2 The Propagation of Rays and Beams

2.0 INTRODUCTION

In this chapter we take up the subject of optical ray propagation through a variety of optical media. These include homogeneous and isotropic materials, thin lenses, dielectric interfaces, and curved mirrors. Since a ray is, by definition, normal to the optical wavefront, an understanding of the ray behavior makes it possible to trace the evolution of optical waves when they are passing through various optical elements. We find that the passage of a ray (or its reflection) through these elements can be described by simple 2×2 matrices. Furthermore, these matrices will be found to describe the propagation of spherical waves and of Gaussian beams such as those characteristic of the output of lasers. Gaussian beam propagation is analyzed in second half of the chapter.

2.1 LENS WAVEGUIDE

Consider a paraxial ray[1] passing through a thin lens of focal length f as shown in Figure 2-1. Taking the cylindrical axis of symmetry as z, denoting the ray distance from the axis by r and its slope dr/dz as r', we can relate

[1]By paraxial ray we mean a ray whose angular deviation from the cylindrical (z) axis is small enough that the sine and tangent of the angle can be approximated by the angle itself.

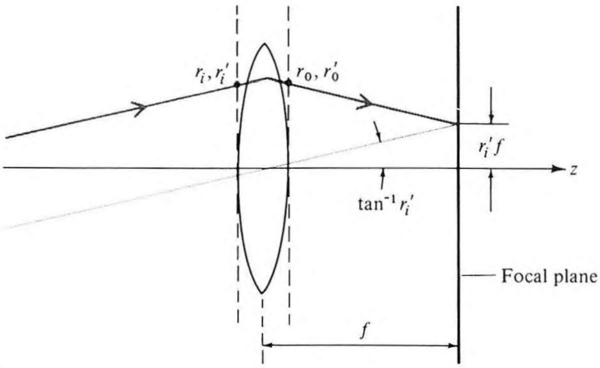


Figure 2-1 Deflection of a ray by a thin lens.

the output ray (r_{out}, r'_{out}) to the input ray (r_{in}, r'_{in}) by means of

$$r_{\text{out}} = r_{\text{in}}$$
$$r'_{\text{out}} = r'_{\text{in}} - \frac{r_{\text{in}}}{f} \tag{2.1-1}$$

where the first of (2.1-1) follows from the definition of a thin lens and the second can be derived from a consideration of the behavior of the undeflected central ray with a slope equal to r'_{in}, as shown in Figure 2-1.

Representing a ray at any position z as a column matrix,

$$\mathbf{r}(z) = \begin{vmatrix} r(z) \\ r'(z) \end{vmatrix}$$

We can rewrite (2.1-1) using the rules for matrix multiplication (see References [1]–[3]) as

$$\begin{vmatrix} r_{\text{out}} \\ r'_{\text{out}} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix} \begin{vmatrix} r_{\text{in}} \\ r'_{\text{in}} \end{vmatrix} \tag{2.1-2}$$

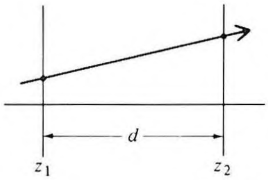
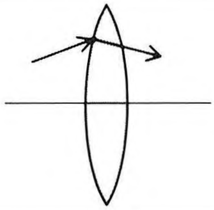
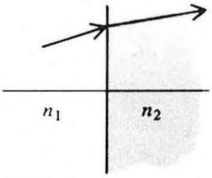
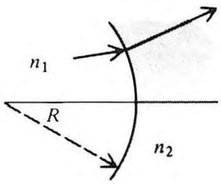
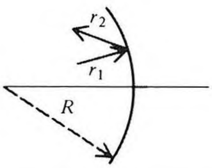
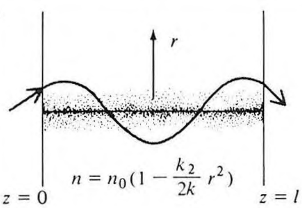
where $f > 0$ for a converging lens and is negative for a diverging one.

The ray matrices for a number of other optical elements are shown in Table 2-1.

Consider as an example the propagation of a ray through a straight section of a homogeneous medium of length d followed by a thin lens of focal length f. This corresponds to propagation between planes a and b in Figure 2-2. Since the effect of the straight section is merely that of increasing r by dr', using (2.1-1) we can relate the output b and input (at a) rays by:

$$\begin{vmatrix} r_{\text{out}} \\ r'_{\text{out}} \end{vmatrix} = \begin{vmatrix} 1 & d \\ -\frac{1}{f} & \left(1 - \frac{d}{f}\right) \end{vmatrix} \begin{vmatrix} r_{\text{in}} \\ r'_{\text{in}} \end{vmatrix} \tag{2.1-3}$$

Table 2-1 Ray Matrices for Some Common Optical Elements and Media

(1) Straight Section: Length d		$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
(2) Thin Lens: Focal length f ($f > 0$, converging; $f < 0$, diverging)		$\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$
(3) Dielectric Interface: Refractive indices n_1, n_2		$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$
(4) Spherical Dielectric Interface: Radius R		$\begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$
(5) Spherical Mirror: Radius of curvature R		$\begin{bmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{bmatrix}$
(6) A medium with a quadratic index profile		$\begin{bmatrix} \cos\left(\sqrt{\frac{k_2}{k}}\, l\right) & \sqrt{\frac{k}{k_2}} \sin\left(\sqrt{\frac{k_2}{k}}\, l\right) \\ -\sqrt{\frac{k_2}{k}} \sin\left(\sqrt{\frac{k_2}{k}}\, l\right) & \cos\left(\sqrt{\frac{k_2}{k}}\, l\right) \end{bmatrix}$

Notice also that the matrix corresponds to the product of the thin lens matrix times the straight section matrix as given in Table 2-1.

We are now in a position to consider the propagation of a ray through a biperiodic lens system made up of lenses of focal lengths f_1 and f_2 separated by d as shown in Figure 2-2. This will be shown in the next chapter to be formally equivalent to the problem of Gaussian-beam propagation inside an optical resonator with mirrors of radii $R_1 = 2f_1$ and $R_2 = 2f_2$ that are separated by d.

The section between the planes s and $s + 1$ can be considered as the basic unit cell of the periodic lens sequence. The matrix relating the ray parameters at the output of a unit cell to those at the input is the product of two matrices, one for each lens, each of which is of the form of the matrix in (2.1-3).

$$\begin{vmatrix} r_{s+1} \\ r'_{s+1} \end{vmatrix} = \begin{vmatrix} 1 & d \\ -\dfrac{1}{f_1} & \left(1 - \dfrac{d}{f_1}\right) \end{vmatrix} \begin{vmatrix} 1 & d \\ -\dfrac{1}{f_2} & \left(1 - \dfrac{d}{f_2}\right) \end{vmatrix} \begin{vmatrix} r_s \\ r'_s \end{vmatrix} \tag{2.1-4}$$

or, in equation form,

$$\begin{aligned} r_{s+1} &= Ar_s + Br'_s \\ r'_{s+1} &= Cr_s + Dr'_s \end{aligned} \tag{2.1-5}$$

where A, B, C, and D are the elements of the matrix resulting from multiplying the two square matrices in (2.1-4) and are given by

$$\begin{aligned} A &= 1 - \frac{d}{f_2} \\ B &= d\left(2 - \frac{d}{f_2}\right) \\ C &= -\left[\frac{1}{f_1} + \frac{1}{f_2}\left(1 - \frac{d}{f_1}\right)\right] \\ D &= -\left[\frac{d}{f_1} - \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right)\right] \end{aligned} \tag{2.1-6}$$

From the first of (2.1-5) we get

$$r'_s = \frac{1}{B}(r_{s+1} - Ar_s) \tag{2.1-7}$$

and thus

$$r'_{s+1} = \frac{1}{B}(r_{s+2} - Ar_{s+1}) \tag{2.1-8}$$

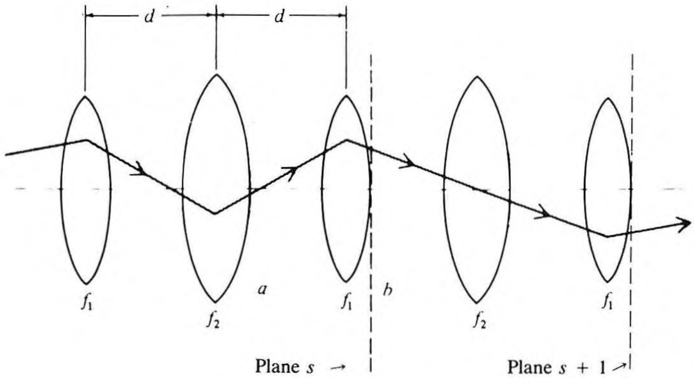


Figure 2-2 Propagation of an optical ray through a biperiodic lens sequence.

Using the second of (2.1-5) in (2.1-8) and substituting for r'_s from (2.1-7) gives

$$r_{s+2} - (A + D)r_{s+1} + (AD - BC)r_s = 0 \tag{2.1-9}$$

for the difference equation governing the evolution through the lens waveguide. Using (2.1-6) we can show that $AD - BC = 1$. We can consequently rewrite (2.1-9) as

$$r_{s+2} - 2br_{s+1} + r_s = 0 \tag{2.1-10}$$

where
$$b = \tfrac{1}{2}(A + D) = \left(1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2f_1f_2}\right) \tag{2.1-11}$$

Equation (2.1-10) is the equivalent, in terms of difference equations, of the differential equation $r'' + Gr = 0$, whose solution is $r(z) = r(0)\exp[\pm i\sqrt{G}z]$. We are thus led to try a solution in the form of

$$r_s = r_0 e^{isq}$$

which, when substituted in (2.1-10), leads to

$$e^{2iq} - 2be^{iq} + 1 = 0 \tag{2.1-12}$$

and therefore

$$e^{iq} = b \pm i\sqrt{1 - b^2} = e^{\pm i\theta} \tag{2.1-13}$$

where $\cos\theta = b$.

The general solution can be taken as a linear superposition of $\exp(is\theta)$ and $\exp(-is\theta)$ solutions or equivalently as

$$r_s = r_{\max} \sin(s\theta + \alpha) \tag{2.1-14}$$

where $r_{\max} = r_0/\sin\alpha$ and α can be expressed using (2.1-8) in terms of r_0 and r_0'.

The condition for a stable, that is, confined, ray is that θ be a real number, since in this case the ray radius r_s oscillates as a function of the cell number s between $r_{\max}$ and $-r_{\max}$. According to (2.1-13), the necessary and sufficient condition for θ to be real is that [5]

$$|b| \le 1 \tag{2.1-15}$$

In terms of the system parameters, we can use (2.1-11) to reexpress (2.1-15)

$$-1 \le 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2f_1f_2} \le 1$$

or

$$0 \le \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) \le 1 \tag{2.1-16}$$

If, on the other hand, the stability condition $|b| \le 1$ is violated, we obtain, according to (2.1-10), a solution in the form of

$$r_s = C_1 e^{(\alpha_+)s} + C_2 e^{(\alpha_-)s} \tag{2.1-17}$$

where $e^{\alpha\pm} = b \pm \sqrt{b^2 - 1}$, and, since the magnitude of either $\exp(\alpha_+)$ or $\exp(\alpha_-)$ exceeds unity, the beam radius will increase without limit as a function of (distance) s.

Identical-Lens Waveguide

The simplest case of a lens waveguide is one in which $f_1 = f_2 = f$; that is, all lenses are identical.

The analysis of this situation is considerably simpler than that used for a biperiodic lens sequence. The reason is that the periodic unit cell (the smallest part of the sequence that can, upon translation, recreate the whole sequence) contains a single lens only. The (A, B, C, D) matrix for the unit cell is given by the square matrix in (2.1-3). Following exactly the steps leading to (2.1-11) through (2.1-14), the stability condition becomes

$$0 \le d \le 4f \tag{2.1-18}$$

and the beam radius at the nth lens is given by

$$r_n = r_{\max}\sin(n\theta + \alpha)$$
$$\cos\theta = \left(1 - \frac{d}{2f}\right) \tag{2.1-19}$$

Because of the algebraic simplicity of this problem we can easily express $r_{\max}$ and α in (2.1-19) in terms of the initial conditions r_0 and r_0', obtaining

$$(r_{\max})^2 = \frac{4f}{4f - d}(r_0^2 + dr_0r_0' + dfr_0'^2) \tag{2.1-20}$$

$$\tan\alpha = \sqrt{\frac{4f}{d} - 1}\bigg/\left(1 + 2f\frac{r_0'}{r_0}\right) \tag{2.1-21}$$

where n corresponds to the plane immediately to the right of the nth lens. The derivation of the last two equations is left as an exercise (Problem 2.1).

