{\dagger T}(Y)\rangle\ (\textit{four equivalent contractions}), \\
&= \int dXdY\ \left\langle\mathcal{T}\left[\Psi^{T}(X)\right]_{1n}(\sigma_{\alpha})_{nm}\left[\Psi(X)\right]_{m1}\left[\Psi^{\dagger T}(Z)\right]_{j1}\left[\Psi^{T}(Z')\right]_{1j}\left[\Psi^{\dagger}(Y)\right]_{1a}\left(\sigma^{\dagger}_{\beta}\right)_{ab}\left[\Psi^{\dagger T}(Y)\right]_{b1}\right\rangle \\
&= \int dXdY\ \left\langle\mathcal{T}\left[\Psi(X)\right]_{n1}\left[\Psi^{\dagger}(Z)\right]_{1j}(\sigma_{\alpha})_{nm}\left[\Psi(X)\right]_{m1}\left[\Psi^{\dagger}(Y)\right]_{1a}\left(\sigma^{\dagger}_{\beta}\right)_{ab}\left[\Psi(Z')\right]_{j1}\left[\Psi^{\dagger}(Y)\right]_{1b}\right\rangle \\
&= \int dXdY\ \left[G(X-Z)\right]_{nj}(\sigma_{\alpha})_{nm}\left[G(X-Y)\right]_{ma}\left(\sigma^{\dagger}_{\beta}\right)_{ab}\left[G(Z'-Y)\right]_{jb} \\
&= \int dXdY\ \left[G^{T}(X-Z)\right]_{jn}(\sigma_{\alpha})_{nm}\left[G(X-Y)\right]_{ma}\left(\sigma^{\dagger}_{\beta}\right)_{ab}\left[G^{T}(Z'-Y)\right]_{bj} \\
&= \int dXdY\ G^{T}(X-Z)\sigma_{\alpha}G(X-Y)\sigma^{\dagger}_{\beta}G^{T}(Z'-Y), \qquad \text{(S41)}
\end{aligned}
$$

where $\int dXdY = \frac{1}{2}\int d(X+Y)d(X-Y)$, we label the field operator $\Psi = (\psi_{\uparrow},\psi_{\downarrow})^{T}$, $X=(x_1,t_1)$, $Y=(x_2,t_2)$, $Z=(x_3,t_3)$, $Z'=(x_4,t_4)$, and we use the definition $G(X-Y)\equiv\langle\mathcal{T}\Psi(X)\Psi^{\dagger}(Y)\rangle$.

Therefore, we have

$$
\begin{aligned}
&\begin{pmatrix} \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\uparrow}(Z)\psi_{\uparrow}(Z')\right|I_{\alpha_i}\right\rangle & \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\uparrow}(Z)\psi_{\downarrow}(Z')\right|I_{s}\right\rangle \\ \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\downarrow}(Z)\psi_{\uparrow}(Z')\right|I_{\alpha_i}\right\rangle & \left\langle O_{\alpha_o}\left|\psi^{\dagger}_{\downarrow}(Z)\psi_{\downarrow}(Z')\right|I_{\alpha_i}\right\rangle \end{pmatrix}^{T} = \left[\left\langle O_{s}\left|\Psi^{\dagger T}(Z)\Psi^{T}(Z')\right|I_{\alpha_i}\right\rangle\right]^{T} \\
&= -D_{\alpha_o\alpha}(E_0,Q)D_{\beta\alpha_i}(E_0,Q)\int dXdY\ G(Z'-Y)\sigma_{\beta}G^{T}(X-Y)\sigma^{\dagger}_{\alpha}G(X-Z), \qquad \text{(S42)}
\end{aligned}
$$

where we use $(ABCDE)^{T} = E^{T}D^{T}C^{T}B^{T}A^{T}$.

