1 Electromagnetic Theory

1.0 INTRODUCTION

In this chapter we derive some of the basic results concerning the propagation of plane, single-frequency, electromagnetic waves in homogeneous isotropic media, as well as in anisotropic crystal media. Starting with Maxwell's equations we obtain expressions for the dissipation, storage, and transport of energy resulting from the propagation of waves in material media. We consider in some detail the phenomenon of birefringence, in which the phase velocity of a plane wave in a crystal depends on its direction of polarization. The two allowed modes of propagation in uniaxial crystals—the "ordinary" and "extraordinary" rays—are discussed using the formalism of the index ellipsoid.

1.1 COMPLEX-FUNCTION FORMALISM

In problems that involve sinusoidally varying time functions, we can save a great deal of manipulation and space by using the complex-function formalism. As an example consider the function

$$a(t) = |A| \cos(\omega t + \phi_a) \tag{1.1-1}$$

where ω is the circular (radian) frequency[1] and ϕ_a is the phase. Defining the complex amplitude of $a(t)$ by

$$A = |A|\, e^{i\phi_a} \tag{1.1-2}$$

[1]The radian frequency ω is to be distinguished from the real frequency $\nu = \omega/2\pi$.

we can rewrite (1.1-1) as

$$a(t) = \mathrm{Re}[Ae^{i\omega t}] \tag{1.1-3}$$

We will often represent $a(t)$ by

$$a(t) = Ae^{i\omega t} \tag{1.1-4}$$

instead of by (1.1-1) or (1.1-3). This, of course, is not strictly correct so that when this happens *it is always understood* that what is meant by (1.1-4) is the *real part of* $A\exp(i\omega t)$. In most situations the replacement of (1.1-3) by the complex form (1.1-4) poses no problems. The exceptions are cases that involve the product (or powers) of sinusoidal functions. In these cases we must use the real form of the function (1.1-3). To illustrate the case where the distinction between the real and complex form is not necessary, consider the problem of taking the derivative of $a(t)$. Using (1.1-1) we obtain

$$\frac{da(t)}{dt} = \frac{d}{dt}[|A| \cos(\omega t + \phi_a)] = -\omega|A| \sin(\omega t + \phi_a) \tag{1.1-5}$$

If we use instead the complex form (1.1-4), we get

$$\frac{da(t)}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = i\omega Ae^{i\omega t}$$

Taking, as agreed, the real part of the last expression and using (1.1-2), we obtain (1.1-5).

As an example of a case in which we have to use the real form of the function, consider the product of two sinusoidal functions $a(t)$ and $b(t)$, where

$$\begin{aligned} a(t) &= |A| \cos(\omega t + \phi_a) \\ &= \frac{|A|}{2}[e^{i(\omega t + \phi_a)} + e^{-i(\omega t + \phi_a)}] \\ &= \mathrm{Re}[Ae^{i\omega t}] \end{aligned} \tag{1.1-6}$$

and

$$\begin{aligned} b(t) &= |B| \cos(\omega t + \phi_b) \\ &= \frac{|B|}{2}[e^{i(\omega t + \phi_b)} + e^{-i(\omega t + \phi_b)}] \\ &= \mathrm{Re}[Be^{i\omega t}] \end{aligned} \tag{1.1-7}$$

with $A = |A| \exp(i\phi_a)$ and $B = |B| \exp(i\phi_b)$. Using the real functions, we get

$$a(t)b(t) = \frac{|A|\,|B|}{2}[\cos(2\omega t + \phi_a + \phi_b) + \cos(\phi_a - \phi_b)] \tag{1.1-8}$$

Were we to evaluate the product $a(t)b(t)$ using the complex form of the functions, we would get

$$a(t)b(t) = ABe^{i2\omega t} = |A|\,|B|\,e^{i(2\omega t + \phi_a + \phi_b)} \tag{1.1-9}$$

Comparing the last result to (1.1-8) shows that the time-independent (dc) term $\frac{1}{2}|A|\,|B|\cos(\phi_a - \phi_b)$ is missing, and thus the use of the complex form led to an error.

Time-Averaging of Sinusoidal Products[2]

Another problem often encountered is that of finding the time average of the product of two sinusoidal functions of the same frequency

$$\overline{a(t)b(t)} = \frac{1}{T}\int_0^T |A|\cos(\omega t + \phi_a)|B|\cos(\omega t + \phi_b)\,dt \tag{1.1-10}$$

where $a(t)$ and $b(t)$ are given by (1.1-6) and (1.1-7) and the horizontal bar denotes time-averaging. $T = 2\pi/\omega$ is the period of the oscillation. Since the integrand in (1.1-10) is periodic in T, the averaging can be performed over a time T. Using (1.1-8) we obtain directly

$$\overline{a(t)b(t)} = \frac{|A|\,|B|}{2}\cos(\phi_a - \phi_b) \tag{1.1-11}$$

This last result can be written in terms of the complex amplitudes A and B, defined immediately following (1.1-7), as

$$\overline{a(t)b(t)} = \tfrac{1}{2}\mathrm{Re}(AB^*) \tag{1.1-12}$$

This important result will find frequent use throughout the book.

1.2 CONSIDERATIONS OF ENERGY AND POWER IN ELECTROMAGNETIC FIELDS

In this section we derive the formal expressions for the power transport, power dissipation, and energy storage that accompany the propagation of electromagnetic radiation in material media. The starting point is Maxwell's equations (in MKS units)

$$\nabla \times \mathbf{h} = \mathbf{i} + \frac{\partial \mathbf{d}}{\partial t} \tag{1.2-1}$$

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t} \tag{1.2-2}$$

and the constitutive equations relating the polarization of the medium to the displacement vectors

$$\mathbf{d} = \varepsilon_0\mathbf{e} + \mathbf{p} \tag{1.2-3}$$

$$\mathbf{b} = \mu_0(\mathbf{h} + \mathbf{m}) \tag{1.2-4}$$

[2]The problem of the time average of the product of two nearly sinusoidal functions is considered in Problems 1.1 and 1.2.

where $\mathbf{i}$ is the current density (amperes per square meter); $\mathbf{e}(\mathbf{r}, t)$ and $\mathbf{h}(\mathbf{r}, t)$ are the electric and magnetic field vectors, respectively; $\mathbf{d}(\mathbf{r}, t)$ and $\mathbf{b}(\mathbf{r}, t)$ are the electric and magnetic displacement vectors; $\mathbf{p}(\mathbf{r}, t)$ and $\mathbf{m}(\mathbf{r}, t)$ are the electric and magnetic polarizations (dipole moment per unit volume) of the medium; and ε_0 and μ_0 are the electric and magnetic permeabilities of vacuum, respectively. We adopt the convention of using lowercase letters to denote the time-varying functions, reserving capital letters for the amplitudes of the sinusoidal time functions. For a detailed discussion of Maxwell's equations, the reader is referred to any standard text on electromagnetic theory such as Reference [1].

Using (1.2-3) and (1.2-4) in (1.2-1) and (1.2-2) leads to

$$\nabla \times \mathbf{h} = \mathbf{i} + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{e} + \mathbf{p}) \tag{1.2-5}$$

$$\nabla \times \mathbf{e} = -\frac{\partial}{\partial t}\mu_0(\mathbf{h} + \mathbf{m}) \tag{1.2-6}$$

Taking the scalar (dot) product of (1.2-5) and $\mathbf{e}$ gives

$$\mathbf{e} \cdot \nabla \times \mathbf{h} = \mathbf{e} \cdot \mathbf{i} + \frac{\varepsilon_0}{2}\frac{\partial}{\partial t}(\mathbf{e} \cdot \mathbf{e}) + \mathbf{e} \cdot \frac{\partial \mathbf{p}}{\partial t} \tag{1.2-7}$$

where we used the relation

$$\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{e} \cdot \mathbf{e}) = \mathbf{e} \cdot \frac{\partial \mathbf{e}}{\partial t}$$

Next we take the scalar product of (1.2-6) and $\mathbf{h}$:

$$\mathbf{h} \cdot \nabla \times \mathbf{e} = -\frac{\mu_0}{2}\frac{\partial}{\partial t}(\mathbf{h} \cdot \mathbf{h}) - \mu_0 \mathbf{h} \cdot \frac{\partial \mathbf{m}}{\partial t} \tag{1.2-8}$$

Subtracting (1.2-8) from (1.2-7) and using the vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{1.2-9}$$

results in

$$\begin{aligned} -\nabla \cdot (\mathbf{e} \times \mathbf{h}) = {} & \mathbf{e} \cdot \mathbf{i} + \frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\mathbf{e} \cdot \mathbf{e} + \frac{\mu_0}{2}\mathbf{h} \cdot \mathbf{h}\right) \\ & + \mathbf{e} \cdot \frac{\partial \mathbf{p}}{\partial t} + \mu_0 \mathbf{h} \cdot \frac{\partial \mathbf{m}}{\partial t} \end{aligned} \tag{1.2-10}$$

We integrate the last equation over an arbitrary volume V and use the Gauss theorem [1]

$$\int_V (\nabla \cdot \mathbf{A})\, dv = \int_S \mathbf{A} \cdot \mathbf{n}\, da$$

where $\mathbf{A}$ is any vector function, $\mathbf{n}$ is the unit vector normal to the surface S

enclosing V, and dv and da are the differential volume and surface elements, respectively. The result is

$$-\int_V \nabla \cdot (\mathbf{e} \times \mathbf{h})\, dv = -\int_S (\mathbf{e} \times \mathbf{h}) \cdot \mathbf{n}\, da$$

$$= \int_V \left[\mathbf{e} \cdot \mathbf{i} + \frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\mathbf{e} \cdot \mathbf{e}\right) + \frac{\partial}{\partial t}\left(\frac{\mu_0}{2}\mathbf{h} \cdot \mathbf{h}\right) + \mathbf{e} \cdot \frac{\partial \mathbf{p}}{\partial t} + \mu_0 \mathbf{h} \cdot \frac{\partial \mathbf{m}}{\partial t} \right] dv \qquad (1.2\text{-}11)$$

According to the conventional interpretation of electromagnetic theory, the left side of (1.2-11), that is,

$$-\int_S (\mathbf{e} \times \mathbf{h}) \cdot \mathbf{n}\, da$$

gives the total power flowing *into* the volume bounded by S. The first term on the right side is the power expended by the field on the moving charges; the sum of the second and third terms corresponds to the rate of increase of the vacuum electromagnetic stored energy $\mathscr{E}_{\text{vac}}$ where

$$\mathscr{E}_{\text{vac}} = \int_V \left[\frac{\varepsilon_0}{2} \mathbf{e} \cdot \mathbf{e} + \frac{\mu_0}{2} \mathbf{h} \cdot \mathbf{h} \right] dv \qquad (1.2\text{-}12)$$

Of special interest in this book is the next-to-last term

$$\mathbf{e} \cdot \frac{\partial \mathbf{p}}{\partial t}$$

which represents the power per unit volume expended by the field *on* the electric dipoles. This power goes into an increase in the potential energy stored by the dipoles as well as into supplying the dissipation that may accompany the change in **p**. We will return to it again in Chapter 5, where we treat the interaction of radiation and atomic systems.