The stability criteria can be demonstrated experimentally by tracing the behavior of a laser beam as it propagates down a sequence of lenses spaced uniformly. One can easily notice the rapid escape of the beam once condition (2.1-18) is violated.

2.2 PROPAGATION OF RAYS BETWEEN MIRRORS [6]

Another important application of the formalism just developed concerns the bouncing of a ray between two curved mirrors. Since the reflection at a mirror with a radius of curvature R is equivalent, except for the folding of the path, to passage through a lens with a focal length $f = R/2$, we can use the formalism of the preceding section to describe the propagation of a ray between two curved reflectors with radii of curvature R_1 and R_2, which are separated by d. Let us consider the simple case of a ray that is injected into a symmetric two-mirror system as shown in Figure 2-3(a). Since the x and y coordinates of the ray are independent variables, we can take them according to (2.1-19) in the form of

$$\begin{aligned} x_n &= x_{\max}\sin(n\theta + \alpha_x) \\ y_n &= y_{\max}\sin(n\theta + \alpha_y) \end{aligned} \tag{2.2-1}$$

where n refers to the ray parameter immediately following the nth reflection. According to (2.2-1), the locus of the points x_n, y_n on a given mirror lies on an ellipse.

Reentrant Rays

If θ in (2.2-1) satisfies the condition

$$2\nu\theta = 2l\pi \tag{2.2-2}$$

where ν and l are any two integers, a ray will return to its starting point following ν round trips and will thus continuously retrace the same pattern on the mirrors. If we consider as an example the simple case of $l = 1$, $\nu = 2$, so that $\theta = \pi/2$, from (2.1-19) we obtain $d = 2f = R$; that is, if the mirrors are separated by a distance equal to their radius of curvature R, the trapped ray will retrace its pattern after two round trips ($\nu = 2$). This situation

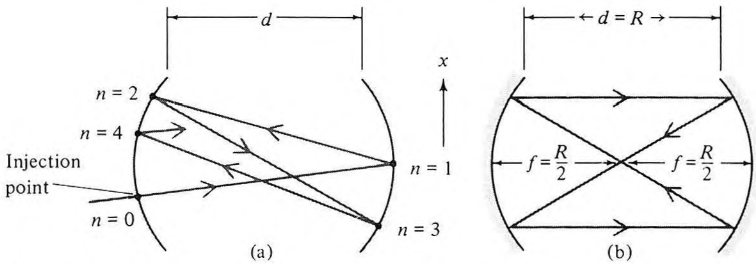


Figure 2-3 (a) Path of a ray injected in plane of figure into the space between two mirrors. (b) Reentrant ray in confocal ($d = R$) mirror configuration repeating its pattern after two round trips.

($d = R$) is referred to as symmetric confocal, since the two mirrors have a common focal point $f = R/2$. It will be discussed in detail in Chapter 4. The ray pattern corresponding to $\nu = 2$ is illustrated in Figure 2-3(b).

2.3 RAYS IN LENSLIKE MEDIA [7]

The basic physical property of lenses that is responsible for their focusing action is the fact that the optical path across them $\int n(r, z)\, dz$ (where n is the index of refraction of the medium) is a quadratic function of the distance r from the z axis. Using ray optics, we account for this fact by a change in the ray's slope as in (2.1-1). This same property can be represented by relating the complex field amplitude of the incident optical field $E_R(x, y)$ immediately to the right of an ideal thin lens to that immediately to the left $E_L(x, y)$ by

$$E_R(x, y) = E_L(x, y) \exp\left[+ ik \frac{x^2 + y^2}{2f}\right] \tag{2.3-1}$$

where f is the focal length and $k = 2\pi n/\lambda_0$.

The effect of the lens, therefore, is to cause a phase shift $k(x^2 + y^2)/2f$, which increases quadratically with the distance from the axis. We consider next the closely related case of a medium whose index of refraction n varies according to[2]

$$n(x, y) = n_0 \left[1 - \frac{k_2}{2k}(x^2 + y^2)\right] \tag{2.3-2}$$

where k_2 is a constant. Since the phase delay of a wave propagating through a section dz of a medium with an index of refraction n is $(2\pi\, dz/\lambda_0)n$, it

[2]Equation (2.3-2) can be viewed as consisting of the first two terms in the Taylor series expansion of $n(x, y)$ for the radial symmetric case.

follows directly that a thin slab of the medium described by (2.3-2) will act as a thin lens, introducing [as in (2.3-1)] a phase shift proportional to $(x^2 + y^2)$. The behavior of a ray in this case is described by the differential equation that applies to ray propagation in an optically inhomogeneous medium [8],

$$\frac{d}{ds}\left(n\,\frac{d\mathbf{r}}{ds}\right) = \nabla n \tag{2.3-3}$$

where s is the distance along the ray measured from some fixed position on it and $\mathbf{r}$ is the position vector of the point at s. For paraxial rays we may replace d/ds by d/dz and, using (2.3-2), obtain

$$\frac{d^2 r}{dz^2} + \left(\frac{k_2}{k}\right) r = 0 \tag{2.3-4}$$

If at the input plane $z = 0$ the ray has a radius r_0 and slope r_0', we can write the solution of (2.3-4) directly as

$$\begin{aligned} r(z) &= \cos\left(\sqrt{\frac{k_2}{k}}\,z\right) r_0 + \sqrt{\frac{k}{k_2}}\,\sin\left(\sqrt{\frac{k_2}{k}}\,z\right) r_0' \\ r'(z) &= -\sqrt{\frac{k_2}{k}}\,\sin\left(\sqrt{\frac{k_2}{k}}\,z\right) r_0 + \cos\left(\sqrt{\frac{k_2}{k}}\,z\right) r_0' \end{aligned} \tag{2.3-5}$$

That is, the ray oscillates back and forth across the axis, as shown in Figure 2-4. A section of the quadratic index medium acts as a lens. This can be proved by showing, using (2.3-5), that a family of parallel rays entering at $z = 0$ at different radii will converge upon emerging at $z = l$ to a common focus at a distance

$$h = \frac{1}{n_0}\sqrt{\frac{k}{k_2}}\,\cot\left(\sqrt{\frac{k_2}{k}}\,l\right) \tag{2.3-6}$$

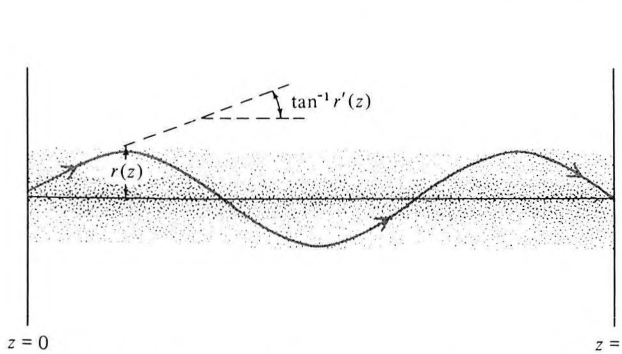


Figure 2-4 Path of a ray in a medium with a quadratic index variation.

from the exit plane. The factor n_0 accounts for the refraction at the boundary, assuming the medium at $z > l$ to possess an index $n = 1$ and a small angle of incidence. The derivation of (2.3-6) is left as an exercise (Problem 2.3).

Equations (2.3-5) apply to a focusing medium with $k_2 > 0$. In a medium where $k_2 < 0$—that is, where the index increases with the distance from the axis—the solutions for $r(z)$ and $r'(z)$ become

$$r(z) = \cosh\left(\sqrt{\frac{|k_2|}{k}}\,z\right) r_0 + \sqrt{\frac{k}{|k_2|}} \sinh\left(\sqrt{\frac{|k_2|}{k}}\,z\right) r_0'$$

$$r'(z) = \sqrt{\frac{|k_2|}{k}} \sinh\left(\sqrt{\frac{|k_2|}{k}}\,z\right) r_0 + \cosh\left(\sqrt{\frac{|k_2|}{k}}\,z\right) r_0' \qquad (2.3\text{-}7)$$

so that $r(z)$ increases with distance and eventually escapes. A section of such a medium acts as a negative lens.

Physical situations giving rise to quadratic index variation include:

1. Propagation of laser beams with Gaussian-like intensity profile in a slightly absorbing medium. The absorption heating gives rise, because of the dependence of n on the temperature T, to an index profile [9]. If $dn/dT < 0$, as is the case for most materials, the index is smallest on the axis where the absorption heating is highest. This corresponds to a $k_2 < 0$ in (2.3-2) and the beam spreads with the distance z. If $dn/dT > 0$, as in certain lead glasses [10], the beams are focused.
2. In optically pumped solid-state laser rods, the portion of the absorbed pump power that is not converted to laser radiation is conducted as heat to the rod surface. This heat conduction requires a temperature gradient in which T is maximum on axis. The dependence of n on T then gives rise to a positive lens effect for $dn/dT > 0$ and a negative lens for $dn/dT < 0$.
3. Dielectric waveguides made by sandwiching a layer of index n_1 between two layers with index $n_2 < n_1$. This situation will be discussed further in Chapter 7 in connection with injection lasers.
4. Optical fibers produced by cladding a thin optical fiber (whose radius is comparable to λ) of an index n_1 with a sheath of index $n_2 < n_1$. Such fibers are used as light pipes.
5. Optical waveguides consisting of glasslike rods or filaments, with radii large compared to λ, whose index decreases with increasing r. Such waveguides can be used for the simultaneous transmission of a number of laser beams, which are injected into the waveguide at different angles. It follows from (2.3-5) that the beams will emerge, each along a unique direction, and consequently can be easily separated. Furthermore, in view of its previously discussed lens properties, the waveguide can be used to transmit optical image information in much the same way as images are transmitted by a multielement lens system to the image plane of a camera [11].

2.4 WAVE EQUATION IN QUADRATIC INDEX MEDIA

The most widely encountered optical beam in quantum electronics is one where the intensity distribution at planes normal to the propagation direction is Gaussian. To derive its characteristics we start with the Maxwell equations in an isotropic charge-free medium.

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0 \tag{2.4-1}$$

Taking the curl of the second of (2.4-1) and substituting the first results in

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla \left(\frac{1}{\varepsilon} \mathbf{E} \cdot \nabla\varepsilon \right) \tag{2.4-2}$$

where we used $\nabla \times \nabla \times \mathbf{E} \equiv \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$. If we assume the field quantities to vary as $\mathbf{E}(x, y, z, t) = \mathrm{Re}[\mathbf{E}(x, y, z)e^{i\omega t}]$ and neglect the right side of (2.4-2),[3] we obtain

$$\nabla^2 \mathbf{E} + k^2(\mathbf{r})\mathbf{E} = 0 \tag{2.4-3}$$

where

$$k^2(\mathbf{r}) = \omega^2 \mu\varepsilon(\mathbf{r}) \left[1 - \frac{i\sigma(\mathbf{r})}{\omega\varepsilon} \right] \tag{2.4-4}$$

thus allowing for a possible dependence of ε on position $\mathbf{r}$. We have also taken k as a complex number to allow for the possibility of losses ($\sigma > 0$) or gain ($\sigma < 0$) in the medium.[4]

We limit our derivation to the case in which $k^2(\mathbf{r})$ is given by

$$k^2(r, \phi, z) = k^2 - kk_2 r^2 \qquad r = \sqrt{x^2 + y^2} \tag{2.4-5}$$

where, according to (2.4-4),

$$k^2 = k^2(0) = \omega^2 \mu\varepsilon(0) \left(1 - i \frac{\sigma(0)}{\omega\varepsilon(0)} \right)$$

so that k_2 is some constant characteristic of the medium. Furthermore, we assume a solution whose transverse dependence is on $r = \sqrt{x^2 + y^2}$ only, so that in (2.4-3) we can replace ∇^2 by

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2.4-6}$$

---

[3]This neglect is justified if the fractional change of ε in one optical wavelength is $\ll 1$.

[4]If k is complex (for example, $k_r + ik_i$), then a traveling electromagnetic plane wave has the form of $\exp[i(\omega t - kz)] = \exp[+k_i z + i(\omega t - k_r z)]$.

