A Unstable Resonators—Electromagnetic Analysis

Unstable resonators are, according to (4.4-2), resonators operating in the region of the stability diagram where

$$\left(1-\frac{l}{R_1}\right)\left(1-\frac{l}{R_2}\right)>1, \quad \left(1-\frac{l}{R_1}\right)\left(1-\frac{l}{R_2}\right)<0$$

which causes the Gaussian beam radii to become infinite. Clearly, in this regime the end reflectors cannot be assumed to be infinite in extent, an assumption highly justified in the "stable" regime where the beam spot radius at the mirror is typically very small (see numerical example in Section 4.3) compared to the mirror radius.

One way to account qualitively for the finite extent of real-life mirrors is to simulate analytically the abrupt drop of the reflectivity to zero at the mirror's edge by some tapering of the reflectivity. If we choose a Gaussian tapering function, we find, not too surprisingly, that we can apply the self-consistent ABCD method of Section 4.5 to the case of unstable resonators.

Consider a Gaussian beam incident on a mirror with a radius of curvature R whose reflectivity is given by

$$\rho(r) = \rho_0 e^{-r^2/a^2} \tag{A-1}$$

where r is the radius measured from the center of the mirror. Let the incident beam possess a spot size ω_i and a radius of curvature R_i at the mirror position and let the medium through which it propagates have an index n. The complex beam parameter at the mirror is, according to (2.6-5)

$$\frac{1}{q_1} = \frac{1}{R_1} - i\frac{\lambda}{\pi\omega_i^2 n} \tag{A-2}$$

The reflected beam has a radius of curvature R_0 and spot size ω_0 where, using (2.6-8)

$$\frac{1}{R_0} = \frac{1}{R_i} - \frac{2}{R} \tag{A-3}$$

By multiplying the incident field distribution $\exp(-r^2/\omega_i^2)$ by $\rho(r)$ we find that the $1/e$ Gaussian spot size is modified from ω_i to ω_0, where

$$\frac{1}{\omega_0^2} = \frac{1}{\omega_i^2} + \frac{1}{a^2} \tag{A-4}$$

The Gaussian tapered mirror thus modifies not only the beam radius of curvature but the spot size ω_0 as well. In a uniform reflectivity mirror ($a^2 = \infty$), the mirror spot size does not change upon reflection.

The transformation properties (A-3) and (A-4) would follow from the ABCD law (2.6-9)

$$q_0 = \frac{Aq_i + B}{Cq_i + D}$$

provided we take the A, B, C, D, matrix as

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\dfrac{2}{R} - i\dfrac{\lambda}{\pi a^2 n} & 1 \end{vmatrix} \equiv \begin{vmatrix} 1 & 0 \\ -\dfrac{2}{r} & 1 \end{vmatrix} \tag{A-5}$$

so that the tapered reflectivity mirror is characterized by a complex radius of curvature r where

$$\frac{1}{r} = \frac{1}{R} + i\frac{\lambda}{2\pi a^2 n} \tag{A-6}$$

To obtain the mode characteristics in this case we apply the self-consistent formalism of Section 4.5, replacing the $ABCD$ parameters representing the mirrors by (A, B, C, D) matrices in the form of (A-5). The tapering function of the left mirror is represented by a_1^2 and that of the right mirror by a_2^2. The $(ABCD)$ matrix relating the beam parameter q_1 following reflection from the left mirror (mirror 1) to the beam parameter at the same position one complete round trip "earlier" is obtained from (2.1-6) after replacing $f_{1,2}$ by $(1/2)r_{1,2}$. The result is

$$A = 1 - \frac{2}{r_2} l \qquad B = 2l\left(1 - \frac{l}{r_2}\right)$$

$$C = \frac{4l}{r_1 r_2} - \frac{2}{r_1} - \frac{2}{r_2} \qquad D = \left(1 - \frac{2l}{r_1}\right)\left(1 - \frac{2l}{r_2}\right) - \frac{2}{r_1} l \tag{A-7}$$