Dipolar Dissipation in Harmonic Fields

According to the discussion in the preceding paragraph, the average power per unit volume expended by the field on the medium electric polarization is

$$\overline{\frac{\text{Power}}{\text{Volume}}} = \overline{\mathbf{e} \cdot \frac{\partial \mathbf{p}}{\partial t}} \qquad (1.2\text{-}13)$$

where the horizontal bar denotes time-averaging. Let us assume for the sake of simplicity that $\mathbf{e}(t)$ and $\mathbf{p}(t)$ are parallel to each other and take their sinusoidally varying magnitudes as

$$e(t) = \text{Re}[Ee^{i\omega t}] \qquad (1.2\text{-}14)$$

$$p(t) = \text{Re}[Pe^{i\omega t}] \qquad (1.2\text{-}15)$$

where E and P are the complex amplitudes. The electric susceptibility χ_e is defined by

$$P = \varepsilon_0 \chi_e E \tag{1.2-16}$$

and is thus a complex number. Substituting (1.2-14) and (1.2-15) in (1.2-13) and using (1.2-16) gives

$$\begin{aligned} \overline{\frac{\text{Power}}{\text{Volume}}} &= \overline{\text{Re}[Ee^{i\omega t}]\ \text{Re}[i\omega P e^{i\omega t}]} \\ &= \tfrac{1}{2}\ \text{Re}[i\omega\varepsilon_0\chi_e E E^*] \\ &= \frac{\omega}{2}\varepsilon_0 |E|^2\ \text{Re}(i\chi_e) \end{aligned} \tag{1.2-17}$$

where in going from the first to the second equality we used (1.1-12). Since χ_e is complex, we can write it in terms of its real and imaginary parts as

$$\chi_e = \chi_e' - i\chi_e'' \tag{1.2-18}$$

which, when used in (1.2-17), gives

$$\overline{\frac{\text{Power}}{\text{Volume}}} = \frac{\omega\varepsilon_0\chi_e''}{2}|E|^2 \tag{1.2-19}$$

which is the desired result.

We leave it as an exercise (Problem 1.3) to show that in anisotropic media in which the complex field components are related by

$$P_i = \varepsilon_0 \sum_j \chi_{ij} E_j \tag{1.2-20}$$

the application of (1.2-13) yields

$$\overline{\frac{\text{Power}}{\text{Volume}}} = \frac{\omega}{2}\varepsilon_0 \sum_{i,j} \text{Re}(i\chi_{ij} E_i^* E_j) \tag{1.2-21}$$

1.3 WAVE PROPAGATION IN ISOTROPIC MEDIA

Here we consider the propagation of electromagnetic plane waves in homogeneous and isotropic media so that ε and μ are scalar constants. Vacuum is, of course, the best example of such a "medium." Liquids and glasses are material media that, to a first approximation, can be treated as homogeneous and isotropic.[3] We choose the direction of propagation as z and, taking the plane wave to be uniform in the x-y plane, put $\partial/\partial x = \partial/\partial y = 0$

[3]The individual molecules making up the liquid or glass are, of course, anisotropic. This anisotropy, however, is averaged out because of the very large number of molecules with random orientations present inside a volume $\sim \lambda^3$.

in (1.2-1) and (1.2-2). Assuming a lossless ($\sigma = 0$) medium, (1.2-1) and (1.2-2) become

$$\nabla \times \mathbf{e} = -\mu \frac{\partial \mathbf{h}}{\partial t} \tag{1.3-1}$$

$$\nabla \times \mathbf{h} = \varepsilon \frac{\partial \mathbf{e}}{\partial t} \tag{1.3-2}$$

$$\frac{\partial e_y}{\partial z} = \mu \frac{\partial h_x}{\partial t} \tag{1.3-3}$$

$$\frac{\partial h_y}{\partial z} = -\varepsilon \frac{\partial e_x}{\partial t} \tag{1.3-4}$$

$$\frac{\partial e_x}{\partial z} = -\mu \frac{\partial h_y}{\partial t} \tag{1.3-5}$$

$$\frac{\partial h_x}{\partial z} = \varepsilon \frac{\partial e_y}{\partial t} \tag{1.3-6}$$

$$0 = \mu \frac{\partial h_z}{\partial t} \tag{1.3-7}$$

$$0 = \varepsilon \frac{\partial e_z}{\partial t} \tag{1.3-8}$$

From (1.3-7) and (1.3-8) it follows that h_z and e_z are both zero; therefore, a uniform plane wave in a homogeneous isotropic medium can have no longitudinal field components. We can obtain a self-consistent set of equations from (1.3-3) through (1.3-8) by taking e_y and h_x (or e_x and h_y) to be zero.[4] In this case the last set of equations reduces to Equations (1.3-4) and (1.3-5). Taking the derivative of (1.3-5) with respect to z and using (1.3-4), we obtain

$$\frac{\partial^2 e_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 e_x}{\partial t^2} \tag{1.3-9}$$

A reversal of the procedure will yield a similar equation for h_y. Since our main interest is in harmonic (sinusoidal) time variation, we postulate a solution in the form of

$$e_x^{\pm} = E_x^{\pm}\, e^{i(\omega t \mp kz)} \tag{1.3-10}$$

where $E_x^{\pm} \exp(\mp ikz)$ are the complex field amplitudes at z. Before substituting (1.3-10) into the wave equation (1.3-9), we may consider the nature of the two functions $e_x^{\pm}$. Taking first e_x^{+}: If an observer were to travel in

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[4]More fundamentally it can be easily shown from (1.3-1) and (1.3-2) (see Problem 1.4) that, for uniform plane harmonic waves, **e** and **h** are normal to each other as well as to the direction of propagation. Thus, **x** and **y** can simply be chosen to coincide with the directions of **e** and **h**.

such a way as to always exercise the same field value, he would have to satisfy the condition

$$\omega t - kz = \text{constant}$$

where the constant is arbitrary and determines the field value "seen" by the observer. By differentiation of the last result, it follows that the observer must travel in the $+z$ direction with a velocity

$$c = \frac{dz}{dt} = \frac{\omega}{k} \tag{1.3-11}$$

This is the *phase velocity* of the wave. If the wave were frozen in time, the separation between two neighboring field peaks—that is, the wavelength—is

$$\lambda = \frac{2\pi}{k} = 2\pi \frac{c}{\omega} \tag{1.3-12}$$

The e_x^- solution differs only in the sign of k, and thus, according to (1.3-11), it corresponds to a wave traveling with a phase velocity c in the $-z$ direction.

The value of c can be obtained by substituting the assumed solution (1.3-10) into (1.3-9), which results in

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{1.3-13}$$

or

$$k = \omega\sqrt{\mu\varepsilon}$$

The phase velocity in vacuum is

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8 \text{ m/s}$$

whereas in material media it has the value

$$c = \frac{c_0}{n}$$

where $n = \sqrt{\varepsilon/\varepsilon_0}$ is the *index of refraction*.

Turning our attention next to the magnetic field h_y, we can express it, in a manner similar to (1.3-10), in the form of

$$h_y^{\pm} = H_y^{\pm} e^{i(\omega t \mp kz)} \tag{1.3-14}$$

Substitution of this equation into (1.3-4) and using (1.3-10) gives

$$-ikH_y^+ e^{i(\omega t - kz)} = -i\omega\varepsilon E_x^+ e^{i(\omega t - kz)}$$

Therefore, from (1.3-13),

$$H_y^+ = \frac{E_x^+}{\eta} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{1.3-15}$$

In vacuum $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \simeq 377$ ohms. Repeating the same steps with H_y^- and E_x^- gives

$$H_y^- = -\frac{E_x^-}{\eta} \tag{1.3-16}$$

so that for negative $(-z)$ traveling waves the relative phase of the electric and magnetic fields is reversed with respect to the wave traveling in the $+z$ direction. Since the wave equation (1.3-9) is a linear differential equation, we can take the solution for the harmonic case as a linear superposition of e_x^+ and e_x^-

$$e_x(z, t) = E_x^+ e^{i(\omega t - kz)} + E_x^- e^{i(\omega t + kz)} \tag{1.3-17}$$

and, similarly,

$$h_y(z, t) = \frac{1}{\eta} [E_x^+ e^{i(\omega t - kz)} - E_x^- e^{i(\omega t + kz)}]$$

where E_x^+ and E_x^- are arbitrary complex constants.

Power Flow in Harmonic Fields

The average power per unit area—that is, the intensity (W/m²)—carried in the direction of propagation by a uniform plane wave is given by (1.2-11) as

$$I = |\overline{\mathbf{e} \times \mathbf{h}}| \tag{1.3-18}$$

where the horizontal bar denotes time averaging. Since $\mathbf{e} \| x$ and $\mathbf{h} \| y$, we can write (1.3-18) as

$$I = \overline{e_x h_y}$$

Taking advantage of the harmonic nature of e_x and h_y, we use (1.3-17) and (1.1-12) to obtain

$$\begin{aligned} I = \tfrac{1}{2} \operatorname{Re}[E_x H_y^*] &= \frac{1}{2\eta} \operatorname{Re}\{[E_x^+ e^{-ikz} + E_x^- e^{ikz}] \\ &\quad \times [(E_x^+)^* e^{ikz} - (E_x^-)^* e^{-ikz}]\} \\ &= \frac{|E_x^+|^2}{2\eta} - \frac{|E_x^-|^2}{2\eta} \end{aligned} \tag{1.3-19}$$

The first term on the right side of (1.3-19) gives the intensity associated with the positive $(+z)$ traveling wave, whereas the second term represents the negative traveling wave, with the minus sign accounting for the opposite direction of power flow.