The kind of propagation we are considering is that of a nearly plane wave in which the flow of energy is predominantly along a single (for example, **z**) direction so that we may limit our derivation to a single transverse field component E. Taking E as

$$E = \psi(x, y, z)e^{-ikz} \tag{2.4-7}$$

we obtain from (2.4-3) and (2.4-5) in a few simple steps,

$$\nabla_t^2\psi - 2ik\psi' - kk_2r^2\psi = 0 \tag{2.4-8}$$

where $\psi' \equiv \partial\psi/\partial z$ and where we assume that the variation is slow enough that $k\psi' \gg \psi'' \ll k^2\psi$.

Next we take ψ in the form of

$$\psi = \exp\left\{-i\left[P(z) + \frac{k}{2q(z)}r^2\right]\right\} \tag{2.4-9}$$

that, when substituted into (2.4-8) and after using (2.4-6), gives

$$-\left(\frac{k}{q}\right)^2 r^2 - 2i\left(\frac{k}{q}\right) - k^2r^2\left(\frac{1}{q}\right)' - 2kP' - kk_2r^2 = 0 \tag{2.4-10}$$

If (2.4-10) is to hold for all r, the coefficients of the different powers of r must be equal to zero. This leads to [7]

$$\left(\frac{1}{q}\right)^2 + \left(\frac{1}{q}\right)' + \frac{k_2}{k} = 0 \qquad P' = -\frac{i}{q} \tag{2.4-11}$$

The wave equation (2.4-3) is thus reduced to (2.4-11).

2.5 GAUSSIAN BEAMS IN A HOMOGENEOUS MEDIUM

We start with Equation (2.4-11)

$$\frac{1}{q^2} + \frac{d}{dz}\left(\frac{1}{q}\right) + \frac{k_2}{k} = 0 \tag{2.5-1}$$

In a homogeneous medium the quadratic coefficient k_2 of (2.4-5) is zero so that

$$\frac{1}{q^2} + \frac{d}{dz}\left(\frac{1}{q}\right) = 0 \tag{2.5-2}$$

or

$$\frac{dq}{dz} = 1 \tag{2.5-3}$$

$$q = z + q_0 \tag{2.5-4}$$

where q_0 is an arbitrary integration constant. From (2.4-11) and (2.5-4) we have

$$P' = -\frac{i}{q} = -\frac{i}{z + q_0} \tag{2.5-5}$$

so that

$$P(z) = -i \ln \left(1 + \frac{z}{q_0}\right) \tag{2.5-6}$$

where the new constant of integration was chosen as $c = i \ln q_0$.

Combining (2.5-5) and (2.5-6) in (2.4-9), we obtain

$$\psi = \exp\left\{-i\left[-i \ln\left(1 + \frac{z}{q_0}\right) + \frac{k}{2(q_0 + z)} r^2\right]\right\} \tag{2.5-7}$$

We take q_0 to be purely imaginary and reexpress it in terms of a new constant ω_0 as

$$q_0 = i\,\frac{\pi\omega_0^2 n}{\lambda} \qquad \lambda = \frac{2\pi n}{k} \tag{2.5-8}$$

The choice of imaginary q_0 will be found to lead to physically meaningful waves whose energy density is confined near the z axis. With this last substitution, let us consider, one at a time, the two factors in (2.5-7). The first one becomes

$$\exp\left[-\ln\left(1 - i\,\frac{\lambda z}{\pi\omega_0^2 n}\right)\right] = \frac{1}{\sqrt{1 + (\lambda^2 z^2/\pi^2\omega_0^4 n^2)}} \exp\left[i \tan^{-1}\left(\frac{\lambda z}{\pi\omega_0^2 n}\right)\right] \tag{2.5-9}$$

where we used $\ln(a + ib) = \ln\sqrt{a^2 + b^2} + i \tan^{-1}(b/a)$. Substituting (2.5-8) in the second term of (2.5-7) and separating the exponent into its real and imaginary parts, we obtain

$$\exp\left[\frac{-ikr^2}{2(q_0 + z)}\right] = \exp\left\{\frac{-r^2}{\omega_0^2[1 + (\lambda z/\pi\omega_0^2 n)^2]} - \frac{ikr^2}{2z[1 + (\pi\omega_0^2 n/\lambda z)^2]}\right\} \tag{2.5-10}$$

If we define the following parameters

$$\omega^2(z) = \omega_0^2\left[1 + \left(\frac{\lambda z}{\pi\omega_0^2 n}\right)^2\right] = \omega_0^2\left(1 + \frac{z^2}{z_0^2}\right) \tag{2.5-11}$$

$$R = z\left[1 + \left(\frac{\pi\omega_0^2 n}{\lambda z}\right)^2\right] = z\left(1 + \frac{z_0^2}{z^2}\right) \tag{2.5-12}$$

$$\eta(z) = \tan^{-1}\left(\frac{z}{\pi\omega_0^2 n}\right) = \tan^{-1}\left(\frac{z}{z_0}\right) \tag{2.5-13}$$

$$z_0 \equiv \frac{\pi\omega_0^2 n}{\lambda}$$

we can combine (2.5-9) and (2.5-10) in (2.5-7) and, recalling that $E(x, y, z) = \psi(x, y, z)\exp(-ikz)$, obtain

$$\begin{aligned} E(x, y, z) &= E_0 \frac{\omega_0}{\omega(z)} \exp\left\{-i[kz - \eta(z)] - i\frac{k^2}{2q(z)}\right\} \\ &= E_0 \frac{\omega_0}{\omega(z)} \exp\left\{-i[kz - \eta(z)] - r^2\left(\frac{1}{\omega^2(z)} + \frac{ik}{2R(z)}\right)\right\} \\ k &= \frac{2\pi n}{\lambda} \end{aligned} \tag{2.5-14}$$

This is our basic result. We refer to it as the fundamental Gaussian-beam solution, since we have excluded the more complicated solutions of (2.4-3) (that is, those with azimuthal variation) by limiting ourselves to transverse dependence involving $r = (x^2 + y^2)^{1/2}$ only. These higher-order modes will be discussed separately.

From (2.5-14) the parameter $\omega(z)$, which evolves according to (2.5-11), is the distance r at which the field amplitude is down by a factor $1/e$ compared to its value on the axis. We will consequently refer to it as the beam *spot size*. The parameter ω_0 is the minimum spot size. It is the beam spot size at the plane $z = 0$. The parameter R in (2.5-14) is the radius of curvature of the very nearly spherical wavefronts at z.[5] We can verify this statement by deriving the radius of curvature of the constant phase surfaces (wavefronts) or, more simply, by considering the form of a spherical wave emitted by a point radiator placed at $z = 0$. It is given by

$$\begin{aligned} E \propto \frac{1}{R} e^{-ikR} &= \frac{1}{R} \exp\left(-ik\sqrt{x^2 + y^2 + z^2}\right) \\ &\simeq \frac{1}{R} \exp\left(-ikz - ik\frac{x^2 + y^2}{2R}\right) \qquad x^2 + y^2 \ll z^2 \end{aligned} \tag{2.5-15}$$

since z is equal to R, the radius of curvature of the spherical wave. Comparing (2.5-15) with (2.5-14), we identify R as the radius of curvature of the Gaussian beam. The convention regarding the sign of $R(z)$ is that it is negative if the center of curvature occurs at $z' > z$ and vice versa.

The form of the fundamental Gaussian beam is, according to (2.5-14), uniquely determined once its minimum spot size ω_0 and its location—that

[5] Actually, it follows from (2.5-14) that, with the exception of the immediate vicinity of the plane $z = 0$, the wavefronts are parabolic since they are defined by $k[z + (r^2/2R)] = \text{const.}$ For $r^2 \ll z^2$, the distinction between parabolic and spherical surfaces is not important.

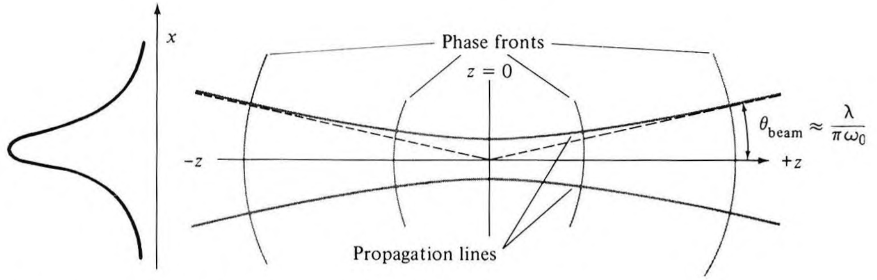


Figure 2-5 Propagating Gaussian beam.

is, the plane $z = 0$—are specified. The spot size ω and radius of curvature R at any plane z are then found from (2.5-11) and (2.5-12). Some of these characteristics are displayed in Figure 2-5. The hyperbolas shown in this figure correspond to the ray direction and are intersections of planes that include the z axis and the hyperboloids

$$x^2 + y^2 = \text{const. } \omega^2(z) \tag{2.5-16}$$

These hyperbolas correspond to the local direction of energy propagation. The spherical surfaces shown have radii of curvature given by (2.5-12). For large z the hyperboloids $x^2 + y^2 = \omega^2$ are asymptotic to the cone

$$r = \sqrt{x^2 + y^2} = \frac{\lambda}{\pi \omega_0 n} z \tag{2.5-17}$$

whose half-apex angle, which we take as a measure of the angular beam spread, is

$$\theta_{\text{beam}} = \tan^{-1}\left(\frac{\lambda}{\pi \omega_0 n}\right) \simeq \frac{\lambda}{\pi \omega_0 n} \qquad \text{for } \theta_{\text{beam}} \ll \pi \tag{2.5-18}$$

This last result is a rigorous manifestation of wave diffraction according to which a wave that is confined in the transverse direction to an aperture of radius ω_0 will spread (diffract) in the far field ($z \gg \pi \omega_0^2 n/\lambda$) according to (2.5-18).

2.6 FUNDAMENTAL GAUSSIAN BEAM IN A LENSLIKE MEDIUM—THE ABCD LAW

We now return to the general case of a lenslike medium so that $k_2 \neq 0$. The P and q functions of (2.4-9) obey, according to (2.4-11)

$$\left(\frac{1}{q}\right)^2 + \left(\frac{1}{q}\right)' + \frac{k_2}{k} = 0$$

$$P' = -\frac{i}{q} \tag{2.6-1}$$

If we introduce the function s defined by

$$\frac{1}{q} = \frac{s'}{s} \tag{2.6-2}$$

we obtain from (2.6-1)

$$s'' + s\,\frac{k_2}{k} = 0$$

so that

$$\begin{aligned} s(z) &= a \sin \sqrt{\frac{k_2}{k}}\, z + b \cos \sqrt{\frac{k_2}{k}}\, z \\ s'(z) &= a \sqrt{\frac{k_2}{k}} \cos \sqrt{\frac{k_2}{k}}\, z - b \sqrt{\frac{k_2}{k}} \sin \sqrt{\frac{k_2}{k}}\, z \end{aligned} \tag{2.6-3}$$

where a and b are arbitrary constants.

Using (2.6-3) in (2.6-2) and expressing the result in terms of an input value q_0 gives the following result for the complex beam radius $q(z)$

$$q(z) = \frac{\cos[(\sqrt{k_2/k})z]q_0 + \sqrt{k/k_2}\,\sin[(\sqrt{k_2/k})z]}{-\sin[(\sqrt{k_2/k})z]\sqrt{k_2/k}q_0 + \cos[(\sqrt{k_2/k})z]} \tag{2.6-4}$$

The physical significance of $q(z)$ in this case can be extracted from (2.4-9). We expand the part of $\psi(r, z)$ that involves r. The result is

$$\psi \propto e^{-ikr^2/2q(z)}$$

If we express the real and imaginary parts of $q(z)$ by means of

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\,\frac{\lambda}{\pi n \omega^2(z)} \tag{2.6-5}$$

we obtain

$$\psi \propto \exp\left[\frac{-r^2}{\omega^2(z)} - i\,\frac{kr^2}{2R(z)}\right]$$

so that $\omega(z)$ is the beam spot size and R its radius of curvature, as in the case of a homogeneous medium, which is described by (2.5-14). For the special case of a homogeneous medium ($k_2 = 0$), (2.6-4) reduces to (2.5-4).

Transformation of the Gaussian Beam—the ABCD Law

We have derived above the transformation law of a Gaussian beam (2.6-4) propagating through a lenslike medium that is characterized by k_2. We note first by comparing (2.6-4) to Table 2-1(6) and to (2.3-5) that the transformation can be described by

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{2.6-6}$$

where A, B, C, D are the elements of the ray matrix that relates the ray (r, r') at a plane 2 to the ray at plane 1. It follows immediately that the propagation through, or reflection from, any of the elements shown in Table 2-1 also obeys (2.6-6), since these elements can all be viewed as special cases of a lenslike medium. For future reference we note that by applying (2.6-6) to a thin lens of focal length f we obtain from (2.6-6) and Table 2-1(2)

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \tag{2.6-7}$$

so that using (2.6-5)

$$\begin{aligned} \omega_2 &= \omega_1 \\ \frac{1}{R_2} &= \frac{1}{R_1} - \frac{1}{f} \end{aligned} \tag{2.6-8}$$

These results apply, as well, to reflection from a mirror with a radius of curvature R if we replace f by $R/2$.

