Intraband radiative transitions and plasma-electromagnetic-wave coupling in periodic semiconductor structure*

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Intraband radiative transitions can occur in a semiconductor with an artificial periodic structure (superlattice). The "lattice momentum" of the periodic structure makes possible the conservation of momentum during the electronic transition. When the electrons in the band are drifting in an electric field, an intraband population inversion may occur, providing optical wave amplification. Under conditions where the Landau damping of the semiconductor carrier's plasma wave is low, phase-matched coupling may occur between the plasma wave and a Floquet component of the electromagnetic wave and result in a high rate of power transfer from one of the waves to the other. These effects are discussed and analyzed quantum mechanically and suggestions are made with regard to possible device applications (amplifier, modulator) in the infrared regime.

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INTRODUCTION

The interaction of electrons in the conduction band of a semiconductor with electromagnetic wave which is not energetic enough to produce interband transitions is usually very small when collisions are negligible. An electron cannot make transitions inside the conduction band involving an absorption or emission of a photon, because the momentum of the photon is always smaller than the momentum change which involves the electronic transition. Appreciable coupling between the electromagnetic wave and the electron collective plasma oscillation is also impossible because the electromagnetic-wave propagation constant is usually much smaller than that of the plasma wave.

Interactions of the kinds just mentioned can be made possible if an artificial periodic structure is imposed on the semiconductor crystal. The "lattice momentum" of the artificial periodic structure may provide the missing momentum and allow the interaction.

Assume a semiconductor structure with some artificial periodicity in the z direction as in Fig. 1. The electromagnetic modes of the structure are given by the Floquet (or Bloch) theorem:

$$\mathbf{E}(\mathbf{r}) = \sum_m \mathbf{E}_m(x,y)\exp i(\omega t - \beta_m z), \qquad (1)$$

where $\beta_m = \beta_0 + m(2\pi/L)(m = 1, 2, \ldots)$, β_0 is approximately the propagation constant of the wave in the absence of periodic perturbation $[\beta_0 \simeq n(\omega/c)\cos\phi]$, n is the index of refraction, and L the periodicity.

If L is short enough, then some components (space harmonics) of the electromagnetic field [Eq. (1)] may have high enough momenta to allow intraband transitions as in Fig. 2 or to couple to a plasma wave. An expression similar to Eq. (1) applies also for the electron waves, and provides additional means of interaction. However, this effect will not be analyzed in the present paper and we assume that the periodic perturbation affects primarily the electromagnetic field.

The analysis that follows can also be viewed as a quantum-mechanical extension of a classical analysis of traveling-wave interaction[1] of semiconductor carriers in the collisionless regime.[2]

THE DISPERSION EQUATION

Following the procedure of Ref. 2, the interaction between the electromagnetic wave and the electrons will be analyzed using a one-dimensional model and coupled-mode technique. We concentrate on one of the space harmonics which propagates in general like $\mathbf{E}_c \exp i(\omega t - \beta z)$. The z component of this field modulates

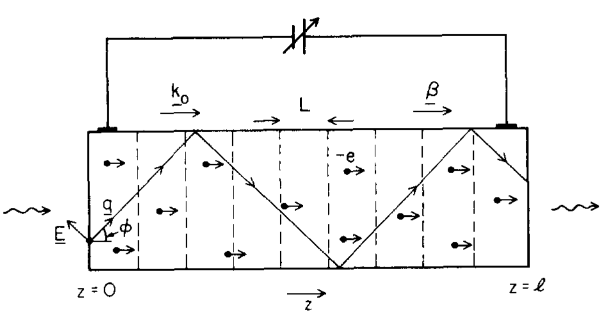

FIG. 1. Example of periodic structure in which interaction occurs between E-type electromagnetic mode and drifting carriers.

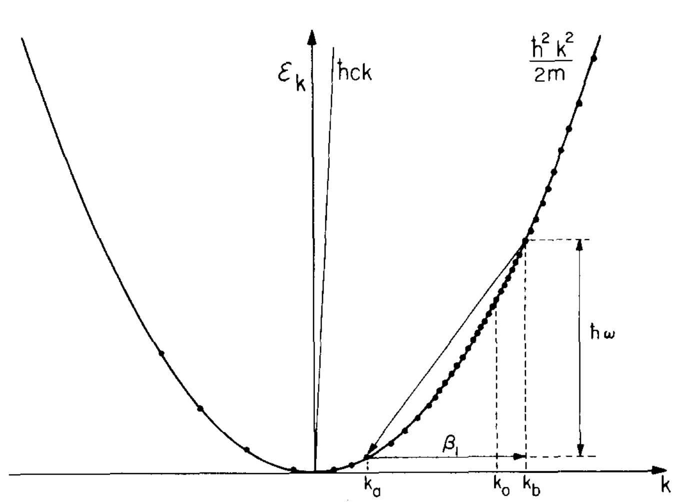

FIG. 2. Schematic intraband radiative transitions in one dimension. The dots represent electron density.

the charge carriers, producing space-charge current density $J_z(z) = J(\omega, \beta)\exp i(\omega t - \beta z)$

$$J_z = i\omega\chi_p(\omega, \beta)E_z, \tag{2}$$

where χ_p is the plasma susceptibility to be found later. E_z is the local field which is experienced by the plasma: $E_z = E_{cz} + E_{pz}$, where E_{pz} is the plasma space-charge field given by the Poisson equation (rationalized mks units system is used throughout this article):

$$-i\beta E_{pz} = (1/\epsilon)\rho = i\beta(\chi_p/\epsilon)E_z. \tag{3}$$

We, hence, obtain

$$E_z = \epsilon_p E_c, \tag{4}$$

$$\epsilon_p^{-1} \equiv 1 + \chi_p/\epsilon, \tag{5}$$

where ϵ is the dielectric constant of the semiconductor (which is real in the present approximation).