C. Derivations of Eq. (18)

Equation (18) in the main text can be calculated by the diagram in Fig. S6 as

$$
\begin{aligned}
&\left\langle O_{\alpha_o} \middle| \mathcal{T}\mathcal{O}_{\sigma 3}(R+x,t)\mathcal{O}^{\dagger}_{\sigma' 3}(R,0) \middle| I_{\alpha_i} \right\rangle \\
&\sim (-i)^2 D_{\alpha_o \alpha}(E_0,Q) D_{\beta\alpha_i}(E_0,Q) \int dXdY \ \left\langle \mathcal{T}\left[\frac{1}{2}\Psi^T(X)\sigma_\alpha \Psi(X)\right]\left[\Psi^\dagger(Z)\Psi(Z)\right]\left[\Psi^\dagger(Z')\Psi(Z')\right]\left[\frac{1}{2}\Psi^\dagger(Y)\sigma^\dagger_\beta \Psi^{\dagger T}(Y)\right]\right\rangle \\
&\sim \frac{1}{4}\int dXdY \Big[\langle \mathcal{T}\Psi^T(X)\sigma_\alpha\Psi(X)\Psi^\dagger(Z)\Psi(Z)\Psi^\dagger(Z')\Psi(Z')\Psi^\dagger(Y)\sigma^\dagger_\beta\Psi^{\dagger T}(Y)\rangle \\
&+\langle \mathcal{T}\Psi^T(X)\sigma_\alpha\Psi(X)\Psi^\dagger(Z)\Psi(Z)\Psi^\dagger(Z')\Psi(Z')\Psi^\dagger(Y)\sigma^\dagger_\beta\Psi^{\dagger T}(Y)\rangle \\
&+\langle \mathcal{T}\Psi^T(X)\sigma_\alpha\Psi(X)\Psi^\dagger(Z)\Psi(Z)\Psi^\dagger(Z')\Psi(Z')\Psi^\dagger(Y)\sigma^\dagger_\beta\Psi^{\dagger T}(Y)\rangle \\
&+\langle \mathcal{T}\Psi^T(X)\sigma_\alpha\Psi(X)\Psi^\dagger(Z)\Psi(Z)\Psi^\dagger(Z')\Psi(Z')\Psi^\dagger(Y)\sigma^\dagger_\beta\Psi^{\dagger T}(Y)\rangle\Big] \\
&= \int dXdY \ \langle \mathcal{T}\Psi^T(X)\sigma_\alpha\Psi(X)\Psi^\dagger(Z)\Psi(Z)\Psi^\dagger(Z')\Psi(Z')\Psi^\dagger(Y)\sigma^\dagger_\beta\Psi^{\dagger T}(Y)\rangle \ (\textit{four equivalent contractions}), \\
&= \int dXdY \left\langle \mathcal{T}\left[\Psi^T(X)\right]_{1a}(\sigma_\alpha)_{ab}\left[\Psi(X)\right]_{b1}\left[\Psi^\dagger(Z)\right]_{1c}\left[\Psi(Z)\right]_{c1}\left[\Psi^\dagger(Z')\right]_{1d}\left[\Psi(Z')\right]_{d1}\left[\Psi^\dagger(Y)\right]_{1m}\left(\sigma^\dagger_\beta\right)_{mn}\left[\Psi^{\dagger T}(Y)\right]_{n1}\right\rangle \\
&= \int dXdY \left\langle \mathcal{T}\left[\Psi(X)\right]_{a1}(\sigma_\alpha)_{ab}\left[G(X-Z)\right]_{bc}\left[G(Z-Z')\right]_{cd}\left[G(Z'-Y)\right]_{dm}\left(\sigma^\dagger_\beta\right)_{mn}\left[\Psi^\dagger(Y)\right]_{1n}\right\rangle \\
&= \int dXdY \left\langle \mathcal{T}\left[\Psi(X)\right]_{a1}\left[\Psi^\dagger(Y)\right]_{1n}\right\rangle(\sigma_\alpha)_{ab}\left[G(X-Z)\right]_{bc}\left[G(Z-Z')\right]_{cd}\left[G(Z'-Y)\right]_{dm}\left(\sigma^\dagger_\beta\right)_{mn} \\
&= \int dXdY \ \left[G(X-Y)\right]_{an}(\sigma_\alpha)_{ab}\left[G(X-Z)\right]_{bc}\left[G(Z-Z')\right]_{cd}\left[G(Z'-Y)\right]_{dm}\left(\sigma^\dagger_\beta\right)_{mn} \\
&= \int dXdY \ \left[G^T(X-Y)\right]_{na}(\sigma_\alpha)_{ab}\left[G(X-Z)\right]_{bc}\left[G(Z-Z')\right]_{cd}\left[G(Z'-Y)\right]_{dm}\left(\sigma^\dagger_\beta\right)_{mn} \\
&= \int dXdY \ \left[G^T(X-Y)\sigma_\alpha G(X-Z)G(Z-Z')G(Z'-Y)\sigma^\dagger_\beta\right] \\
&\sim G_0(Z-Z')\int dXdY \ \left[G^T(X-Y)\sigma_\alpha G(X-Z)G(Z'-Y)\sigma^\dagger_\beta\right] \\
&= G_0(Z-Z')\int dXdY \ \left[G(Z'-Y)\sigma^\dagger_\beta G^T(X-Y)\sigma_\alpha G(X-Z)\right].
\end{aligned} \tag{S43}
$$