Consider next the propagation of a Gaussian beam through two lenslike media that are adjacent to each other. The ray matrix describing the first one is (A_1, B_1, C_1, D_1) while that of the second one is (A_2, B_2, C_2, D_2). Taking the input beam parameter as q_1 and the output beam parameter as q_3, we have from (2.6-6)

$$q_2 = \frac{A_1 q_1 + B_1}{C_1 q_1 + D_1}$$

for the beam parameter at the output of medium 1 and

$$q_3 = \frac{A_2 q_2 + B_2}{C_2 q_2 + D_2}$$

and after combining the last two equations,

$$q_3 = \frac{A_T q_1 + B_T}{C_T q_1 + D_T} \tag{2.6-9}$$

where (A_T, B_T, C_T, D_T) are the elements of the ray matrix relating the output plane (3) to the input plane (1), that is,

$$\begin{vmatrix} A_T & B_T \\ C_T & D_T \end{vmatrix} = \begin{vmatrix} A_2 & B_2 \\ C_2 & D_2 \end{vmatrix} \begin{vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{vmatrix} \tag{2.6-10}$$

It follows by induction that (2.6-9) applies to the propagation of a Gaussian beam through any arbitrary number of lenslike media and elements. The matrix (A_T, B_T, C_T, D_T) is the ordered product of the matrices characterizing the individual members of the chain.

The great power of the ABCD law is that it enables us to trace the Gaussian beam parameter $q(z)$ through a complicated sequence of lenslike elements. The beam radius $R(z)$ and spot size $\omega(z)$ at any plane z can be recovered through the use of (2.6-5). The application of this method will be made clear by the following example.

Example: Gaussian Beam Focusing

As an illustration of the application of the ABCD law, we consider the case of a Gaussian beam that is incident at its waist on a thin lens of focal length f, as shown in Figure 2-6. We will find the location of the waist of the output beam and the beam radius at that point.

At the input plane 1 $\omega = \omega_{01}$, $R_1 = \infty$ so that

$$\frac{1}{q_1} = \frac{1}{R_1} - i\frac{\lambda}{\pi\omega_{01}^2 n} = -i\frac{\lambda}{\pi\omega_{01}^2 n}$$

using (2.6-8) leads to

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} = -\frac{1}{f} - i\frac{\lambda}{\pi\omega_{01}^2 n}$$

$$q_2 = \frac{1}{-1/f - i(\lambda/\pi\omega_{01}^2 n)} = \frac{-a + ib}{a^2 + b^2}$$

$$a \equiv \frac{1}{f} \qquad b \equiv \frac{\lambda}{\pi\omega_{01}^2 n}$$

At plane 3 we obtain, using (2.5-4),

$$q_3 = q_2 + l = \frac{-a}{a^2 + b^2} + \frac{ib}{a^2 + b^2} + l$$

$$\frac{1}{q_3} = \frac{1}{R_3} - i\frac{\lambda}{\pi\omega_3^2 n}$$

$$= \frac{[-a/(a^2 + b^2) + l] - ib/(a^2 + b^2)}{[-a/(a^2 + b^2) + l]^2 + b^2/(a^2 + b^2)^2}$$

Since plane 3 is, according to the statement of the problem, to correspond to the output beam waist, $R_3 = \infty$. Using this fact in the last equation leads to

$$l = \frac{a}{a^2 + b^2} = \frac{f}{1 + (f/\pi\omega_{01}^2 n/\lambda)^2} = \frac{f}{1 + (f/z_{01})^2} \tag{2.6-11}$$

as the location of the new waist, and to

$$\frac{\omega_3}{\omega_{01}} = \frac{f\lambda/\pi\omega_{01}^2 n}{\sqrt{1 + (f\lambda/\pi\omega_{01}^2 n)^2}} = \frac{f/z_{01}}{\sqrt{1 + (f/z_{01})^2}} \tag{2.6-12}$$

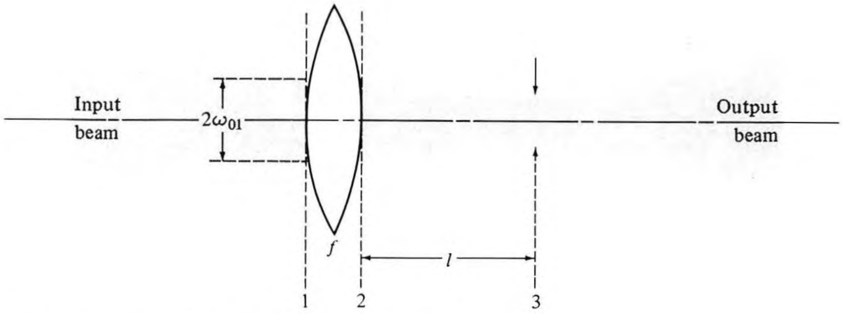


Figure 2-6 Focusing of a Gaussian beam.

for the output beam waist. The confocal beam parameter

$$z_{01} \equiv \frac{\pi\omega_{01}^2 n}{\lambda}$$

is, according to (2.5-11), the distance from the waist in which the input beam spot size increases by $\sqrt{2}$ and is a convenient measure of the convergence of the input beam. The smaller z_{01}, the "stronger" the convergence.

2.7 A GAUSSIAN BEAM IN LENS WAVEGUIDE

As another example of the application of the ABCD law, we consider the propagation of a Gaussian beam through a sequence of thin lenses, as shown in Figure 2-2. The matrix, relating a ray in plane $s + 1$ to the plane $s = 1$ is

$$\begin{vmatrix} A_T & B_T \\ C_T & D_T \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}^s \tag{2.7-1}$$

where (A, B, C, D) is the matrix for propagation through a single two-lens, unit cell ($\Delta s = 1$) and is given by (2.1-6). We can use a well-known formula for the sth power of a matrix with a unity determinant (unimodular) to obtain

$$\begin{aligned} A_T &= \frac{A \sin(s\theta) - \sin[(s-1)\theta]}{\sin\theta} \\ B_T &= \frac{B \sin(s\theta)}{\sin\theta} \\ C_T &= \frac{C \sin(s\theta)}{\sin\theta} \\ D_T &= \frac{D \sin(s\theta) - \sin[(s-1)\theta]}{\sin\theta} \end{aligned} \tag{2.7-2}$$

where

$$\cos\theta = \tfrac{1}{2}(A + D) = \left(1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2f_1f_2}\right) \tag{2.7-3}$$

and then use (2.7-2) in (2.6-9) with the result

$$q_{s+1} = \frac{\{A\sin(s\theta) - \sin[(s-1)\theta]\}q_1 + B\sin(s\theta)}{C\sin(s\theta)q_1 + D\sin(s\theta) - \sin[(s-1)\theta]} \tag{2.7-4}$$

The condition for the confinement of the Gaussian beam by the lens sequence is, from (2.7-4), that θ be real; otherwise, the sine functions will yield growing exponentials. From (2.7-3), this condition becomes $|\cos\theta| \leq 1$, or

$$0 \leq \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) \leq 1 \tag{2.7-5}$$

that is, the same as condition (2.1-16) for stable-ray propagation.

2.8 HIGH-ORDER GAUSSIAN BEAM MODES IN A HOMOGENEOUS MEDIUM

The Gaussian mode treated up to this point has a field variation that depends only on the axial distance z and the distance r from the axis. If we do not impose the condition $\partial/\partial\phi = 0$ (where ϕ is the azimuthal angle in a cylindrical coordinate system (r, ϕ, z)) and take $k_2 = 0$, the wave equation (2.4-3) has solutions in the form of [12]

$$\begin{aligned} E_{l,m}(x, y, z) &= E_0 \frac{\omega_0}{\omega(z)} H_l\left(\sqrt{2}\frac{x}{\omega(z)}\right) H_m\left(\sqrt{2}\frac{y}{\omega(z)}\right) \\ &\quad \times \exp\left[-ik\frac{x^2 + y^2}{2q(z)} - ikz + i(l + m + 1)\eta\right] \\ &= E_0 \frac{\omega_0}{\omega(z)} H_l\left(\sqrt{2}\frac{x}{\omega(z)}\right) H_m\left(\sqrt{2}\frac{y}{\omega(z)}\right) \\ &\quad \times \exp\left[-\frac{x^2 + y^2}{\omega^2(z)} - \frac{ik(x^2 + y^2)}{2R(z)} - ikz + i(l + m + 1)\eta\right] \end{aligned} \tag{2.8-1}$$

where H_l is the Hermite polynomial of order l, and $\omega(z)$, $R(z)$, $q(z)$, and η are defined as in (2.5-11) through (2.5-13).

We note for future reference that the phase shift on the axis is

$$\eta = kz - (l + m + 1)\tan^{-1}\left(\frac{z}{z_0}\right)$$

$$z_0 = \frac{\pi\omega_0^2 n}{\lambda} \tag{2.8-2}$$

The transverse variation of the electric field along x (or y) is seen to be of the form $H_l(\xi)\exp(-\xi^2/2)$ where $\xi = \sqrt{2}x/\omega$. This function has been studied extensively, since it corresponds, also, to the quantum mechanical wavefunction $u_l(\xi)$ of the harmonic oscillator [13]. Some low-order functions normalized to represent the same amount of total beam power are shown in Figure 2-7. Photographs of actual field patterns are shown in Figure 2-8. Note that the first four correspond to the intensity $|u_l(\xi)|^2$ plots ($l = 0, 1, 2, 3$) of Figure 2-7.

2.9 HIGH-ORDER GAUSSIAN BEAM MODES IN QUADRATIC INDEX MEDIA

In Section 2.6 we treated the propagation of a circularly symmetric Gaussian beam in lenslike media. Here we extend the treatment to higher-order modes and limit our attention to steady-state (that is, $q(z)$ = const.) solutions in media whose index of refraction can be described by

$$n^2(\mathbf{r}) = n^2\left(1 - \frac{n_2}{n}r^2\right) \qquad r^2 = x^2 + y^2 \tag{2.9-1a}$$

that is consistent with (2.4-5) if we put $k_2 = 2\pi n_2/\lambda$.

The vector-wave equation (2.4-3) takes the form

$$\nabla^2\mathbf{E} + k^2\left(1 - \frac{n_2}{n}r^2\right)\mathbf{E} = 0 \tag{2.9-1b}$$

We consider some (scalar) component E of the last equation and assume a solution in the form

$$E(x, y) = \psi(x, y)\exp(-i\beta z)$$

Taking $\psi(x, y) = f(x)g(y)$ the wave equation (2.9-1b) becomes

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} + \frac{1}{g}\frac{\partial^2 g}{\partial y^2} + k^2 - k^2\frac{n_2}{n}(x^2 + y^2) - \beta^2 = 0 \tag{2.9-2}$$

Since (2.9-2) is the sum of a y dependent part and an x dependent part, it follows that

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \left(k^2 - \beta^2 - k^2\frac{n_2}{n}x^2\right) = C \tag{2.9-3}$$

$$\frac{1}{g}\frac{d^2 g}{dy^2} - k^2\frac{n_2}{n}y^2 = -C \tag{2.9-4}$$

where C is some constant. Consider first (2.9-4). Defining a variable ξ by

$$\xi = \alpha y \qquad \alpha \equiv k^{1/2}\left(\frac{n_2}{n}\right)^{1/4} \tag{2.9-5}$$

(2.9-4) becomes

$$\frac{d^2g}{d\xi^2} + \left(\frac{C}{\alpha^2} - \xi^2\right) g = 0 \tag{2.9-6}$$