An important relation that will be used in a number of later chapters relates the intensity of the plane wave to the stored electromagnetic energy density. We start by considering the second and fourth terms on the right of (1.2-11)

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\mathbf{e}\cdot\mathbf{e}\right)+\mathbf{e}\cdot\frac{\partial \mathbf{p}}{\partial t}$$

Using the relations

$$\begin{aligned}\mathbf{p} &= \varepsilon_0\chi_e\mathbf{e}\\ \varepsilon &= \varepsilon_0(1+\chi_e)\end{aligned} \tag{1.3-20}$$

we obtain

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\mathbf{e}\cdot\mathbf{e}\right)+\mathbf{e}\cdot\frac{\partial \mathbf{p}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}\mathbf{e}\cdot\mathbf{e}\right) \tag{1.3-21}$$

Since we assumed the medium to be lossless, the last term must represent the rate of change of electric energy density stored in the vacuum as well as in the electric dipoles; that is,

$$\frac{\mathscr{E}_{\text{electric}}}{\text{Volume}}=\frac{\varepsilon}{2}\mathbf{e}\cdot\mathbf{e} \tag{1.3-22}$$

The magnetic energy density is derived in a similar fashion using the relations

$$\begin{aligned}\mathbf{m} &= \chi_m\mathbf{h}\\ \mu &= \mu_0(1+\chi_m)\end{aligned}$$

resulting in

$$\frac{\mathscr{E}_{\text{magnetic}}}{\text{Volume}}=\frac{\mu}{2}\mathbf{h}\cdot\mathbf{h} \tag{1.3-23}$$

Considering only the positive traveling wave in (1.3-17), we obtain from (1.3-22) and (1.3-23)

$$\begin{aligned}\frac{\overline{\mathscr{E}_{\text{magnetic}}}+\overline{\mathscr{E}_{\text{electric}}}}{\text{Volume}} &= \left(\frac{\varepsilon}{2}\right)\overline{(e_x^+)^2}+\left(\frac{\mu}{2}\right)\overline{(h_y^+)^2}\\ &= \frac{\varepsilon}{4}|E_x^+|^2+\frac{\mu}{4}|H_y^+|^2\\ &= \frac{\varepsilon}{4}|E_x^+|^2+\frac{\mu}{4}\frac{|E_x^+|^2}{\eta^2}\\ &= \tfrac{1}{2}\varepsilon|E_x^+|^2\end{aligned} \tag{1.3-24}$$

where the second equality is based on (1.1-12), and the third and fourth use (1.3-15). Comparing (1.3-24) to (1.3-19), we get

$$\frac{I}{\overline{\mathscr{E}}/\text{Volume}}=\frac{1}{\sqrt{\mu\varepsilon}}=c \tag{1.3-25}$$

where $\overline{\mathscr{E}} = \overline{\mathscr{E}_{\text{magnetic}}} + \overline{\mathscr{E}_{\text{electric}}}$ is the electromagnetic field energy and c is the phase velocity of light in the medium. In terms of the electric field we get

$$I = \frac{c\varepsilon|E|^2}{2} \tag{1.3-26}$$

1.4 WAVE PROPAGATION IN CRYSTALS—THE INDEX ELLIPSOID

In the discussion of electromagnetic wave propagation up to this point, we have assumed that the medium was isotropic. This causes the induced polarization to be parallel to the electric field and to be related to it by a (scalar) factor that is independent of the direction along which the field is applied. This situation does not apply in the case of dielectric crystals. Since the crystal is made up of a regular periodic array of atoms (or ions), we may expect that the induced polarization will depend in its magnitude and direction, on the direction of the applied field. Instead of the simple relation (1.3-20) linking **p** and **e**, we have

$$\begin{aligned} P_x &= \varepsilon_0(\chi_{11}E_x + \chi_{12}E_y + \chi_{13}E_z) \\ P_y &= \varepsilon_0(\chi_{21}E_x + \chi_{22}E_y + \chi_{23}E_z) \\ P_z &= \varepsilon_0(\chi_{31}E_x + \chi_{32}E_y + \chi_{33}E_z) \end{aligned} \tag{1.4-1}$$

where the capital letters denote the complex amplitudes of the corresponding time-harmonic quantities. The 3×3 array of the χ_{ij} coefficients is called the electric susceptibility tensor. The magnitude of the χ_{ij} coefficients depends, of course, on the choice of the x, y, and z axes relative to that of the crystal structure. It is always possible to choose x, y, and z in such a way that the off-diagonal elements vanish, leaving

$$\begin{aligned} P_x &= \varepsilon_0\chi_{11}E_x \\ P_y &= \varepsilon_0\chi_{22}E_y \\ P_z &= \varepsilon_0\chi_{33}E_z \end{aligned} \tag{1.4-2}$$

These directions are called the *principal dielectric axes of the crystal*. In this book we will use only the principal coordinate system. We can, instead of using (1.4-2), describe the dielectric response of the crystal by means of the electric permeability tensor ε_{ij}, defined by

$$\begin{aligned} D_x &= \varepsilon_{11}E_x \\ D_y &= \varepsilon_{22}E_y \\ D_z &= \varepsilon_{33}E_z \end{aligned} \tag{1.4-3}$$

From (1.4-2) and the relation

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

we have

$$\begin{aligned}\varepsilon_{11} &= \varepsilon_0(1 + \chi_{11})\\ \varepsilon_{22} &= \varepsilon_0(1 + \chi_{22})\\ \varepsilon_{33} &= \varepsilon_0(1 + \chi_{33})\end{aligned} \tag{1.4-4}$$

Birefringence

One of the most important consequences of the dielectric anisotropy of crystals is the phenomenon of birefringence in which the phase velocity of an optical beam propagating in the crystal depends on the direction of polarization of its **e** vector. Before treating this problem mathematically, we may pause and ponder its physical origin. In an isotropic medium the induced polarization is independent of the field direction so that $\chi_{11} = \chi_{22} = \chi_{33}$, and, using (1.4-4), $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon$. Since $c = (\mu\varepsilon)^{-1/2}$, the phase velocity is independent of the direction of polarization. In an anisotropic medium the situation is different. Consider, for example, a wave propagating along z. If its electric field is parallel to x, it will induce, according to (1.4-2), only P_x and will consequently "see" an electric permeability ε_{11}. Its phase velocity will thus be $c_x = (\mu\varepsilon_{11})^{-1/2}$. If, on the other hand, the wave is polarized parallel to y, it will propagate with a phase velocity $c_y = (\mu\varepsilon_{22})^{-1/2}$.

Birefringence has some interesting consequences. Consider, as an example, a wave propagating along the crystal z direction and having at some plane, say $z = 0$, a linearly polarized field with equal components along x and y. Since $k_x \neq k_y$, as the wave propagates into the crystal the x and y components get out of phase and the wave becomes elliptically polarized. This phenomenon is discussed in detail in Section 9.2 and forms the basis of the electrooptic modulation of light.

Returning to the example of a wave propagating along the crystal z direction, let us assume, as in Section 1.3, that the only nonvanishing field components are e_x and h_y. Maxwell's curl equations (1.3-5) and (1.3-4) reduce, in a self-consistent manner, to

$$\begin{aligned}\frac{\partial e_x}{\partial z} &= -\mu \frac{\partial h_y}{\partial t}\\ \frac{\partial h_y}{\partial z} &= -\varepsilon_{11} \frac{\partial e_x}{\partial t}\end{aligned} \tag{1.4-5}$$

Taking the derivative of the first of Equations (1.4-5) with respect to z and then substituting the second equation for $\partial h_y/\partial z$ gives

$$\frac{\partial^2 e_x}{\partial z^2} = \mu\varepsilon_{11} \frac{\partial^2 e_x}{\partial t^2} \tag{1.4-6}$$

If we postulate, as in (1.3-10), a solution in the form

$$e_x = E_x e^{i(\omega t - k_x z)} \tag{1.4-7}$$

then Equation (1.4-6) becomes

$$k_x^2 E_x = \omega^2 \mu \varepsilon_{11} E_x$$

Therefore, the propagation constant of a wave polarized along x and traveling along z is

$$k_x = \omega\sqrt{\mu\varepsilon_{11}} \tag{1.4-8}$$

Repeating the derivation but with a wave polarized along the y axis, instead of the x axis, yields $k_y = \omega\sqrt{\mu\varepsilon_{22}}$.

Index Ellipsoid

As shown above, in a crystal the phase velocity of a wave propagating along a given direction depends on the direction of its polarization. For propagation along z, as an example, we found that Maxwell's equations admitted two solutions: one with its linear polarization along x and the second along y. If we consider the propagation along some arbitrary direction in the crystal, the problem becomes more difficult. We have to determine the directions of polarization of the two allowed waves, as well as their phase velocities. This is done most conveniently using the so-called index ellipsoid

$$\frac{x^2}{\varepsilon_{11}/\varepsilon_0} + \frac{y^2}{\varepsilon_{22}/\varepsilon_0} + \frac{z^2}{\varepsilon_{33}/\varepsilon_0} = 1 \tag{1.4-9}$$

This is the equation of a generalized ellipsoid with major axes parallel to x, y, and z whose respective lengths are $2\sqrt{\varepsilon_{11}/\varepsilon_0}$, $2\sqrt{\varepsilon_{22}/\varepsilon_0}$, and $2\sqrt{\varepsilon_{33}/\varepsilon_0}$. The procedure for finding the polarization directions and the corresponding phase velocities for a *given* direction of propagation is as follows: Determine the ellipse formed by the intersection of a plane through the origin and normal to the direction of propagation and the index ellipsoid (1.4-9). The directions of the major and minor axes of this ellipse are those of the two allowed polarizations,[5] and the lengths of these axes are $2n_1$ and $2n_2$, where n_1 and n_2 are the indices of refraction of the two allowed solutions. The two waves propagate, thus, with phase velocities c_0/n_1 and c_0/n_2, respectively, where $c_0 = (\mu_0\varepsilon_0)^{-1/2}$ is the phase velocity in vacuum. A formal proof of this procedure is given in References [2–4].