D. T matrix for s wave without SOC

As shown in Fig. S7(a), the two-body T matrix for s wave is given by [7]

$$
-iT_s(E_0,Q) = \frac{-ig_S}{1-(-ig_S)\Pi_s(E_0,Q)}. \tag{S44}
$$

As shown in Fig. S7(c), the dimer propagator matrix for s wave is given by

$$
D_{SS}(E_0,Q) = \frac{ig_S}{1-(ig_S)\Pi_{SS}(E_0,Q)}, \tag{S45}
$$

With $D_{SS}(E_0,Q) = iT_s(E_0,Q)$, one can get $\Pi_{SS}(E_0,Q) = -\Pi_s(E_0,Q)$.

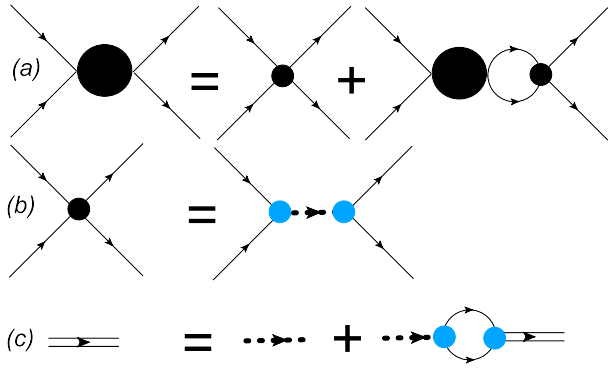

FIG. S7. (a) Diagrams for the T matrix of s-wave without SOC. (b) Diagrams for the interaction vertex of s-wave. (c) Diagrams for the dimer propagator of s-wave. The black disk represents the s-wave T matrix: $-iT_s$, the single line denotes the bare atom propagator G, the black dot represents the interaction vertex: $-ig_S$, the black dotted line denotes the bare dimer propagator ig_S, the double lines denote the dimer propagator D, and the blue dot represents the interaction vertex: $-i$.

E. T matrix for p wave without SOC

The two-body T matrix for p wave is given by [7]

$$\frac{-iT_p(E_0,Q)}{qq'} = \frac{-ig_P}{1-(-ig_P)\Pi_p(E_0,Q)}. \tag{S46}$$

The dimer propagator matrix for p wave is given by

$$D_{PP}(E_0,Q) = \frac{ig_P}{1-(ig_P)\Pi_{PP}(E_0,Q)}, \tag{S47}$$

With $D_{PP}(E_0,Q) = iT_p(E_0,Q)/(qq')$, one can get $\Pi_{PP}(E_0,Q) = -\Pi_p(E_0,Q)$.

[1] M. Olshanii, "Atomic scattering in the presence of an external confinement and a gas of impenetrable bosons," Phys. Rev. Lett. **81**, 938–941 (1998).
[2] Marcus Barth and Wilhelm Zwerger, "Tan relations in one dimension," Annals of Physics **326**, 2544 – 2565 (2011).
[3] Xiaoling Cui and Huifang Dong, "High-momentum distribution with a subleading k^{-3} tail in odd-wave interacting one-dimensional fermi gases," Phys. Rev. A **94**, 063650 (2016).
[4] Pengfei Zhang and Ning Sun, "Universal relations for spin-orbit-coupled fermi gas near an s-wave resonance," Phys. Rev. A **97**, 040701 (2018).
[5] Eric Braaten, Daekyoung Kang, and Lucas Platter, "Universal relations for a strongly interacting fermi gas near a feshbach resonance," Phys. Rev. A **78**, 053606 (2008).
[6] Xiaoling Cui, "Universal one-dimensional atomic gases near odd-wave resonance," Phys. Rev. A **94**, 043636 (2016).
[7] V. Gurarie and L. Radzihovsky, "Resonantly paired fermionic superfluids," Annals of Physics **322**, 2 – 119 (2007), january Special Issue 2007.