This is a well-known differential equation and is identical to the Schrödinger equation of the harmonic oscillator [13]. The eigenvalue C/α^2 must satisfy

$$\frac{C}{\alpha^2} = (2m + 1) \qquad m = 1, 2, 3 \ldots \tag{2.9-7}$$

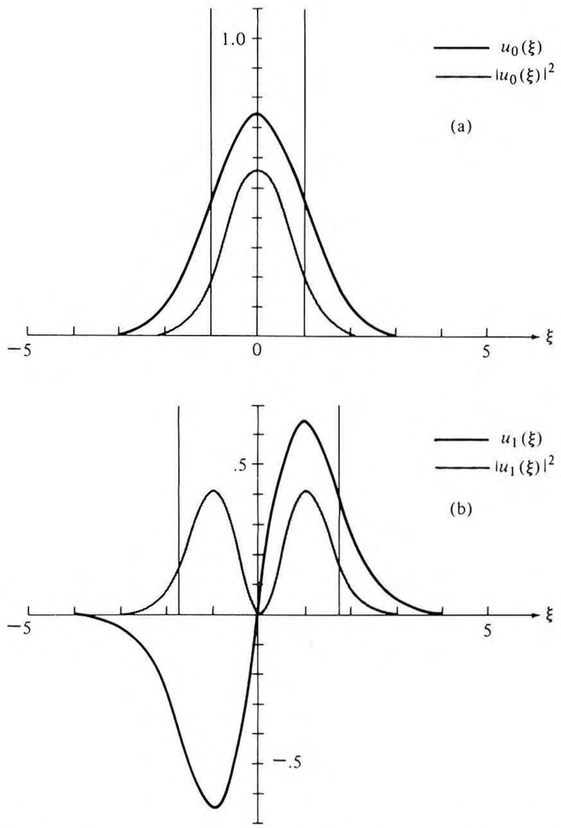


Figure 2-7 Hermite–Gaussian functions $u_l(\xi) = (\pi^{1/2}l!2^l)^{-1/2}H_l(\xi)e^{-\xi^2/2}$ corresponding to higher-order beam solutions (Equation 2.8-1). The curves are normalized so as to represent a fixed amount of total beam power in all the modes

$$\left(\int_{-\infty}^{\infty} u_l^2(\xi)d\xi = 1\right)$$

The solid curves are the functions $u_l(\xi)$ for $l = 0, 1, 2, 3,$ and 10. The dashed curves are $u_l^2(\xi)$.

and corresponding to an integer m, the solution is

$$g_m(\xi) = H_m(\xi)e^{-\xi^2/2} \tag{2.9-8}$$

where H_m is the Hermite polynomial of order m.

We now repeat the procedure with (2.9-3). Substituting

$$\zeta = \alpha x$$

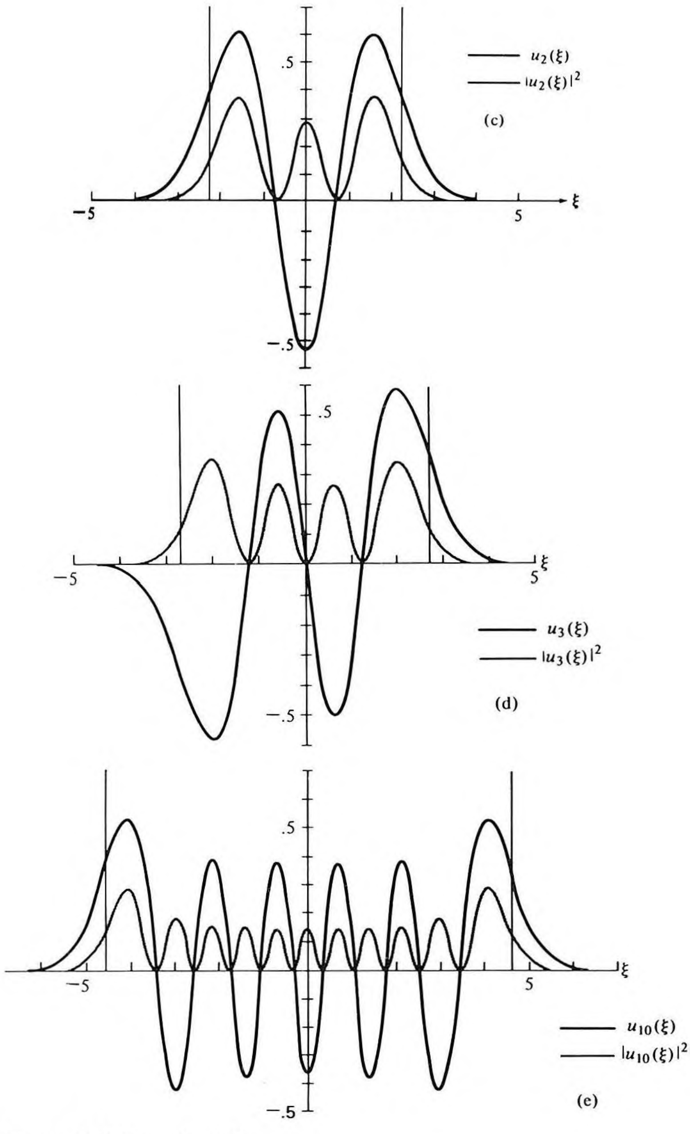


Figure 2-7 *(continued)*

it becomes

$$\frac{\partial^2 f}{\partial \zeta^2} + \left[\frac{k^2 - \beta^2 - C}{\alpha^2} - \zeta^2\right] f = 0$$

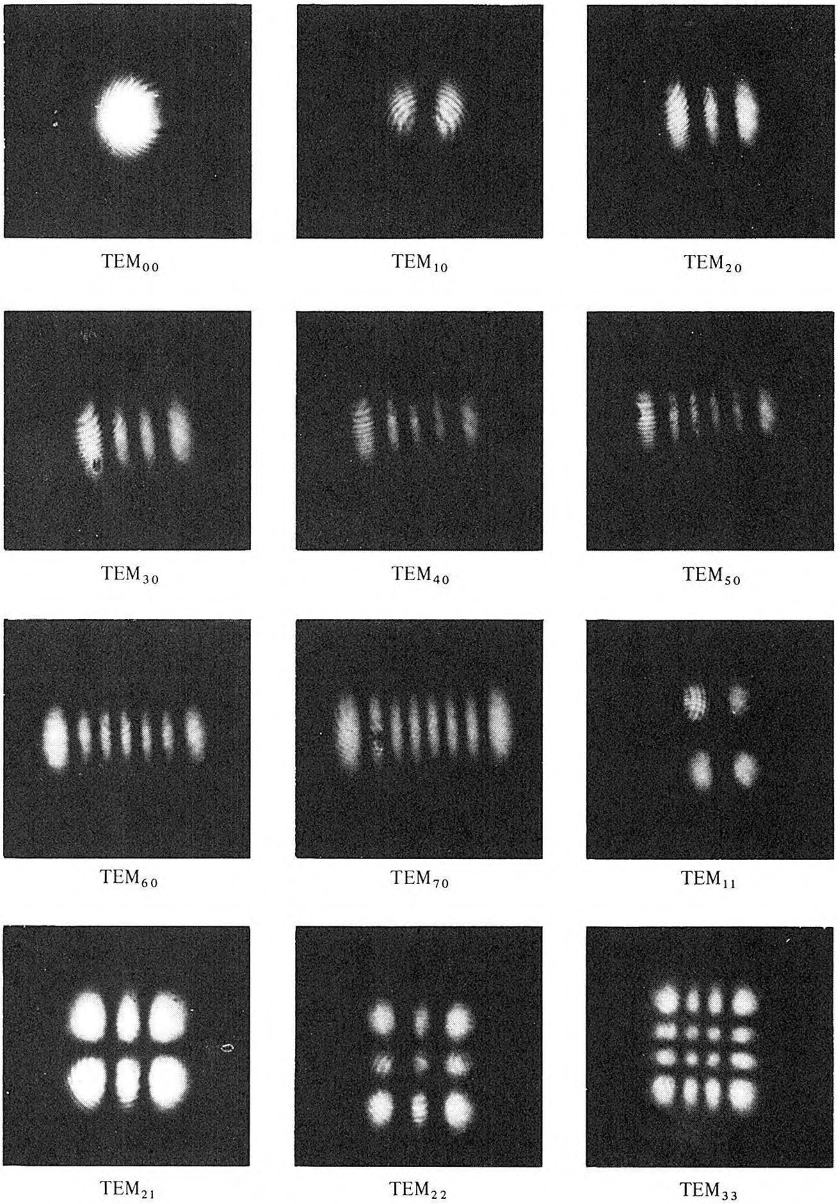


Figure 2-8 Intensity photographs of some low-order Gaussian beam modes. (After Reference [14].)

so that, as in (2.9-7),

$$\frac{k^2 - \beta^2 - C}{\alpha^2} = (2l + 1) \qquad l = 1, 2, 3, \ldots \tag{2.9-9}$$

and

$$f_l(\zeta) = H_l(\zeta)e^{-\zeta^2/2} \tag{2.9-10}$$

The total solution for ψ is thus

$$\psi(x, y) = H_l\left(\frac{\sqrt{2}x}{\omega}\right) H_m\left(\frac{\sqrt{2}y}{\omega}\right) e^{-(x^2+y^2)/\omega^2}$$

where the "spot size" ω is, according to (2.9-5),

$$\omega = \frac{\sqrt{2}}{\alpha} = \sqrt{\frac{2}{k}}\left(\frac{n}{n_2}\right)^{1/4} = \sqrt{\frac{\lambda}{\pi}}\left(\frac{1}{nn_2}\right)^{1/4} \tag{2.9-11}$$

The total (complex) field is

$$\begin{aligned} E_{l,m}(x, y, z) &= \psi_{l,m}(x, y)e^{-i\beta_{l,m}z} \\ &= E_0 H_l\left(\sqrt{2}\,\frac{x}{\omega}\right) H_m\left(\sqrt{2}\,\frac{y}{\omega}\right) \exp\left(-\frac{x^2 + y^2}{\omega^2}\right) \exp\left(-i\beta_{l,m}z\right) \end{aligned} \tag{2.9-12}$$

The propagation constant $\beta_{l,m}$ of the l, m mode is obtained from (2.9-7) and (2.9-9)

$$\beta_{l,m} = k\left[1 - \frac{2}{k}\sqrt{\frac{n_2}{n}}\,(l + m + 1)\right]^{1/2} \tag{2.9-13}$$

Two features of the mode solutions are noteworthy. (1) Unlike the homogeneous medium solution ($n_2 = 0$), the mode spot size ω is independent of z. This can be explained by the focusing action of the index variation ($n_2 > 0$), which counteracts the natural tendency of a confined beam to diffract (spread). In the case of an index of refraction which increases with $r(n_2 < 0)$, it follows from (2.9-11) and (2.9-12) that $\omega^2 < 0$ and no confined solutions exist. The index profile in this case leads to defocusing, thus reinforcing the diffraction of the beam. (2) The dependence of β on the mode indices l, m causes the different modes to have phase velocities $v_{l,m} = \omega/\beta_{l,m}$ as well as group velocities $(v_g)_{l,m} = d\omega/d\beta_{l,m}$ that depend on l and m.

Let us consider the modal dispersion (that is, the dependence on l and m) of the group velocity of mode l, m

$$(v_g)_{l,m} = \frac{d\omega}{d\beta_{l,m}} \tag{2.9-14}$$

If the index variation is small so that

$$\frac{1}{k}\sqrt{\frac{n_2}{n}}\,(l + m + 1) \ll 1 \tag{2.9-15}$$

we can approximate (2.9-13) as

$$\beta_{l,m} \cong k - \sqrt{\frac{n_2}{n}}\,(l + m + 1) - \frac{n_2}{2kn}\,(l + m + 1)^2 \tag{2.9-16}$$

so that, according to (2.9-14),

$$(v_g)_{l,m} = \frac{c/n}{\left[1 + \dfrac{(n_2/n)}{2k^2}\,(l + m + 1)^2\right]} \tag{2.9-17}$$

The effect of the group velocity dispersion on pulse propagation is considered next.

Pulse Spreading in Quadratic Index Glass Fibers

Glass fibers with quadratic index profiles (2.9-1a) are excellent channels for optical communication systems [15, 16]. The information is coded onto trains of optical pulses and the channel information capacity is thus fundamentally limited by the number of pulses that can be transmitted per unit time [17, 18].

There are two ways in which the group velocity dispersion limits the pulse repetition rate of the quadratic index channel.

1. Modal Dispersion If the optical pulses fed into the input end of the fiber excite a large number of modes (this will be the case if the input light is strongly focused so that the "rays" subtend a large angle), then each mode will travel with a group velocity $(v_g)_{l,m}$, as given by (2.9-17). If all the modes from (0, 0) to $(l_{\max}, m_{\max})$ are excited, the output pulse at $z = L$ will broaden to

$$\Delta\tau \cong L\left[\frac{1}{(v_g)_{l_{\max},m_{\max}}} - \frac{1}{(v_g)_{0,0}}\right] \tag{2.9-18}$$

We can use (2.9-17) and the condition $(n_2/n)(l + m + 1)^2/2k^2 \ll 1$ to obtain

$$\Delta\tau \cong \frac{n_2 L}{2ck^2}\left[(l_{\max} + m_{\max} + 1)^2 - 1\right] \tag{2.9-19}$$

The maximum number of pulses per second that can be transmitted without serious overlap of adjacent output pulses is thus $f_{\max} \sim 1/\Delta\tau$. High data rate transmission will thus require the use of single mode excitation, which can be achieved by the use of coherent single mode laser excitation [16, 17, 18, 24].