To illustrate the use of the index ellipsoid, consider the case of a uniaxial crystal (that is, a crystal with a single axis of threefold, fourfold, or sixfold symmetry). Taking the direction of this axis as z, symmetry considerations dictate that $\varepsilon_{11} = \varepsilon_{22}$.[6] Defining the principal indices of refraction n_o and n_e

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[5]These are actually the directions of the **D**, not of the **E**, vector. In a crystal these two are separated, in general, by a small angle; see References [2] and [3].

[6]See, for example, J. F. Nye, *Physical Properties of Crystals*. New York: Oxford University Press, 1957.

by

$$n_o^2 \equiv \frac{\varepsilon_{11}}{\varepsilon_0} = \frac{\varepsilon_{22}}{\varepsilon_0} \qquad n_e^2 \equiv \frac{\varepsilon_{33}}{\varepsilon_0} \tag{1.4-10}$$

the equation of the index ellipsoid (1.4-9) becomes

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1 \tag{1.4-11}$$

This is an ellipsoid of revolution with the circular symmetry axis parallel to z. The z major axis of the ellipsoid is of length $2n_e$, whereas that of the x and y axes is $2n_o$. The procedure of using the index ellipsoid is illustrated by Figure 1-1.

The direction of propagation is along s and is at an angle θ to the (optic) z axis. Because of the circular symmetry of (1.4-11) about z, we can choose, without any loss of generality, the y axis to coincide with the projection of s on the x-y plane. The intersection ellipse of the plane normal to s with the ellipsoid is shaded in the figure. The two allowed polarization directions are parallel to the axes of the ellipse and thus correspond to the line segments OA and OB. They are consequently perpendicular to s as well as to each other. The two waves polarized along these directions have, respectively,

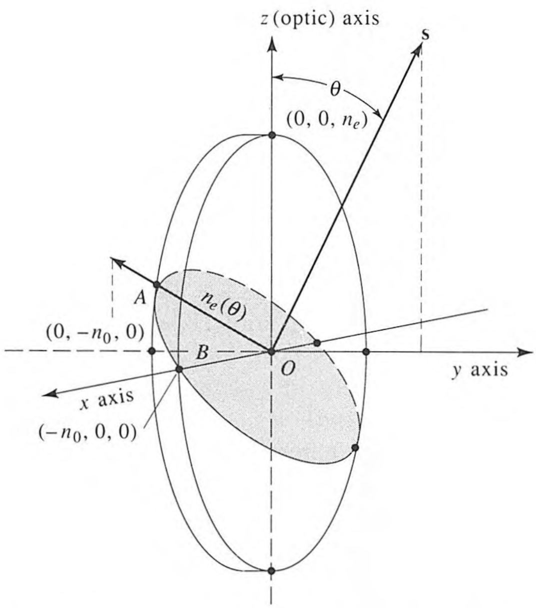


Figure 1-1 Construction for finding indices of refraction and allowed polarization for a given direction of propagation s. The figure shown is for a uniaxial crystal with $n_x = n_y = n_o$.

indices of refraction given by $n_e(\theta) = |OA|$ and $n_o = |OB|$. The first of these two waves, which is polarized along OA, is called the *extraordinary wave*. Its direction of polarization varies with θ following the intersection point A. Its index of refraction is given by the length of OA. It can be determined using Figure 1-2, which shows the intersection of the index ellipsoid with the y-z plane.

Using the relations

$$n_e^2(\theta) = z^2 + y^2$$

$$\frac{z}{n_e(\theta)} = \sin\theta$$

and the equation of the ellipse

$$\frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1$$

we obtain

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2} \qquad (1.4\text{-}12)$$

Thus, for $\theta = 0°$, $n_e(0°) = n_o$, and for $\theta = 90°$, $n_e(90°) = n_e$.

The ordinary wave remains, according to Figure 1-1, polarized along the same direction OB independent of θ. It has an index of refraction n_o. The amount of birefringence $n_e(\theta) - n_o$ thus varies from zero for $\theta = 0°$ (that is, propagation along the optic axis) to $n_e - n_o$ for $\theta = 90°$.

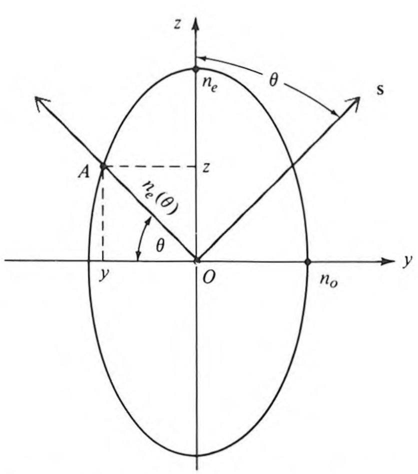


Figure 1-2 Intersection of the index ellipsoid with the z-y plane. $|OA| = n_e(\theta)$ is the index of refraction of the extraordinary wave propagating in the direction s.

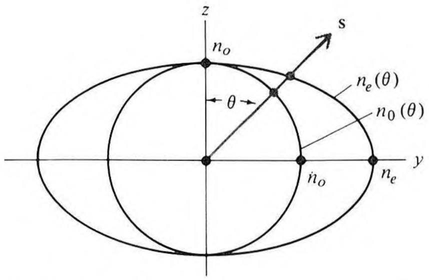


Figure 1-3 Intersection of s-z plane with normal surfaces of a positive uniaxial crystal ($n_e > n_o$).

Normal (index) Surfaces

Consider the surface in which the distance of a given point from the origin is equal to the index of refraction of a wave propagating along this direction. This surface, not to be confused with the index ellipsoid, is called the normal surface. It is constructed using the index ellipsoid (Figure 1-1). The normal surface of the extraordinary ray is constructed by measuring along each direction $\mathbf{s}(\theta, \phi)$ the corresponding index n_e (θ, ϕ), which is the distance OA in Figure 1-1. For a uniaxial crystal, this results in an ellipsoid of revolution about the z axis as illustrated by the outer line in Figure 1-3. For the ordinary ray we plot the distance $OB = n_0$ (which is independent of θ, ϕ), resulting in the inner sphere of Figure 1-3.

1.5 JONES CALCULUS AND ITS APPLICATION TO PROPAGATION IN OPTICAL SYSTEMS WITH BIREFRINGENT CRYSTALS

Many sophisticated optical systems, such as electrooptic modulators (to be discussed in Chapter 9) involve the passage of light through a train of polarizers and birefringent (retardation) plates. The effect of each individual element, either polarizer or retardation plate, on the polarization state of the transmitted light can be described by simple means. However, when an optical system consists of many such elements, each oriented at a different azimuthal angle, the calculation of the overall transmission becomes complicated and is greatly facilitated by a systematic approach. The Jones calculus, invented in 1940 by R. C. Jones [5], is a powerful matrix method in which the state of polarization is represented by a two-component vector, while each optical element is represented by a 2 × 2 matrix. The overall transfer matrix for the whole system is obtained by multiplying all the individual element matrices, and the polarization state of the transmitted light is computed by multiplying the vector representing the input beam by the overall matrix. We will first develop the mathematical formulation of the Jones matrix method and then apply it to some cases of practical interest.

We have shown in the previous section that a unidirectional light propagation in a birefringent crystal generally consists of a linear superposition of two orthogonally polarized waves—the eigenwaves. These eigenwaves, for a given direction of propagation, have well-defined phase velocities and directions of polarization. The birefringent crystals may be either uniaxial ($n_x = n_y, n_z$) or biaxial ($n_x \neq n_y \neq n_z$). However, the most commonly used materials, such as calcite and quartz, are uniaxial. In a uniaxial crystal, these eigenwaves are the so-called *ordinary* and *extraordinary* waves, whose properties were derived in Section 1.4. The directions of polarization for these eigenwaves are mutually orthogonal and are called the *slow* and *fast* axes of the crystal for the given direction of propagation. Retardation plates are usually cut in such a way that the c axis lies in the plane of the plate surfaces. Thus the propagation direction of normally incident light is perpendicular to the c axis.

Retardation plates (also called wave plates) are polarization-state converters, or transformers. The polarization state of a light beam can be converted to any other polarization state by using a suitable retardation plate. In formulating the Jones matrix method, we assume that there is no reflection of light from either surface of the plate and the light is totally transmitted through the plate surfaces. In practice, there is some reflection, though most retardation plates are coated to reduce the surface reflection loss. Referring to Figure 1.4, we consider a light beam that is incident normally on a re-

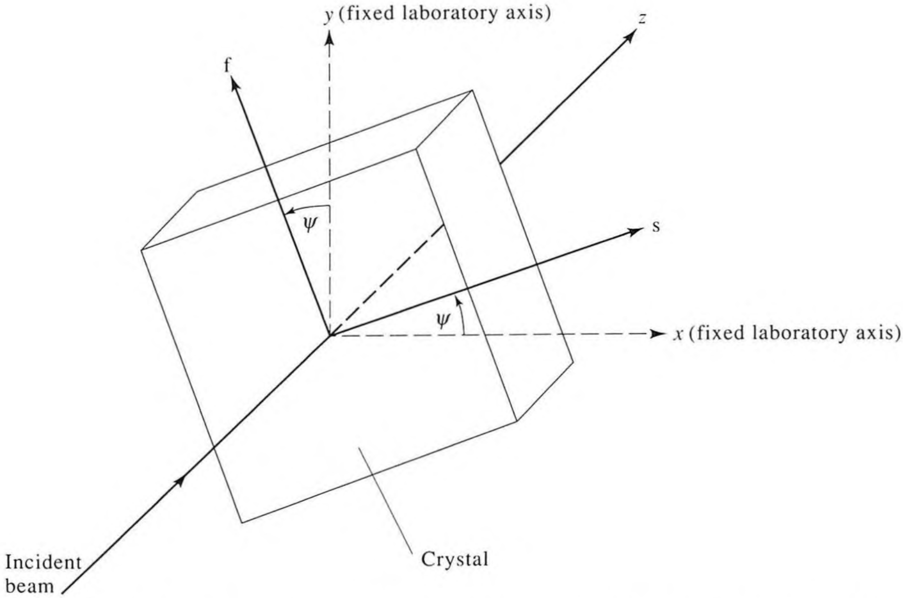


Figure 1-4 A retardation plate rotated at an angle ψ about the z axis. f("fast") and s("slow") are the two principal dielectric axes of the crystal for light propagating along z (see Section 1.4). The x and y axes are fixed in the laboratory frame.

tardation plate along the z axis with a polarization state described by the Jones column vector

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} \tag{1.5-1}$$

where V_x and V_y are two complex numbers representing the complex field amplitudes along x and y. The x, y and z axes are *fixed* laboratory axes. To determine how the light propagates in the retardation plate, we need to resolve it into a linear combination of the fast and slow eigenwaves of the crystal. This is done by the coordinate transformation

$$\begin{pmatrix} V_s \\ V_f \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} \equiv R(\psi) \begin{pmatrix} V_x \\ V_y \end{pmatrix} \tag{1.5-2}$$

V_s is the slow component of the polarization vector $\mathbf{V}$, whereas V_f is the fast component. The slow and fast axes are fixed in the crystal. The angle between the fast axis and the y direction is ψ. These two components are eigenwaves of the retardation plate and will propagate with their own phase velocities and polarizations as discussed in Section 1.4. Because of the difference in phase velocity, the two components undergo a different phase delay in passage through the crystal. This retardation changes the polarization state of the emerging beam.