The solution for the steady-state beam parameter $\underset{\rightarrow}{q_1}$ immediately following reflection from mirror 1 is given by (4.5-3) as

$$\frac{1}{\underset{\rightarrow}{q_1}} = \frac{D - A}{2B} + i\frac{\sin\theta}{B} \tag{A-8}$$

where the arrow denotes the direction of beam travel and

$$\cos\theta \equiv \cos(\alpha + i\beta) = \tfrac{1}{2}(D + A) \tag{A-9}$$

α and β are real.

Using (A-7) and (A-9) we have

$$\cos\theta = (2g_1g_2 - 2t_1t_2 - 1) - i(2g_1t_2 + 2g_2t_1) \tag{A-10}$$

so that

$$\frac{1}{\underset{\rightarrow}{q_1}} = \frac{1}{l}\left(-\frac{l}{R_1} - it_1 + i\frac{\sin\alpha\cosh\beta + i\cos\alpha\sinh\beta}{2(g_2 - it_2)}\right) \tag{A-11}$$

and

$$\operatorname{Im}\left(\frac{l}{\underset{\rightarrow}{q_1}}\right) = -\frac{l\lambda}{\pi\underset{\rightarrow}{\omega_1^2}n} = -t_1 + \frac{g_2\sin\alpha\cosh\beta - t_2\cos\alpha\sinh\beta}{2(g_2^2 + t_2^2)} \tag{A-12}$$

where

$$g_{1,2} \equiv 1 - \frac{l}{R_{1,2}} \tag{A-13}$$

$$t_{1,2} \equiv \frac{l\lambda}{2\pi a_{1,2}^2 n} \tag{A-14}$$

According to (A-10) and (A-11) the parameters α and β are determined implicitly by means of the equations

$$\begin{aligned}\cos\alpha\cosh\beta &= 2g_1g_2 - 2t_1t_2 - 1\\ \sin\alpha\sinh\beta &= 2g_1t_2 + 2g_2t_1\end{aligned} \tag{A-15}$$

Once α and β are determined, we substitute them in (A-11) to determine the beam parameter $\underset{\rightarrow}{q_1}$. The resulting expression is extremely complicated so it may be advantageous to consider the special case where only one mirror, say 1, has a positive reflectivity taper ($t_1 > 0$), and to assume a uniform reflectivity for the second mirror ($t_2 = 0$). By putting $t_2 = 0$ in (A-15) and using the result in (A-13) we obtain

$$\frac{l\lambda}{\pi \underset{\rightarrow}{\omega_1^2 n}} = t_1(1 - \coth \beta) > 2t_1 \qquad \text{for } \beta < 0 \tag{A-16}$$

In a confined beam it is necessary that $\underset{\leftarrow}{\omega_1}^{-2}$ be finite and positive (that is, the left-going beam before reflection from mirror 1 has a finite spot size). Using (A-4) and (A-14) this last requirement translates into $l\lambda/\pi\underset{\rightarrow}{\omega_1^2 n} > 2t_1$. This condition is indeed satisfied, according to (A-16) by $\underset{\rightarrow}{\omega_1^2}$ provided $\beta < 0$. We are always free to choose $\beta < 0$, since for each (α, β) satisfying (A-9) (α and β real), the pair $(-\alpha, -\beta)$ is an equally valid solution. The solution with $\beta < 0$ is thus seen to lead to a confined beam *regardless* of $g_1(= 1 - L/R_1)$ and $g_2(=1 - L/R_2)$.[1]

We note, in contrast, that the stability (confinement) condition (4.4-2) for uniform reflectivity mirrors ($t_1 = t_2 = 0$) can be written as

$$0 \le g_1 g_2 \le 1 \tag{A-17}$$

[1] An exception is the case $g_2 = 0$ for which $\omega_1^2 = 0$.