Example: Numerical Example

Consider a 1-km-long quadratic index fiber with $n = 1.5$, $n_2 = 5.1 \times 10^3$ cm^{-2}. Let the input optical pulses at $\lambda = 1$ μm excite the modes up to $l_{\max} = m_{\max} = 30$. Substitution in (2.9-19) gives

$$\Delta\tau = 8 \times 10^{-9} \text{ s}$$

and $f_{\max} \sim (\Delta\tau)^{-1} = 1.25 \times 10^8$ pulses per second for the maximum pulse rate.

2. Group Velocity Dispersion The pulse spreading (2.9-19) due to multimode excitation can be eliminated if one were to excite a single mode, say l, m only. In this case pulse spreading would still result from the dependence of $(v_g)_{l,m}$ on frequency. This spreading can be explained by the fact that a pulse with a spectral width $\Delta\omega$ will spread in a distance L by

$$\Delta\tau \approx 2L \left| \frac{d}{d\omega}\left(\frac{1}{v_g}\right) \right| \Delta\omega = \frac{2L}{v_g^2} \left| \frac{dv_g}{d\omega} \right| \Delta\omega \tag{2.9-20}$$

If the pulse is derived, say, by gating, from a coherent continuous source with a negligible spectral width, the pulse spectral width is related to the pulse duration τ by $\Delta\omega \sim 2/\tau$ and (2.9-20) becomes

$$\Delta\tau \approx \frac{4L}{v_g^2 \tau} \left(\frac{dv_g}{d\omega} \right) \tag{2.9-21}$$

If the source bandwidth $\Delta\omega_s$ exceeds τ^{-1}, then we need to replace $\Delta\omega$ in (2.9-20) by $\Delta\omega_s$.

In order to check our semi-intuitive derivation of (2.9-21) and also as an instructive exercise in the mathematics of dispersive propagation, we will, in what follows, consider the problem of an optical pulse with a Gaussian envelope propagating in a dispersive channel.

The input pulse is taken as the product of a slowly varying envelope function $\exp(-\alpha t^2)$ and a carrier $\exp(i\omega_0 t)$

$$\begin{aligned} E(z = 0, t) &= e^{-\alpha t^2} e^{i\omega_0 t} \\ &= e^{i\omega_0 t} \int_{-\infty}^{\infty} F(\Omega) e^{i\Omega t}\, d\Omega \end{aligned} \tag{2.9-22}$$

where $F(\Omega)$, the Fourier transform of the envelope $\exp(-\alpha t^2)$, is

$$F(\Omega) = \sqrt{\frac{1}{4\pi\alpha}}\, e^{-\Omega^2/4\alpha} \tag{2.9-23}$$

The field at a distance z is obtained by multiplying each frequency component

$(\omega_0 + \Omega)$ in (2.9-22) by $\exp[-i\beta(\omega_0 + \Omega)z]$. If we expand $\beta(\omega_0 + \Omega)$ near ω_0 as

$$\beta(\omega_0 + \Omega) = \beta(\omega_0) + \left.\frac{d\beta}{d\omega}\right|_{\omega_0} \Omega + \frac{1}{2} \left.\frac{d^2\beta}{d\omega^2}\right|_{\omega_0} \Omega^2 + \cdots$$

we obtain

$$E(z, t) = e^{i(\omega_0 t - \beta_0 z)} \int_{-\infty}^{\infty} d\Omega F(\Omega) \exp\left\{i\left[\Omega t - \frac{\Omega z}{v_g} - \frac{1}{2}\frac{\partial}{\partial\omega}\left(\frac{1}{v_g}\right)\Omega^2 z\right]\right\} \tag{2.9-24}$$

where

$$\beta_0 \equiv \beta(\omega_0), \qquad \left.\frac{d\beta}{d\omega}\right|_{\omega_0} = \frac{1}{v_g} = \frac{1}{\text{group velocity}}$$

The field envelope is given by the integral in (2.9-24)

$$\begin{aligned}\mathscr{E}(z, t) &= \int_{-\infty}^{\infty} d\Omega F(\Omega) \exp\left\{i\Omega\left[\left(t - \frac{z}{v_g}\right) - \frac{1}{2}\frac{d}{d\omega}\left(\frac{1}{v_g}\right)\Omega z\right]\right\} \\ &= \int_{-\infty}^{\infty} d\Omega F(\Omega) \exp\left\{i\Omega\left[\left(t - \frac{z}{v_g}\right) - a\Omega z\right]\right\}\end{aligned}$$

where

$$a \equiv \frac{1}{2}\frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{2v_g^2}\frac{dv_g}{d\omega} = \frac{1}{2}\left.\frac{d^2\beta}{d\omega^2}\right|_{\omega=\omega_0} \tag{2.9-25}$$

After substituting for $F(\Omega)$ from (2.9-23), the last equation becomes

$$\mathscr{E}(z, t) = \sqrt{\frac{1}{4\pi\alpha}} \int_{-\infty}^{\infty} \exp\left\{-\left[\Omega^2\left(\frac{1}{4\alpha} + iaz\right) + i\left(t - \frac{z}{v_g}\right)\Omega\right]\right\} d\Omega$$

Carrying out the integration yields

$$\mathscr{E}(z, t) = \frac{1}{\sqrt{1 + i4a\alpha z}} \exp\left(-\frac{(t - z/v_g)^2}{1/\alpha + 16a^2z^2\alpha}\right) \exp\left(i\frac{4az(t - z/v_g)^2}{1/\alpha^2 + 16a^2z^2}\right) \tag{2.9-26}$$

The pulse duration τ at z can be taken as the separation between the two times when the pulse envelope squared (intensity) is smaller by a factor of $\frac{1}{2}$ from its peak value, that is,

$$\tau(z) = \sqrt{2 \ln 2}\sqrt{\frac{1}{\alpha} + 16a^2z^2\alpha} \tag{2.9-27}$$

The initial pulse width, defined as the full width at half-maximum intensity, is given by

$$\tau_0 = \left(\frac{2 \ln 2}{\alpha}\right)^{1/2} \tag{2.9-28}$$

The pulse width after propagating a distance L can thus be expressed as

$$\tau(L) = \tau_0 \sqrt{1 + \left(\frac{8aL \ln 2}{\tau_0^2}\right)^2} \tag{2.9-29}$$

At large distances such that $aL \gg \tau_0^2$ we obtain

$$\tau(L) \sim \frac{(8 \ln 2)aL}{\tau_0} \tag{2.9-30}$$

If we use the definition of the factor a [see line following (2.9-25)], the last expression becomes

$$\tau(L) = \frac{4 \ln 2}{v_g^2} \frac{dv_g}{d\omega} \frac{L}{\tau_0} \tag{2.9-31}$$

which, within a factor of ln 2, is in agreement with (2.9-21).

The group velocity dispersion is often characterized by $D \equiv L^{-1}(dT/d\lambda)$, where T is the pulse transmission time through length L of the fiber. This definition is related to the second-order derivative of β with respect to ω as

$$D = -\frac{2\pi c}{\lambda^2}\left(\frac{d^2\beta}{d\omega^2}\right) \tag{2.9-32}$$

and is related to the parameter a used above by

$$D = -\frac{4\pi c}{\lambda^2} a \tag{2.9-33}$$

With this new definition, the pulse-width expression (2.9-29) can be written as

$$\tau(L) = \tau_0 \sqrt{1 + \left(\frac{2 \ln 2}{\pi c} \frac{DL\lambda^2}{\tau_0^2}\right)^2} \tag{2.9-34}$$

If DL is in units of picoseconds per nanometer, λ is in units of micrometers, and τ is in units of picoseconds, the pulse width can be written as

$$\tau(L) = \tau_0 \sqrt{1 + \left(\frac{1.47DL\lambda^2}{\tau_0^2}\right)^2} \tag{2.9-35}$$

The group velocity dispersion, i.e., the dependence of v_g on ω, which according to (2.9-31) leads to pulse broadening, is due to two mechanisms:

1. v_g depends, according to (2.9-17), on ω, since $k = \omega n/c$.
2. v_g depends implicitly on ω through the dependence of the material index of refraction n on ω. We thus write $dv_g/d\omega = \partial v_g/\partial\omega + \partial v_g/\partial n(dn/d\omega)$ and obtain from (2.9-17)

$$\Delta\tau = \frac{2L}{c}\left|\frac{nn_2}{ck^3}(l + m + 1)^2 - \frac{dn}{d\omega}\right| \Delta\omega \tag{2.9-36}$$

where in the second term we assumed $(n_2/2k^2n)(l + m + 1)^2 \ll 1$. In most fibers the pulse spreading is dominated by the material dispersion term $dn/d\omega$.

The actual index variation in a parabolic index SiO_2–B_2O_3 fiber with a value of $n_2 \simeq 5 \times 10^3$ cm^{-2} is shown in Figure 2-9. The broadening of an optical pulse after a propagation distance of ~2.5 km is illustrated by Figure 2-10. The input optical pulse excites a large number of (l, m) modes, and the broadening is of the type described by (2.9-19). The data are reproduced from Reference [17], which also describes the important consequence of intermode mixing on pulse broadening.

If we combine (2.9-26) with (2.9-24), we find that the total field at z is

$$E(z, t) = \mathscr{E}(z, t)e^{i(\omega_0 t - \beta_0 z)}$$
$$= \frac{e^{-i\beta_0 z}}{\sqrt{1 + i4a\alpha z}} \exp\left\{i\left[\omega_0 t + \frac{4az(t - z/v_g)^2}{\alpha^{-2} + 16a^2z^2}\right] - \frac{(t - z/v_g)^2}{1/\alpha + 16a^2z^2\alpha}\right\} \quad (2.9\text{-}37)$$

The oscillation phase is thus

$$\Phi(z, t) = \omega_0 t + \frac{4az(t - z/v_g)^2}{\alpha^{-2} + 16a^2z^2} - \beta_0 z \quad (2.9\text{-}38)$$

The local "frequency" $\omega(z, t)$ is then

$$\omega(z, t) = \frac{\partial\Phi}{\partial t} = \omega_0 + \frac{8az(t - z/v_g)}{\alpha^{-2} + 16a^2z^2} \quad (2.9\text{-}39)$$

and consists of the original frequency ω_0 and a linear frequency sweep (chirp), which is proportional to the group velocity dispersion term a. The chirp can be understood intuitively by the fact that, due to group velocity dispersion,

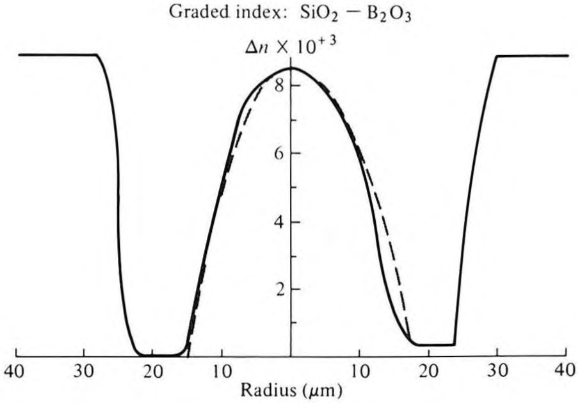


Figure 2-9 Graded refractive-index profile across the core of the SiO_2–B_2O_3 fiber. The dashed lines describe quadratic profiles. (After Reference [17].)