Let n_s and n_f be the refractive indices of the slow and fast eigenwaves, respectively. The polarization state of the emerging beam in the crystal coordinate system is thus given by

$$\begin{pmatrix} V_s' \\ V_f' \end{pmatrix} = \begin{pmatrix} \exp\left(-in_s \dfrac{\omega}{c} l\right) & 0 \\ 0 & \exp\left(-in_f \dfrac{\omega}{c} l\right) \end{pmatrix} \begin{pmatrix} V_s \\ V_f \end{pmatrix} \tag{1.5-3}$$

where l is the thickness of the plate and ω is the radian frequency of the light beam. The phase retardation is defined as the difference of the phase delays (exponents) in (1.5-3)

$$\Gamma = (n_s - n_f) \frac{\omega l}{c} \tag{1.5-4}$$

Notice that the phase retardation Γ is a measure of the relative change in phase, not the absolute change. The birefringence of a typical crystal retardation plate is small, that is, $|n_s - n_f| << n_s, n_f$. Consequently, the absolute change in phase caused by the plate may be hundreds of times greater than the phase retardation. Let ϕ be the mean absolute phase change

$$\phi = \tfrac{1}{2}(n_s + n_f) \frac{\omega l}{c} \tag{1.5-5}$$

Then Equation (1.5-3) can be written in terms of ϕ and Γ as

$$\begin{pmatrix} V'_s \\ V'_f \end{pmatrix} = e^{-i\phi} \begin{pmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{pmatrix} \begin{pmatrix} V_s \\ V_f \end{pmatrix} \tag{1.5-6}$$

The Jones vector of the polarization state of the emerging beam in the xy coordinate system is given by transforming back from the crystal to the laboratory coordinate system

$$\begin{pmatrix} V'_x \\ V'_y \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} V'_s \\ V'_f \end{pmatrix} \tag{1.5-7}$$

By combining Equations (1.5-2), (1.5-6), and (1.5-7), we can write the transformation due to the retardation plate as

$$\begin{pmatrix} V'_x \\ V'_y \end{pmatrix} = R(-\psi)\, W_0 R(\psi) \begin{pmatrix} V_x \\ V_y \end{pmatrix} \tag{1.5-8}$$

where $R(\psi)$ is the rotation matrix of (1.5-2) and W_0 is the Jones matrix of (1.5-6) for the retardation plate. These are given, respectively, by

$$R(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \tag{1.5-9}$$

and

$$W_0 = e^{-i\phi} \begin{pmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{pmatrix} \tag{1.5-10}$$

The phase factor $e^{-i\phi}$ can usually be left out.[7] A retardation plate, characterized by its phase retardation Γ and its azimuth angle ψ, is represented by the product of three matrices

$$\begin{aligned} W &= R(-\psi) W_0 R(\psi) \\ &= \begin{vmatrix} e^{-i(\Gamma/2)} \cos^2\psi + e^{i(\Gamma/2)} \sin^2\psi & -i \sin\frac{\Gamma}{2} \sin(2\psi) \\ -i \sin\frac{\Gamma}{2} \sin(2\psi) & e^{-i(\Gamma/2)} \sin^2\psi + e^{i(\Gamma/2)} \cos^2\psi \end{vmatrix} \end{aligned} \tag{1.5-11}$$

Note that the Jones matrix of a wave plate is a unitary matrix, that is,

$$W^\dagger W = 1$$

where the dagger † signifies the Hermitian conjugate ($W^*_{ij} = (W^\dagger)_{ji}$). The passage of a polarized light beam through a wave plate is described mathematically as a unitary transformation. Many physical properties are invar-

[7]The overall phase factor exp $(-i\phi)$ is only important when the output field $\mathbf{V}'$ is combined coherently with another field.

iant under unitary transformations; these include the orthogonal relation between the Jones vectors and the magnitude of the Jones vectors. Thus, if the polarization states of two beams are mutually orthogonal, they will remain orthogonal after passing through an arbitrary wave plate.

The Jones matrix of an ideal, homogeneous, linear, thin platelet polarizer oriented with its transmission axis parallel to the laboratory x axis is

$$P_0 = e^{-i\phi}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.5-12}$$

where ϕ is the absolute phase accumulated due to the finite optical thickness of the polarizer. The Jones matrix of a polarizer rotated by an angle ψ about z is given by

$$P = R(-\psi)P_0R(\psi) \tag{1.5-13}$$

Thus, if we neglect the (in this case unimportant) absolute phase ϕ, the Jones matrix representations of the polarizers oriented so as to transmit light with electric field vectors parallel to the x and y laboratory axes, respectively, are given by

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.5-14}$$

To find the effect of an arbitrary train of retardation plates and polarizers on the polarization state of polarized light, we multiply the Jones vector of the incident beam by the ordered product of the matrices of the various elements.

Example: A Half-Wave Retardation Plate

A half-wave plate has a phase retardation of $\Gamma = \pi$. According to Equation (1.5-4), an x-cut[8] (or y-cut) uniaxial crystal will act as a half-wave plate, provided the thickness is $t = \lambda/2(n_e - n_0)$. We will determine the effect of a half-wave plate on the polarization state of a transmitted light beam. The azimuth angle of the wave plate is taken as 45° and the incident beam as vertically (y) polarized. The Jones vector for the incident beam can be written as

$$\mathbf{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.5-15}$$

[8] A crystal plate is called *x-cut* if its facets are perpendicular to the principal x axis.

and the Jones matrix for the half-wave plate is obtained by using Equation (1.5-11) with $\Gamma = \pi$, $\psi = \pi/4$

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{1.5-16}$$

The Jones vector for the emerging beam is obtained by multiplying Equations (1.5-16) and (1.5-15); the result is

$$\mathbf{V}' = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.5-17}$$

which corresponds to horizontally (x) polarized light. The effect of the half-wave plate is thus to rotate the input polarization by 90°. It can be shown that for a general azimuth angle ψ, the half-wave plate will rotate the polarization by an angle 2ψ (see Problem 1.7a). In other words, linearly polarized light remains linearly polarized, except that the plane of polarization is rotated by an angle of 2ψ.

When the incident light is circularly polarized, a half-wave plate will convert right-hand circularly polarized light into left-hand circularly polarized light and vice versa, regardless of the azimuth angle. The proof is left as an exercise (see Problem 1.7). Figure 1.5 illustrates the effect of a half-wave plate.

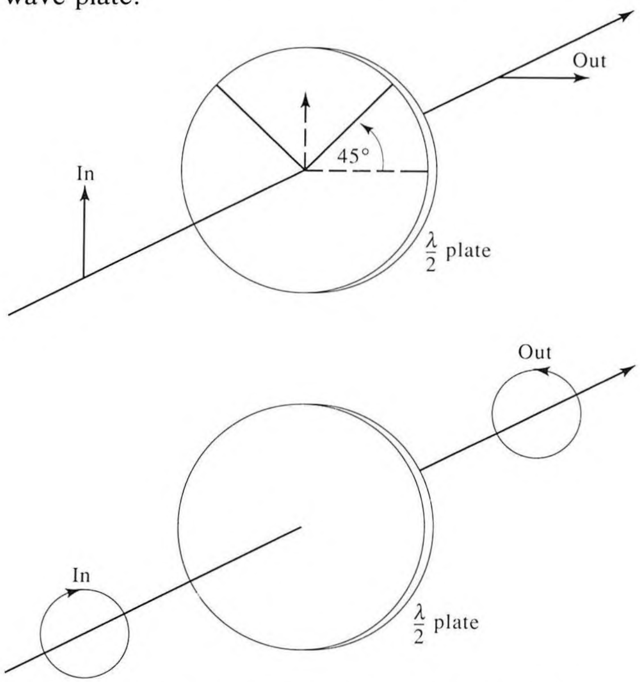


Figure 1-5 The effect of a half-wave plate on the polarization state of a beam.

Example: A Quarter-Wave Plate

A quarter-wave plate has a phase retardation of $\Gamma = \pi/2$. If the plate is made of an x-cut (or y-cut) uniaxially anistropic crystal, the thickness is $t = \lambda/4\ (n_e - n_0)$ (or odd multiples thereof). Suppose again that the azimuth angle of the plate is $\psi = 45°$ and the incident beam is vertically polarized. The Jones vector for the incident beam is given by Equation (1.5-15). The Jones matrix for this quarter-wave plate is

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{1.5-18}$$

The Jones vector of the emerging beam is obtained by multiplying Equations (1.5-18) and (1.5-15) and is given by

$$\mathbf{V}' = -\frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{1.5-19}$$

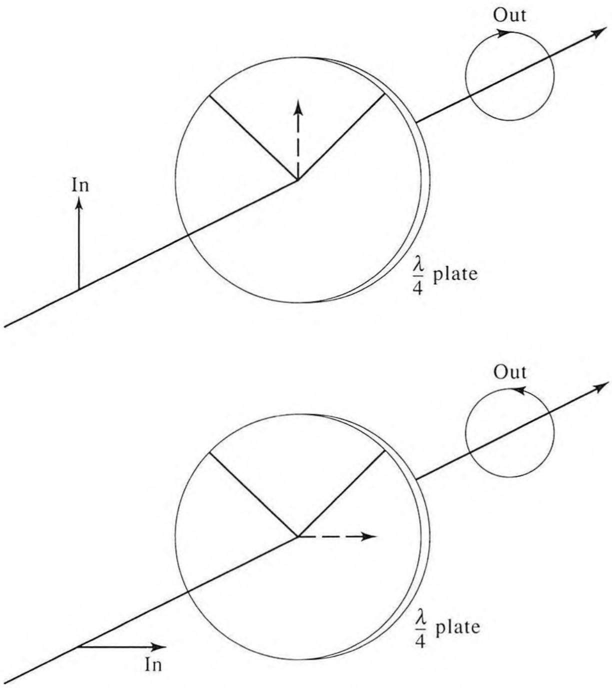


Figure 1-6 The effect of a quarter-wave plate on the polarization state of a linearly polarized input wave.