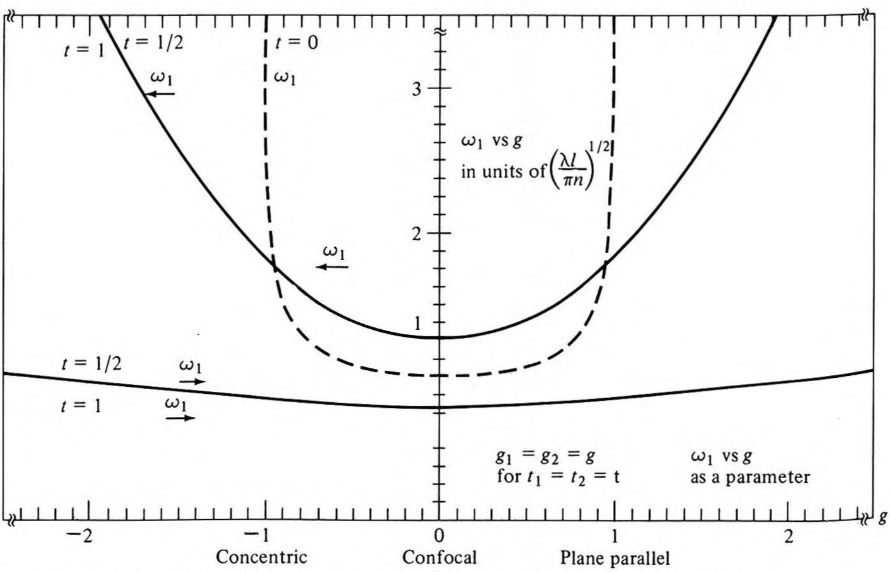

Figure A-1 Behavior of mirror spot size as a function of g in a symmetric resonator ($g_1 = g_2 = g$) for various degrees of reflectivity tapering (t).

and is to be contrasted with the unconditional stability of the resonator with positive tapered ($t_{1,2} > 0$) mirrors.

It can be shown that for $t_1 > 0$ and $t_2 > 0$ the beam modes are stable for any combination of g_1 and g_2.[2] The beam parameters are then determined from (A-1). A plot using (A-12) and (A-15) of $\underset{\leftarrow}{\omega_1}$ versus g with t as a parameter in a symmetric ($g_1 = g_2$, $t_1 = t_2$) resonator is shown in Figure A-1. We note that when $t = 0$, that is, both mirrors possess uniform reflectivity, ω_1 becomes infinite at $g = \pm 1$, while for $t > 0$, $\underset{\leftarrow}{\omega_1}$ is finite everywhere.

PERTURBATION STABILITY

The perturbation stability analysis of Section 4.5 can be used to derive the stability of the modes of the tapered mirror resonators. Using (A-9) in (4.5-11) leads to

$$\left.\frac{dq_{\text{out}}^{-1}}{dq_{\text{in}}^{-1}}\right|_{q_{\text{in}}=q_0} = e^{-2i\theta} = e^{-2i\alpha}e^{2\beta} \tag{A-18}$$

It was shown in the discussion following (A-16) that confined ($\underset{\leftarrow}{\omega_1^{-2}}$ finite and positive) modes solutions require that we choose the $\beta < 0$ branch of (A-9). With this choice it follows from (A-18) that

$$|\Delta q_{\text{out}}^{-1}| < |\Delta q_{\text{in}}^{-1}| \tag{A-19}$$

so that the beam perturbation decays progressively with each passage. The effect of positive reflectivity tapering ($\alpha^2 > 0$) is thus to replace the neutral stability of (4.5-12) by the absolute stability of (A-19).

[2] A. Yariv and P. Yeh, "Confinement and stability in optical resonators employing mirrors with Gaussian reflectivity tapers," *Opt. Commun.* 13:370–374, April 1975.