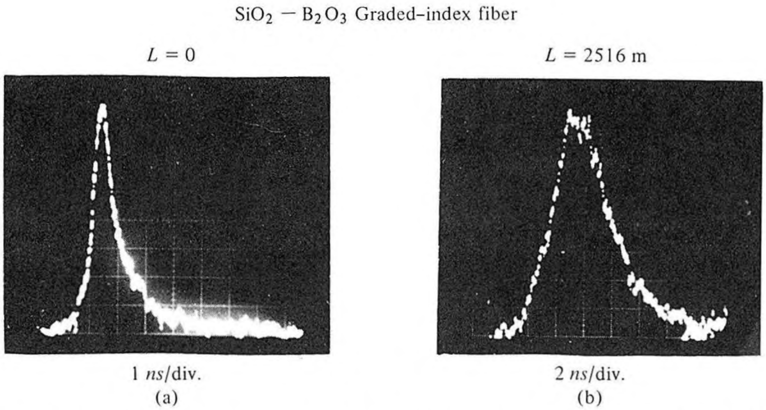


Figure 2-10 (a) Pulse shape at the input of the quadratic index fiber of Figure 2-9. (b) The output pulse after a propagation through 2516 m (note the change in the time scales). (After Reference [17].)

different frequencies travel with different (group) velocity. According to (2.9-35), the broadening ratio $\tau(L)/\tau_0$ for a given laser pulse will be small (i.e., $\sim$ 1) provided the group dispersion–length product DL is very small compared to $(\tau_0/\lambda)^2$. It is, of course, desirable to transmit light pulses at the condition when the group velocity dispersion is zero ($D = 0$). This may happen in optical fibers when the frequency dispersion and the material dispersion cancel each other at some wavelength. A nearly distortionless propagation of a 5-ps (5×10^{-12} s) Fourier transform limited pulse, in two fused-silica, single-mode fibers of 0.76 and 2.5 km length, respectively, has

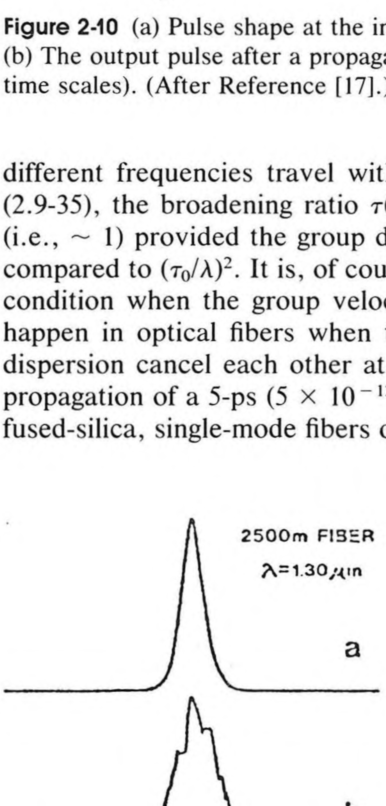


Figure 2-11 Pulse broadening in 2.5-km-long fiber resulting from linear dispersion [18].

been demonstrated at $\lambda = 1.30\ \mu m$ where the group velocity dispersion is zero (or almost zero) [24]. Figure 2-11 shows the measured pulse broadening. Some additional discussion of pulse broadening by group velocity dispersion in optical silica-based fibers is presented in Section 3.3.

2.10 PROPAGATION IN MEDIA WITH A QUADRATIC GAIN PROFILE

In many laser media the gain is a strong function of position. This variation can be due to a variety of causes, among them: (1) the radial distribution of energetic electrons in the plasma region of gas lasers [19], (2) the variation of pumping intensity in solid state lasers, and (3) the dependence of the degree of gain saturation on the radial position in the beam.

We can account for an optical medium with quadratic gain (or loss) variation by taking the complex propagation constant $k(r)$ in (2.4-5) as

$$k(r) = k \pm i(\alpha_0 - \tfrac{1}{2}\alpha_2 r^2) \tag{2.10-1}$$

where the plus (minus) sign applies to the case of gain (loss). Assuming $k_2 r^2 \ll k$ in (2.4-5), we have $k_2 = i\alpha_2$. Using this value in (2.4-11) to obtain the steady-state $((1/q)' = 0)$ solution of the complex beam radius yields[6]

$$\frac{1}{q} = -i\sqrt{\frac{k_2}{k}} = -i\sqrt{\frac{i\alpha_2}{k}} \tag{2.10-2}$$

[6] "Steady state" here refers not to the intensity, which according to (2.10-1) is growing or decaying with z, but to the beam radius of curvature and spot size.

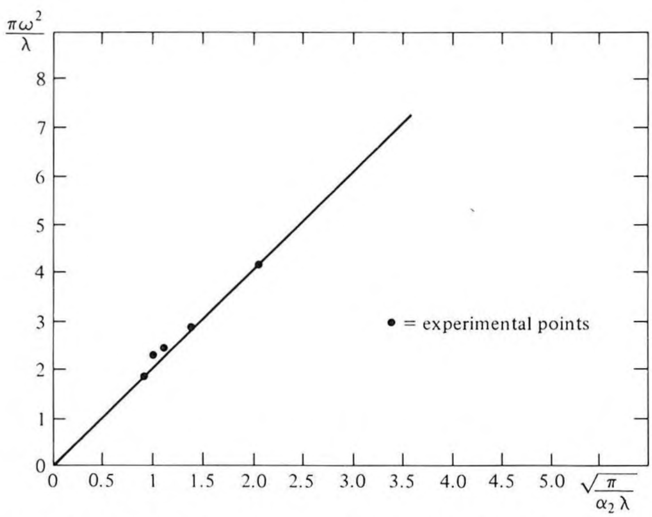


Figure 2-12 Theoretical curve showing the dependence of beam radius on quadratic gain constant α_2. Experimental points were obtained in a xenon 3.39-μm laser in which α_2 was varied by controlling the unsaturated laser gain. (After Reference [20].)

The steady-state beam radius and spot size are obtained from (2.6-5) and (2.10-2)

$$\omega^2 = 2\sqrt{\frac{\lambda}{\pi n \alpha_2}}$$
$$R = 2\sqrt{\frac{\pi n}{\lambda \alpha_2}} \tag{2.10-3}$$

We thus find that the steady-state solution corresponds to beams with a constant spot size but with a finite radius of curvature.

The general (nonsteady state) behavior of the Gaussian beam in a quadratic gain medium is described by (2.6-4), where $k_2 = i\alpha_2$.

Experimental data showing a decrease of the beam spot size with increasing gain parameter α_2 in agreement with (2.10-3) are shown in Figure 2-12.

2.11 ELLIPTIC GAUSSIAN BEAMS

All the beam solutions considered up to this point have one feature in common. The field drops off as in (2.8-1), according to

$$E_{m,n} \propto \exp\left[-\frac{x^2 + y^2}{\omega^2(z)}\right] \tag{2.11-1}$$

so that the locus in the x-y plane of the points where the field is down by a factor of e^{-1} from its value on the axis is a circle of radius $\omega(z)$. We will refer to such beams as *circular Gaussian beams*.

The wave equation (2.4-8) also admits solutions in which the variation in the x and y directions is characterized by

$$E_{m,n} \propto \exp\left[-\frac{x^2}{\omega_x^2(z)} - \frac{y^2}{\omega_y^2(z)}\right] \tag{2.11-2}$$

with $\omega_x \neq \omega_y$. Such beams, which we name *elliptic Gaussian,* result, for example, when a circular Gaussian beam passes through a cylindrical lens or when a laser beam emerges from an astigmatic resonator—that is, one whose mirrors possess different radii of curvature in the z-y and z-x planes.

We will not repeat the whole derivation for this case, but will indicate the main steps.

Instead of (2.4-9) we assume a solution

$$\psi = \exp\left\{-i\left[P(z) + \frac{k}{2q_x(z)}x^2 + \frac{k}{2q_y(z)}y^2\right]\right\} \tag{2.11-3}$$

that results, in a manner similar to (2.4-11), in[7]

$$\left(\frac{1}{q_x}\right)^2 + \left(\frac{1}{q_x}\right)' + \frac{k_{2x}}{k} = 0$$

$$\left(\frac{1}{q_y}\right)^2 + \left(\frac{1}{q_y}\right)' + \frac{k_{2y}}{k} = 0 \tag{2.11-4}$$

and

$$\frac{dP}{dz} = -i\left(\frac{1}{q_x} + \frac{1}{q_y}\right) \tag{2.11-5}$$

In the case of a homogeneous ($k_{2x} = k_{2y} = 0$) beam we obtain as in (2.5-4),

$$q_x(z) = z + C_x \tag{2.11-6}$$

where C_x is an arbitrary constant of integration. We find it useful to write C_x as

$$C_x = -z_x + q_{0x} \tag{2.11-7}$$

where z_x is real and q_{0x} is imaginary. The physical significance of these two constants will become clear in what follows. A similar result with $x \to y$ is obtained for $q_y(z)$. Using the solutions of $q_x(z)$ and $q_y(z)$ in (2.11-5) gives

$$P = -\frac{i}{2}\left[\ln\left(1 + \frac{z - z_x}{q_{0x}}\right) + \ln\left(1 + \frac{z - z_y}{q_{0y}}\right)\right]$$

Proceeding straightforwardly, as in the derivation connecting (2.5-6, . . . 14), results in

$$\begin{aligned} E(x, y, z) &= E_0 \frac{\sqrt{\omega_{0x}\omega_{0y}}}{\sqrt{\omega_x(z)\omega_y(z)}} \exp\left\{-i[kz - \eta(z)] - \frac{ikx^2}{2q_x(z)} - \frac{iky^2}{2q_y(z)}\right\} \\ &= E_0 \frac{\sqrt{\omega_{0x}\omega_{0y}}}{\sqrt{\omega_x(z)\omega_y(z)}} \exp\left\{-i[kz - \eta(z)] - x^2\left(\frac{1}{\omega_x^2(z)} + \frac{ik}{2R_x(z)}\right)\right. \\ &\quad \left. - y^2\left(\frac{1}{\omega_y^2(z)} + \frac{ik}{2R_y(z)}\right)\right\} \end{aligned} \tag{2.11-8}$$

where

$$\begin{aligned} q_{0x} &= i\,\frac{\pi\omega_{0x}^2 n}{\lambda} \\ \omega_x^2(z) &= \omega_{0x}^2\left[1 + \left(\frac{\lambda(z - z_x)}{\pi\omega_{0x}^2 n}\right)^2\right] \\ R_x(z) &= z\left[1 + \left(\frac{\pi\omega_{0x}^2 n}{\lambda(z - z_x)}\right)^2\right] \end{aligned} \tag{2.11-9}$$

with similar expressions in which $x \to y$ for q_{0y}, ω_y, R_y.

---

[7]The parameters k_{2x} and k_{2y} are defined by

$$k^2(x, y) = k^2 - kk_{2x}x^2 - kk_{2y}y^2$$

which is a generalization of (2.4-5).

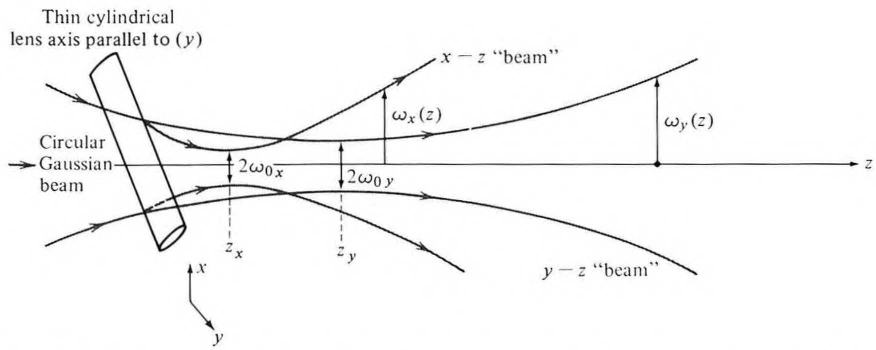


Figure 2-13 Illustration of an elliptic beam produced by cylindrical focusing of a circular Gaussian beam.

The phase delay $\eta(z)$ in (2.11-8) is now given by

$$\eta(z) = \tfrac{1}{2}\tan^{-1}\left(\frac{\lambda(z - z_x)}{\pi\omega_{0x}^2 n}\right) + \tfrac{1}{2}\tan^{-1}\left(\frac{\lambda(z - z_y)}{\pi\omega_{0y}^2 n}\right) \tag{2.11-10}$$

It follows that *all* the results derived for the case of circular Gaussian beams apply, separately, to the x-z and to the y-z behavior of the elliptic Gaussian beam. For the purpose of analysis the elliptic beam can be considered as two independent "beams." The position of the waist is not necessarily the same for these two beams. It occurs at $z = z_x$ for the x-z beam and at $z = z_y$ for the y-z beam in the example of Figure 2-13, where z_x and z_y are arbitrary.