This is a left-hand circularly polarized light. The effect of a 45°-oriented quarter-wave plate is thus to convert vertically polarized light into left-hand circularly polarized light. If the incident beam is horizontally polarized, the emerging beam will be right-hand circularly polarized. The effect of this quarter-wave plate is illustrated in Figure 1-6.

Intensity Transmission

Up to this point our development of the Jones calculus was concerned with the polarization state of the light beam. In many cases, we need to determine the transmitted intensity. The combination of retardation plates and polarizers is often used to control or modulate the transmitted optical intensity. Because the phase retardation of each wave plate is wavelength-dependent, the polarization state of the emerging beam and its intensity (when polarizers are present) depend on the wavelength of the light. Let us represent the field as a Jones vector

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} \tag{1.5-20}$$

The intensity is taken using (1.1-12) and (1.3-24) as proportional to:

$$I = \mathbf{V} \cdot \mathbf{V}^* = |V_x|^2 + |V_y|^2 \tag{1.5-21}$$

If the output beam is given by

$$\mathbf{V}' = \begin{pmatrix} V'_x \\ V'_y \end{pmatrix} \tag{1.5-22}$$

the transmissivity of the optical system is calculated as

$$\frac{|V'_x|^2 + |V'_y|^2}{|V_x|^2 + |V_y|^2} \tag{1.5-23}$$

Example: A Birefringent Plate Sandwiched between Parallel Polarizers

Referring to Figure 1-7, we consider a birefringent plate sandwiched between a pair of parallel polarizers. The plate is oriented so that the slow and fast axes are at 45° with respect to the polarizer. Let the birefringence be $n_e - n_0$ and the plate thickness be d. The phase retardation is then given by

$$\Gamma = 2\pi(n_e - n_0)\frac{d}{\lambda} \tag{1.5-24}$$

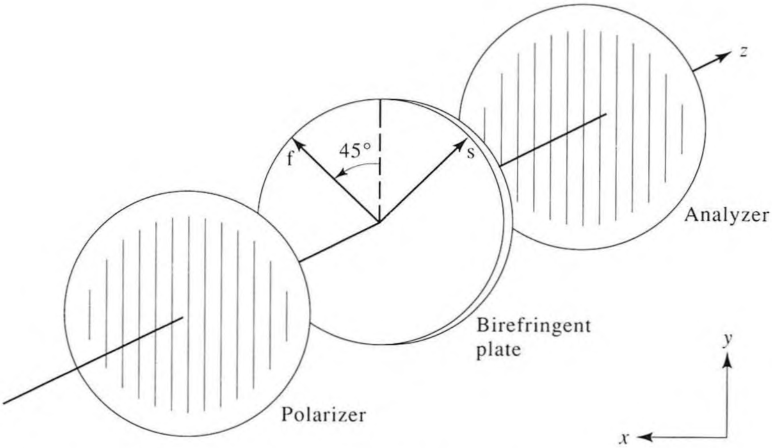


Figure 1-7 A birefringent plate sandwiched between a pair of parallel polarizers.

and the corresponding Jones matrix is, according to Equation (1.5-11), with $\psi = 45°$

$$W = \begin{pmatrix} \cos\frac{1}{2}\Gamma & -i\sin\frac{1}{2}\Gamma \\ -i\sin\frac{1}{2}\Gamma & \cos\frac{1}{2}\Gamma \end{pmatrix} \tag{1.5-25}$$

The incident beam, after it passes through the front polarizer, is polarized parallel to y and can be represented by

$$\mathbf{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.5-26}$$

we shall take, arbitrarily, the intensity corresponding to (1.5-26) as unity. The Jones vector representation of the electric field vector of the transmitted beam is obtained as follows:

$$\begin{aligned} \mathbf{V}' &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\Gamma & -i\sin\frac{1}{2}\Gamma \\ -i\sin\frac{1}{2}\Gamma & \cos\frac{1}{2}\Gamma \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \cos\frac{1}{2}\Gamma \end{pmatrix} \end{aligned} \tag{1.5-27}$$

The transmitted beam is y polarized with an intensity given by

$$I = \cos^2\tfrac{1}{2}\Gamma = \cos^2\left[\frac{\pi(n_e - n_0)d}{\lambda}\right] \tag{1.5-28}$$

It can be seen from Equation (1.5-27) that the transmitted intensity is a sinusoidal function of the wave number (λ^{-1}) and peaks at $\lambda = (n_e - n_0)d$, $(n_e - n_0)d/2$, $(n_e - n_0)d/3, \ldots$. The wave-number separation between transmission maxima increases with decreasing plate thickness.

Example: A Birefringent Plate Sandwiched between a Pair of Crossed Polarizers

If we rotate the analyzer shown in Figure 1-7 by 90°, then the input and output polarizers are crossed. The transmitted beam for this case is obtained as follows:

$$\mathbf{V}' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\Gamma & -i\sin\frac{1}{2}\Gamma \\ -i\sin\frac{1}{2}\Gamma & \cos\frac{1}{2}\Gamma \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i \begin{pmatrix} \sin\frac{\Gamma}{2} \\ 0 \end{pmatrix} \tag{1.5-29}$$

The transmitted beam is horizontally (x) polarized with an intensity relative to the input value given by

$$\frac{I_{\text{out}}}{I_{\text{in}}} = \sin^2\tfrac{1}{2}\Gamma = \sin^2\left[\frac{\pi(n_e - n_0)\,d}{\lambda}\right] \tag{1.5-30}$$

This is again a sinusoidal function of the wave number. The transmission spectrum consists of a series of maxima at $\lambda = 2(n_e - n_0)d$, $2(n_e - n_0)d/3, \ldots$. These wavelengths correspond to phase retardations of $\pi, 3\pi, 5\pi, \ldots$, that is, when the wave plate becomes a "half-wave" plate or odd integral multiples of a half-wave plate.

Circular Polarization Representation

Up to this point we represented the state of the propagating field as a vector $\mathbf{V}$ [Equation (1.5-1)] with components V_x and V_y.

$$\mathbf{V} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} \tag{1.5-31}$$

The orthogonal unit vectors (*basis* vector set) in this representation are

$$\mathbf{V}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{V}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.5-32}$$

The above choice is most convenient when dealing with birefringent crystals, since the propagating eigenmodes in this case are linearly and orthogonally polarized. It is often more convenient to express the field in terms of "basis" vectors that are circularly polarized [6]. This is the case, for example, when we propagate through a magnetic medium. We define a wave of unit amplitude seen rotating in the CCW sense by an observer gazing along the $+z$ axis as $\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, while $\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ denotes a CW rotating wave. As in the case of the linearly polarized basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, $\begin{Bmatrix}1\\0\end{Bmatrix}$ and $\begin{Bmatrix}0\\1\end{Bmatrix}$ constitute a complete set that can be used to describe a transverse field of arbitrary polarization. Let $\mathbf{V}$ be some such field. We can write

$$\mathbf{V} = V_x \begin{pmatrix}1\\0\end{pmatrix} + V_y \begin{pmatrix}0\\1\end{pmatrix} \equiv \begin{pmatrix}V_x\\V_y\end{pmatrix} \tag{1.5-33}$$

or alternatively

$$\mathbf{V} = V_+ \begin{Bmatrix}1\\0\end{Bmatrix} + V_- \begin{Bmatrix}0\\1\end{Bmatrix} \equiv \begin{Bmatrix}V_+\\V_-\end{Bmatrix} \tag{1.5-34}$$

The $\begin{pmatrix}V_x\\V_y\end{pmatrix}$ and $\begin{Bmatrix}V_+\\V_-\end{Bmatrix}$ representations of a given vector can be derived from each other by a 2×2 matrix[9]

$$\begin{Bmatrix}V_+\\V_-\end{Bmatrix} = \frac{1}{2}\begin{vmatrix}1 & i\\1 & -i\end{vmatrix}\begin{pmatrix}V_x\\V_y\end{pmatrix} \equiv T\begin{pmatrix}V_x\\V_y\end{pmatrix} \tag{1.5-35}$$

$$\begin{pmatrix}V_x\\V_y\end{pmatrix} = \begin{vmatrix}1 & 1\\-i & i\end{vmatrix}\begin{Bmatrix}V_+\\V_-\end{Bmatrix} \equiv S\begin{Bmatrix}V_+\\V_-\end{Bmatrix} \tag{1.5-36}$$

so that $T = S^{-1}$. As an example, consider a (unit) field polarized along x. Its rectangular representation is $\begin{pmatrix}1\\0\end{pmatrix}$, while its rotating representation is

$$\begin{Bmatrix}V_+\\V_-\end{Bmatrix} = \begin{vmatrix}1 & i\\1 & -i\end{vmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{Bmatrix}1\\1\end{Bmatrix} \tag{1.5-37}$$

i.e., equal and in-phase admixture of the two counter-rotating eigenmodes. Conversely, a clockwise, circularly polarized unit wave $\begin{Bmatrix}0\\1\end{Bmatrix}$, for example, is expressed in the rectangular representation by

$$\begin{pmatrix}V_x\\V_y\end{pmatrix} = \begin{vmatrix}1 & 1\\-i & i\end{vmatrix}\begin{Bmatrix}0\\1\end{Bmatrix} = \begin{pmatrix}1\\i\end{pmatrix} \tag{1.5-38}$$