It also follows from the similarity between (2.11-4) and (2.4-11) that the ABCD transformation law (2.6-9) can be applied separately to $q_x(z)$ and $q_y(z)$ which, according to (2.11-8), are given by

$$\begin{aligned} \frac{1}{q_x(z)} &= \frac{1}{R_x(z)} - i\,\frac{\lambda}{\pi n \omega_x^2(z)} \\ \frac{1}{q_y(z)} &= \frac{1}{R_y(z)} - i\,\frac{\lambda}{\pi n \omega_y^2(z)} \end{aligned} \tag{2.11-11}$$

Elliptic Gaussian Beams in a Quadratic Lenslike Medium

Here we consider the *steady-state* elliptic beam propagating in a medium whose index of refraction is given by

$$n^2(\mathbf{r}) = n^2\left(1 - \frac{n_{2x}}{n}x^2 - \frac{n_{2y}}{n}y^2\right) \tag{2.11-12}$$

The derivation is identical to that presented in Section 2.9, resulting in

$$E_{l,m}(\mathbf{r}) = E_0 e^{-i\beta_{l,m}z} H_l\left(\sqrt{2}\,\frac{x}{\omega_x}\right) H_m\left(\sqrt{2}\,\frac{y}{\omega_y}\right)\exp\left(-\frac{x^2}{\omega_x^2} - \frac{y^2}{\omega_y^2}\right) \tag{2.11-13}$$

$$\text{where} \quad \omega_x = \left(\frac{\lambda}{\pi}\right)^{1/2}\left(\frac{1}{nn_{2x}}\right)^{1/4}$$

$$\omega_y = \left(\frac{\lambda}{\pi}\right)^{1/2}\left(\frac{1}{nn_{2y}}\right)^{1/4} \tag{2.11-14}$$

$$\beta_{l,m} = k\left\{1 - \frac{2}{k}\left[\sqrt{\frac{n_{2x}}{n}}\left(l+\frac{1}{2}\right) + \sqrt{\frac{n_{2y}}{n}}\left(m+\frac{1}{2}\right)\right]\right\}^{1/2} \tag{2.11-15}$$

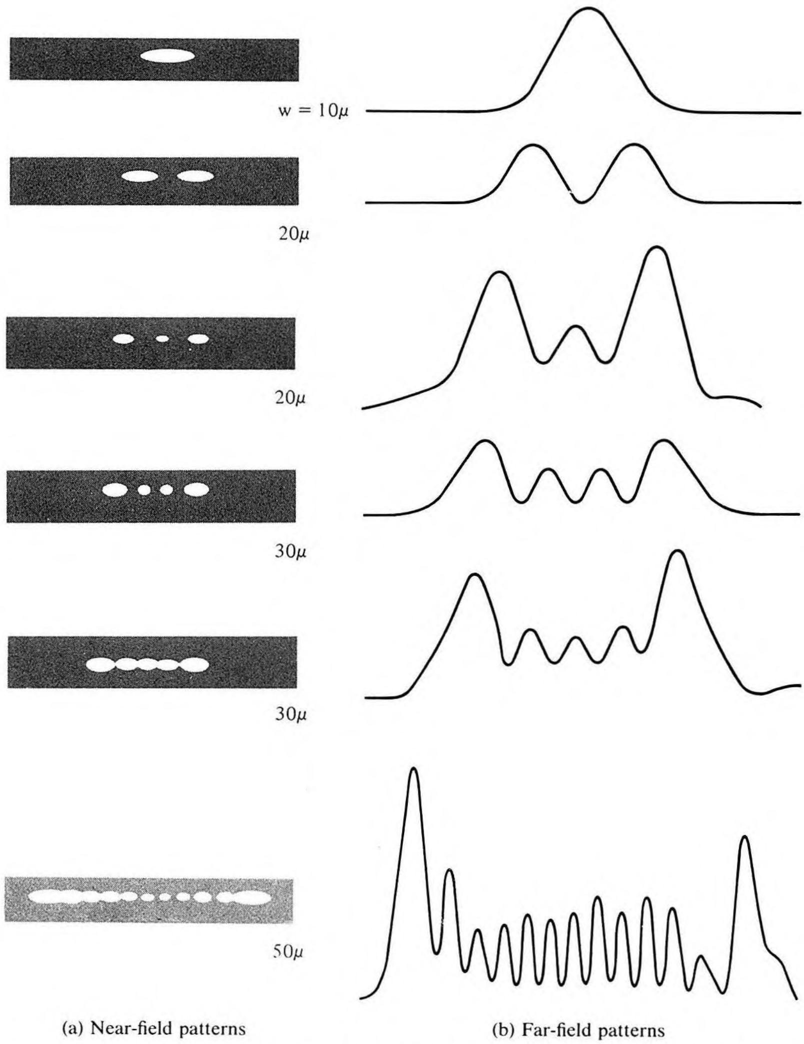


Figure 2-14 (a) Near-field and (b) far-field intensity distributions of the output of stripe contact GaAs-GaAlAs lasers. (After Reference [21].)

The beam, as in the solution of the homogeneous case (2.11-8), possesses different spot sizes and radii of curvature in the y-z and x-z planes. The beam parameters, however, are independent of z.

Elliptic Gaussian beams have been observed experimentally in the output of stripe geometry gallium arsenide junction lasers [21–23]. Near-field and far-field experimental intensity distributions corresponding to some $(0, m)$ modes are shown in Figure 2-14.

Problems

2.1 Derive Equations (2.1-19) through (2.1-21).

2.2 Show that the eigenvalues λ of the equation

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} r_s \\ r'_s \end{vmatrix} = \lambda \begin{vmatrix} r_s \\ r'_s \end{vmatrix}$$

are $\lambda = e^{\pm i\theta}$ with $\exp(\pm i\theta)$ given by Equation (2.1-13). Note that, according to Equation (2.1-5), the foregoing matrix equation can also be written as

$$\begin{vmatrix} r_{s+1} \\ r'_{s+1} \end{vmatrix} = \lambda \begin{vmatrix} r_s \\ r'_s \end{vmatrix}$$

2.3 Derive Equation (2.3-6).

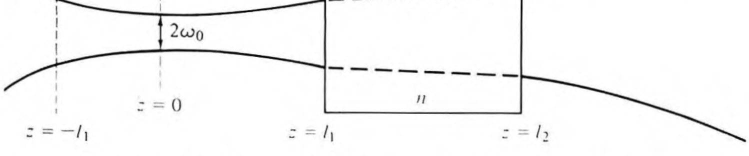
2.4 Make a plausibility argument to justify Equation (2.3-1) by showing that it holds for a plane wave incident on a lens.

2.5 Show that a lenslike medium occupying the region $0 \le z \le l$ will image a point on the axis at $z < 0$ onto a single point. (If the image point occurs at $z < l$, the image is virtual.)

2.6 Derive the ray matrices of Table 2-1.

2.7 Solve the problem leading up to Equations (2.6-11) and (2.6-12) for the case where the lens is placed in an arbitrary position relative to the input beam (that is, not at its waist).

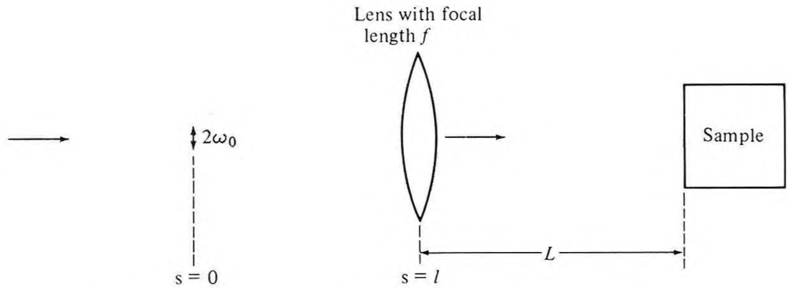
2.8



a. Assume a Gaussian beam incident normally on a solid prism with an index of refraction n as shown. What is the far-field diffraction angle of the output beam?

b. Assume that the prism is moved to the left until its input face is at $z = -l_1$. What is the new beam waist and what is its location? (Assume that the crystal is long enough that the beam waist is inside the crystal.)

2.9 A Gaussian beam with a wavelength λ is incident on a lens placed at $z = l$ as shown. Calculate the lens focal length, f, so that the output beam has a waist at the front surface of the sample crystal. Show that (given l and L) two solutions exist. Sketch the beam behavior for each of these solutions.



2.10 Complete all the missing steps in the derivation of Section 2.11.

2.11 Find the beam spot size and the maximum number of pulses per second that can be carried by an optical beam ($\lambda = 1\ \mu m$) propagating in a quadratic index glass fiber with $n = 1.5$, $n_2 = 5 \times 10^2\ cm^{-2}$. (a) in the case of a single mode excitation $l = m = 0$; (b) in the case where all the modes with $l, m < 5$ are excited. Using dispersion data of any typical commercial glass and taking $n_2 = 5 \times 10^3\ cm^{-2}$, $l_{max} = 30$, compare the relative contributions of modal and glass dispersion to pulse broadening.

2.12 Given a thick lens with radii of curvature R_1 and R_2 on its entrance and exit surfaces, an index of refraction n, and a thickness d,

a. Obtain the *ABCD* matrix of the lens.
b. What is its focal distance for light incident from the left?

References

1. Pierce, J. R., *Theory and Design of Electron Beams*, 2d ed. Princeton, N.J.: Van Nostrand, 1954, Chapter 11.
2. Ramo, S., J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*. New York: Wiley, 1965, p. 576.
3. Yariv, A., *Quantum Electronics*, 2d ed. New York: Wiley, 1975.
4. Siegman, A. E., *An Introduction to Lasers and Masers*. New York: McGraw-Hill, 1968.
5. Kogelnik, H., and T. Li, "Laser beams and resonators," *Proc. IEEE* 54:1312, 1966.
6. Herriot, D., H. Kogelnik, and R. Kompfner, "Off-axis paths in spherical mirror interferometers," *Appl. Opt.* 3:523, 1964.

7. Kogelnik, H., "On the propagation of Gaussian beams of light through lenslike media including those with a loss and gain variation," *Appl. Opt.* 4:1562, 1965.
8. Born, M., and E. Wolf, *Principles of Optics*, 3d ed. New York: Pergamon, 1965, p. 121.
9. Gordon, J. P., R. C. C. Leite, R. S. Moore, S. P. S. Porto, and J. R. Whinnery, "Long-transient effects in lasers with inserted liquid samples," *J. Appl. Phys.* 36:3, 1965.
10. Dabby, F. W., and J. R. Whinnery, "Thermal self-focusing of laser beams in lead glasses," *Appl. Phys. Lett.* 13:284, 1968.
11. Yariv, A., "Three dimensional pictorial transmission in optical fibers," *Appl. Phys. Lett.* 2:88, 1976.
12. Marcuse, D., *Light Transmission Optics*. Princeton, N.J.: Van Nostrand, 1972.
13. Yariv, A., *Quantum Electronics*, 2d ed. New York: Wiley, 1975, Section 2.2.
14. Kogelnik, H., and W. Rigrod, *Proc. IRE* 50:230, 1962.
15. Kawakami, S., and J. Nishizawa, "An optical waveguide with the optimum distribution of the refractive index with reference to waveform distortion," *IEEE Trans. Microwave Theory and Technique*, MTT-16, 10:814, 1968.
16. Miller, S. E., E. A. J. Marcatili, and T. Li, "Research toward optical fiber transmission systems," *Proc. IEEE* 61:1703, 1973.
17. Cohen, L. G., and H. M. Presby, "Shuttle pulse measurement of pulse spreading in a low loss graded index fiber," *Appl. Opt.* 14:1361, 1975.
18. Bloom, D. M., L. F. Mollenauer, Chinlon Lin, D. W. Taylor, and A. M. DelGaudio, "Direct demonstration of distortionless picosecond-pulse propagation in kilometer-length optical fibers," *Opt. Lett.* 4:297, 1979.
19. Bennett, W. R., "Inversion Mechanisms in Gas Lasers," *Appl. Opt.*, Suppl. 2 (*Chemical Lasers*):3, 1965.
20. Casperson, L., and A. Yariv, "The Gaussian mode in optical resonators with a radial gain profile," *Appl. Phys. Lett.* 12:355, 1968.
21. Zachos, T. H., "Gaussian beams from GaAs junction lasers," *Appl. Phys. Lett.* 12:318, 1969.
22. Zachos, T. H., and J. E. Ripper, "Resonant modes of GaAs junction lasers," *IEEE J. of Quantum Electron. QE*-5:29, 1969.
23. H. Yonezu et al., "A GaAs–Al_xGa_{1-x} as double heterostructure planar stripe laser," *Jpn. J. Appl. Phys.* 12:1585, 1973.
24. Suematsu, Y., "Long Wavelength Optical Fiber Communication," *Proc. IEEE* 71:692, 1983.