Faraday Rotation In certain optical materials containing magnetic atoms or ions, the natural modes of propagation are the two counter-rotating, circularly-polarized (CP) waves described above. The z direction is usually that of an applied magnetic field or that of the spontaneous magnetization. As in the case of a birefringent crystal, the two CP modes propagate with different phase velocities or, equivalently, have different indices of refrac-

[9]The form of T implies that at $t=0$ the rotating waves $\begin{Bmatrix}1\\0\end{Bmatrix}$ and $\begin{Bmatrix}0\\1\end{Bmatrix}$ are parallel to the x axis.

tion. This difference is due to the fact that the individual atomic magnetic moments precess in a unique sense about the z axis and thus interact differently (have slightly displaced resonances) with the two CP waves. Using the notation of (1.5-34), we can describe the propagation of a wave with arbitrary transverse polarization by first resolving it, at $z = 0$, into its components $\begin{Bmatrix} V_+(0) \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} 0 \\ V_-(0) \end{Bmatrix}$ and propagating each component with its appropriate phase delay through the magnetic medium

$$\begin{Bmatrix} V_+(z) \\ V_-(z) \end{Bmatrix} = \begin{Bmatrix} V_+(0) \\ 0 \end{Bmatrix} e^{-i(\omega/c)n_+z} + \begin{Bmatrix} 0 \\ V_-(0) \end{Bmatrix} e^{-i(\omega/c)n_-z}$$

$$= e^{-(i/2)(\theta_+ + \theta_-)} \begin{vmatrix} e^{(i/2)(\theta_- - \theta_+)} & 0 \\ 0 & e^{-(i/2)(\theta_- - \theta_+)} \end{vmatrix} \begin{Bmatrix} V_+(0) \\ V_-(0) \end{Bmatrix} \tag{1.5-39}$$

where $\theta_\pm \equiv (\omega/c)\, n_\pm z$ is the phase delay for the $(+)$ or $(-)$ circularly polarized wave. Ignoring the prefactor $\exp[-(i/2)(\theta_+ + \theta_-)]$ (it is only relative phase delays that are of interest here) we rewrite (1.5-39) as

$$\begin{Bmatrix} V_+\ (z) \\ V_-\ (z) \end{Bmatrix} = \begin{vmatrix} e^{i\theta_F(z)} & 0 \\ 0 & e^{-i\theta_F(z)} \end{vmatrix} \begin{Bmatrix} V_+(0) \\ V_-(0) \end{Bmatrix} \tag{1.5-40}$$

$$\theta_F(z) \equiv \frac{1}{2}(\theta_- - \theta_+) = \frac{\omega}{2c}(n_- - n_+)z \tag{1.5-41}$$

$$\equiv \text{Faraday rotation angle}$$

The reason for calling θ_F the *Faraday rotation angle* becomes clear if we consider the effect of a magnetic medium on an incident wave that is described in the rectangular component representation

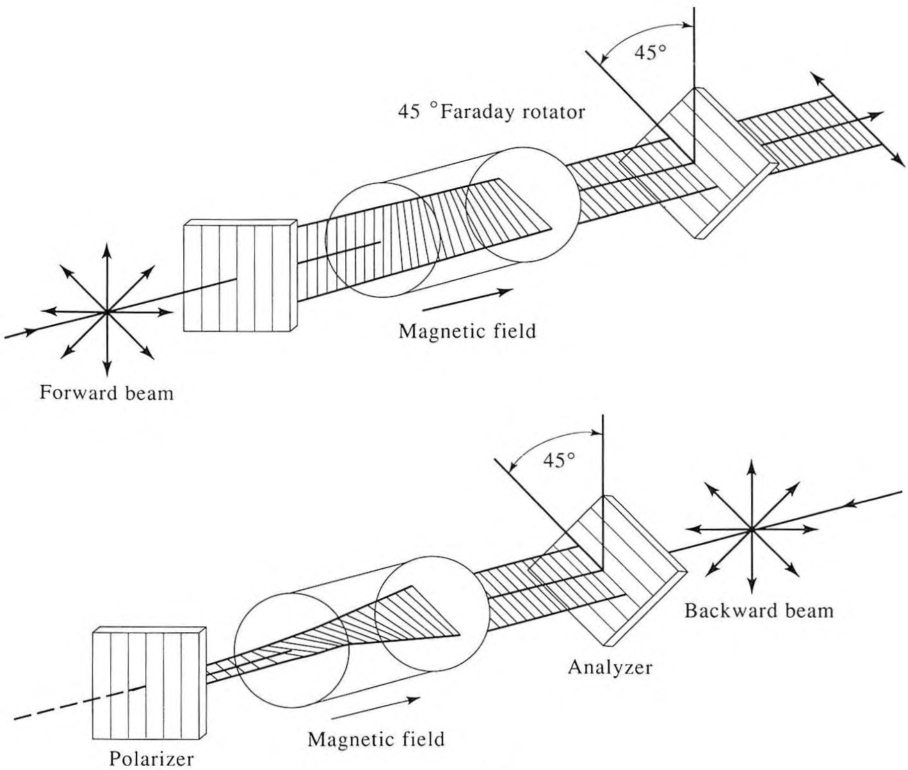
$$\begin{pmatrix} V_x(z) \\ V_y(z) \end{pmatrix} = T^{-1} \begin{vmatrix} e^{i\theta_F(z)} & 0 \\ 0 & e^{-i\theta_F(z)} \end{vmatrix} T \begin{pmatrix} V_x(0) \\ V_y(0) \end{pmatrix}$$

$$= \begin{vmatrix} \cos\theta_F & -\sin\theta_F \\ \sin\theta_F & \cos\theta_F \end{vmatrix} \begin{pmatrix} V_x(0) \\ V_y(0) \end{pmatrix} \tag{1.5-42}$$

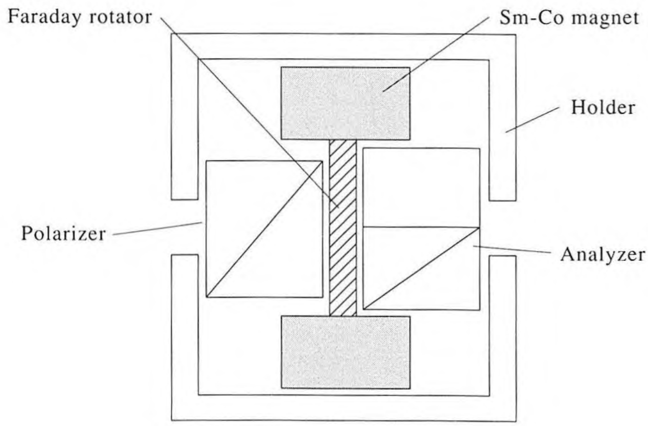
$$= \mathrm{R}(-\theta_F) \begin{pmatrix} V_x(0) \\ V_y(0) \end{pmatrix} \tag{1.5-43}$$

where $R(-\theta_F)$ is, according to (1.5-2), the matrix representing a *rotation* by $-\theta_F$ about the z axis. The output field is thus rotated by $-\theta_F$ with respect to the input field.

There exists a basic difference between propagation in a magnetic medium and in a dielectric birefringent medium. Consider the latter case first. An x'-polarized eigenwave, for instance, propagating along the z direction in a birefringent crystal has a phase velocity c/n_x, where x' is a principal dielectric axis. The same applies to the wave propagating in the reverse direction. The medium is *reciprocal*. In a magnetic medium the story is quite



(a)



(b)

Figure 1-8 (a) A Faraday isolator comprised of two polarizers rotated by 45° relative to each other on either side of a magnetic medium with $\theta_F = 45°$. (b) A cross-sectional view of a practical commercial isolator. (Courtesy of Namiki Precision Jewel Company) different. Let a linearly polarized wave traveling from left to right a distance L (in the $+z$ direction) undergo a (Faraday) rotation of its plane of polarization of $+\theta$ (the sign signifies the sense of the rotation about the *direction of propagation*). A wave traveling in the $-z$ direction in the crystal will experience a rotation of $-\theta(L)$ *about the new direction* ($-z$) *of propagation*. This is because the magnetic field or, equivalently, the magnetic polarization now points in the opposite direction relative to the direction of propagation. (The wave can differentiate between $+z$ and $-z$—something that it cannot do in a birefringent crystal). The medium is termed nonreciprocal. The net effect of a round trip through the medium of length L is that the plane of polarization of the beam returning to the starting, $z = 0$ plane, is rotated by $2\theta_F(L)$. This Faraday rotation is used to make optical isolators to block off back-reflected radiation. The basic configuration of a Faraday isolator is illustrated in Figure 1-8 (a) and (b). A linearly polarized incident wave is rotated by 45° in passage through the Faraday medium and then passed fully by the output polarizer. A reflected wave is rotated an additional 45° in the return trip and is thus blocked off by the input polarizer. Faraday isolators now form an integral part of most optical communication systems employing semiconductor diode lasers since such lasers are extremely sensitive to even small amounts of reflected light that cause instabilities in their power and frequency characteristics.

Problems

1.1 Consider the problem of finding the time average

$$\overline{a^2(t)} = \frac{1}{T}\int_0^T a^2(t)\, dt$$

of

$$\begin{aligned} a(t) &= |A_1| \cos(\omega_1 t + \phi_1) + |A_2| \cos(\omega_2 t + \phi_2) \\ &= \mathrm{Re}[V_a(t)] \end{aligned}$$

where

$$V_a(t) = A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}$$

and $A_{1,2} = |A_{1,2}|e^{i\phi_{1,2}}$. $V_a(t)$ is called the *analytical signal* of $a(t)$. Assume that $(\omega_1 - \omega_2) \ll \omega_1$ and integrate over a time T, which is long compared to the period $2\pi/\omega_{1,2}$ but short compared to the beat period $2\pi/(\omega_1 - \omega_2)$.[10] Show that

$$\overline{a^2(t)} = \tfrac{1}{2}[V_a(t)V_a^*(t)]$$

[10]When this condition is fulfilled, $a(t)$ consists of a sinusoidal function with a "slowly" varying amplitude and is often called a *quasi-sinusoid*.

1.2 Show how we can use the analytic functions as defined by Problem 1.1 to find the time average

$$\overline{a(t)b(t)} = \frac{1}{T}\int_0^T a(t)b(t)dt$$

where $a(t)$ is the same as in Problem 1.1, and the analytic function of $b(t)$ is

$$V_b(t) = [A_3 e^{i\omega_3 t} + A_4 e^{i\omega_4 t}]$$

so that $b(t) = \text{Re}[V_b(t)]$. Assume that the difference between any two of the frequencies ω_1, ω_2, ω_3, and ω_4 is small compared to the frequencies themselves. (*Answer:* $\overline{a(t)b(t)} = \frac{1}{2}\,\text{Re}[V_a(t)V_b^*(t)]$.)

1.3 Derive Equation (1.2-21).

1.4 Starting with Maxwell's curl equations [(1.2-1), (1.2-2)] and taking $\mathbf{i} = 0$, show that in the case of a harmonic (sinusoidal) uniform plane wave, the field vectors $\mathbf{e}$ and $\mathbf{h}$ are normal to each other as well as to the direction of propagation. [*Hint:* Assume the wave to have the form $e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$ and show by actual differentiation that we can formally replace the operator ∇ in Maxwell's equations by $-i\mathbf{k}$.]

1.5 Derive Equation (1.3-19).

1.6 A linearly polarized electromagnetic wave is incident normally at $z = 0$ on the x-y face of a crystal so that it propagates along its z axis. The crystal electric permeability tensor referred to x, y, and z is diagonal with elements ε_{11}, ε_{22}, and ε_{33}. If the wave is polarized initially so that it has equal components along x and y, what is the state of its polarization at the plane z, where

$$(k_x - k_y)z = \frac{\pi}{2}$$

Plot the position of the electric field vector in this plane at times $t = 0$, $\pi/6\omega$, $\pi/3\omega$, $\pi/2\omega$, $2\pi/3\omega$, $5\pi/6\omega$.

1.7 *Half-wave plate.* A half-wave plate has a phase retardation of $\Gamma = \pi$. Assume that the plate is oriented so that the azimuth angle (i.e., the angle between the x axis and the slow axis of the plate) is ψ.

a. Find the polarization state of the transmitted beam, assuming that the incident beam is linearly polarized in the y direction.
b. Show that a half-wave plate will convert right-hand circularly polarized light into left-hand circularly polarized light, and vice versa, regardless of the azimuth angle of the plate.
c. Lithium tantalate ($LiTaO_3$) is a uniaxial crystal with $n_0 = 2.1391$ and $n_e = 2.1432$ at $\lambda = 1\ \mu$m. Find the half-wave-plate thickness at this wave-

length, assuming the plate is cut in such a way that the surfaces are perpendicular to the x axis of the principal coordinate (i.e., x-cut).

1.8 *Quarter-wave plate.* A quarter-wave plate has a phase retardation of $\Gamma = \pi/2$. Assume that the plate is oriented in a direction with azimuth angle ψ.

a. Find the polarization state of the transmitted beam, assuming that the incident beam is polarized in the y direction.
b. If the polarization state resulting from (a) is represented by a complex number on the complex plane, show that the focus of these points as ψ varies from 0 to $\frac{1}{2}\pi$ is a branch of a hyperbola. Obtain the equation of the hyperbola.
c. Quartz ($\alpha = SiO_2$) is a uniaxial crystal with $n_0 = 1.53283$ and $n_e = 1.54152$ at $\lambda = 1.1592$ μm. Find the thickness of an x-cut quartz quarter-wave plate at this wavelength.

1.9 A matrix $\mathbf{A}$ is called unitary if

$$\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}\mathbf{A}^{\dagger} = \mathbf{1}$$

where $\mathbf{1}$ is the unity matrix and the Hermitian conjugate $A^{\dagger}$ of matrix $\mathbf{A}$ is defined by $(A^{\dagger})_{ij} = A_{ji}^*$. Show that if $\mathbf{A}$ is unitary

$$\sum_j A_{ji}^* A_{jk} = \delta_{ik}$$

This property will be needed in Problem 1.10f.

1.10 *Polarization transformation by a wave plate.* A wave plate is characterized by its phase retardation Γ and azimuth angle ψ.

a. Find the polarization state of the emerging beam, assuming that the incident beam is polarized in the x direction.
b. Use a complex number to represent the resulting polarization state obtained in (a).
c. The polarization state of the incident x-polarized beam is represented by a point at the origin of the complex plane. Show that the transformed polarization state can be anywhere on the complex plane, provided Γ can be varied from 0 to 2π and ψ can be varied from 0 to $\frac{1}{2}\pi$. Physically, this means that any polarization state can be produced from linearly polarized light, provided a proper wave plate is available.
d. Show that the focus of these points in the complex plane obtained by rotating a wave plate from $\psi = 0$ to $\psi = \frac{1}{2}\pi$ is a hyperbola. Derive the equation of this hyperbola.
e. Show that the Jones matrix $\mathbf{W}$ of a wave plate is unitary, that is,

$$W^{\dagger}W = 1,\ (W^{\dagger})_{ij} \equiv W_{ji}^*$$

where the dagger indicates Hermitian conjugation [see Equation (1.5-11)].

f. Let $\mathbf{V}_1'$ and $\mathbf{V}_2'$ be the transformed Jones vectors of $\mathbf{V}_1$ and $\mathbf{V}_2$, respectively. Show that if $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal, so are $\mathbf{V}_1'$ and $\mathbf{V}_2'$. ($\mathbf{A}$ and $\mathbf{B}$ are orthogonal if $\mathbf{A} \cdot \mathbf{B}^* = 0$.)

1.11 Show that the (Jones) matrix (in the rectangular eigenwave representation) of a birefringent plate with a retardation Γ that is rotated by an angle ψ from the x axis is

$$W(\Gamma,\psi) = \begin{vmatrix} \cos^2\psi \exp(-i\Gamma/2) + \sin^2\psi \exp(+i\Gamma/2) & -i\sin 2\psi \sin\Gamma/2 \\ -i\sin 2\psi \sin\Gamma/2 & \sin^2\psi \exp(-i\Gamma/2) + \cos^2\psi \exp(+i\Gamma/2) \end{vmatrix}$$

Derive an expression for the intensity transmission through a system consistent of a polarizer $\parallel$ to $\hat{x}$, a Faraday rotator θ, wave plate with retardation Γ rotated an angle ψ from the x axis, and a crossed output polarizer ($\parallel$ to $\hat{y}$).

1.12

a. Show that $(AB)^\dagger = B^\dagger A^\dagger$.

b. Show that if an optical element is represented by a unitary matrix, the intensity of an incident wave of arbitrary polarization is preserved in passage through the element.

c. Show that the matrix representing a train of arbitrary retardation plates is unitary.

1.13

a. Show that in an isotropic medium we can take the general solution of the wave equation of a monochromatic field

$$\nabla^2\mathbf{E} + \mathrm{k}^2\mathbf{E} = 0$$

as

$$\mathbf{E}(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{\infty}^{\infty} dk_x dk_y \mathbf{A}(\mathbf{k}) e^{-i(k_x x + k_y y + \sqrt{k^2 - k_x^2 - k_y^2}z)} \tag{1}$$

where $\mathbf{A}(\mathbf{k})$ is an arbitrary vector lying on a plane normal to $\mathbf{k}$ and $k^2 = \omega^2\mu\epsilon$.

b. Show that if $\mathbf{E}(\mathbf{r})$ is specified at some plane S, say the plane $z = 0$, as $\mathbf{E}(x,y,0)$, then

$$\mathbf{A}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^2 \iint_S dxdy \mathbf{E}(x,y,0) e^{i(k_x x + k_y y)} \tag{2}$$

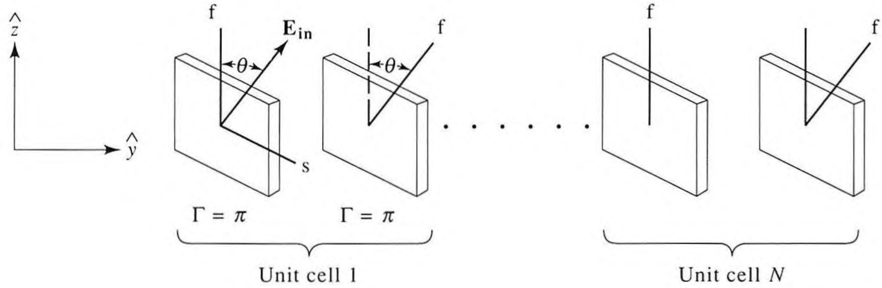
where $\mathbf{k} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}\sqrt{k^2 - k_x^2 - k_y^2}$. [*Hint:* Compare Equations (1) and (2) with $z = 0$ to the integral Fourier transform relationships.]

c. Assume that at $z = 0$ the field is given by

$$E(x,y,0) = E_0 \begin{cases} -a/2 \le x \le a/2 \\ -a/2 \le y \le a/2 \end{cases}$$

and is zero everywhere else. Find the spreading angle of the beam far away to the right of the aperture. [*Hint:* Each $\mathbf{k}$ signifies a direction of propagation so that $|A(\mathbf{k})|^2$ contains information about the angular spread of the beam to the right of the aperture.]

1.14 Consider light propagating through a sequence of $\lambda/2$ retardation plates ($\Gamma = \pi$) as shown:



Each unit cell consists of two plates whose surfaces are normal to $\hat{y}$—one with its f (fast) axis $\parallel$ to $\hat{z}$ and one rotated by ϕ about $\hat{x}$. Find the effect of propagation through N cells on a beam initially polarized as shown. Solve the problem first by simple considerations, if possible, then formally.

1.15 Show that if we define

$$\mathbf{v}_g \equiv \nabla_{\mathbf{k}}\omega(\mathbf{k})$$

$$\mathbf{v}_e \equiv \frac{\mathbf{E} \times \mathbf{H}}{\frac{1}{2}[\mathbf{E} \cdot \boldsymbol{\epsilon}\mathbf{E} + \mathbf{H} \cdot \mu\mathbf{H}]}$$

in a crystal, then

$$\mathbf{v}_g = \mathbf{v}_e$$

a. Recall that $\boldsymbol{\epsilon}$ is a tensor.
b. After giving the problem a real try, you may consult *Optical Waves in Crystals,* A. Yariv and P. Yeh, New York: Wiley, p. 79, 1983.

1.16 Derive the transfer matrix of a polarizer whose transmission direction is rotated by α from the laboratory x axis.

1.17 Prove relation (1.5-33).

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