Equations (2), (4), and (5) give

$$J_z = \frac{i\omega\chi_p}{1 + \chi_p/\epsilon}E_{cz}. \tag{6}$$

This last result represents the linear plasma response to an external field E_{cz}. It includes as a special case the plasma dispersion relation $\epsilon_p^{-1} \equiv 1 + \chi_p(\omega, \beta)/\epsilon = 0$. This follows from requiring that J_z be finite with zero external field $E_c = 0$ in Eq. (6), which can occur only when the denominator $1 + \chi_p/\epsilon$ vanishes.

For completing the coupled modes analysis we need an expression for the electric field E_{cz} which would be induced in the structure by a current J_z. Following Refs. 1 and 2 we use the heuristic Pierce expression

$$E_{cz} = \frac{\beta^2\beta_1K_1S}{\beta_1^2 - \beta^2}J_z, \tag{7}$$

where S is the interaction cross-section area and $\beta_1 = \beta_0 + 2\pi/L$ is the propagation constant of the first-order space harmonic in the absence of charge carriers (real number). K_1 is the interaction impedance[1] which is characteristic to the electromagnetic mode

$$K_1 = \frac{|E_{z1}|^2}{2\beta_1^2 P} \tag{8}$$

and P is the total power in the electromagnetic mode.

The dispersion equation of the coupled modes is obtained by imposing self-consistency on Eqs. (6) and (7). This yields

$$\frac{K_1S\beta_1^2\omega\beta^2}{\beta^2 - \beta_1^2}\frac{\chi_p(\omega, \beta)}{1 + \chi_p(\omega, \beta)/\epsilon} = 1. \tag{9}$$

QUANTUM-MECHANICAL TREATMENT

From this point on, our analysis departs from previous treatments[2-4] due to our use of quantum-mechanical response theory to find the susceptibility $\chi_p(\omega, \beta)$ instead of different classical approximations. This will result in new expressions for the traveling-wave interaction gain, which agrees with the classical expression only at the limit $\beta_1 \to 0$. Indeed, the quantum-mechanical limit is of practical importance since the classical limit and the Boltzmann equation do not apply in the case of the large momentum transfer $\hbar\beta_1$ achievable with present state of the art of periodic structures fabrication.[5,6]

Standard quantum-mechanical linear response theory is utilized to calculate the plasma susceptibility $\chi_p(\omega, \beta)$. The carriers (electrons) in the conduction band are approximated as free carriers with an effective mass m. Collisions are negligible and $\omega\tau \gg 1$, where $\tau \approx 10^{-12}$ sec is the collision relaxation time. In our model the periodic structure affects electromagnetic wave only and not the electrons. This can be achieved if there is only periodic optical index modulation, and the donor impurities doping is such that the conduction band is flat. In the general case also the electrons will have a Floquet-Bloch expansion and provide an additional mechanism for interaction with the electromagnetic wave, which will not be elaborated on in the present article. We start by solving the Liouville equation

$$i\hbar\frac{\partial}{\partial t}\rho = [\mathcal{H}, \rho] \tag{10}$$

where ρ is the electron density matrix and $\mathcal{H}$ the total single body Hamiltonian. The unperturbed Hamiltonian is simple $\mathcal{H}^{(0)} = p^2/2m$ and its eigenfunction solutions are $|\mathbf{k}\rangle = V^{-1/2}\exp(i\mathbf{kr})$, so that $\mathcal{H}^{(0)}|\mathbf{k}\rangle = \mathcal{E}(\mathbf{k})|\mathbf{k}\rangle = \hbar^2k^2/2m|\mathbf{k}\rangle$ and $\rho^{(0)}|\mathbf{k}\rangle = f_0(\mathbf{k})|\mathbf{k}\rangle$, where $f_0(\mathbf{k})$ is the statistical distribution function.

The electromagnetic linear perturbation Hamiltonian is $\mathcal{H}^{(1)} = -(e/2m)\cdot(\mathrm{pA} + \mathrm{Ap})$, where $\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(\beta)\exp i(\omega t - \beta z)$ is the field vector potential, and we chose a scalar potential gauge $\phi = 0$. A perturbation expansion is assumed for ρ and Eq. (10) is solved for the first two terms $\rho^{(0)} + \rho^{(1)}$. This results in

$$\langle\mathbf{k}|\rho^{(1)}|\mathbf{k}+\mathbf{q}\rangle = \frac{e\hbar}{m}\frac{f_0(\mathbf{k}+\beta) - f_0(\mathbf{k})}{\hbar\omega - (\mathcal{E}_{\mathbf{k}+\beta} - \mathcal{E}_{\mathbf{k}}) - i\eta}(\mathbf{k} + \tfrac{1}{2}\beta)\mathbf{A}(\beta)\exp(i\omega t)\,\delta_{q\beta} \tag{11}$$

where η is positive and infinitesimal. (The analysis can be extended to include collisions, in which case $\eta = \hbar/\tau$ is finite and kept throughout the rest of the analysis. In the present paper we will simplify the presentation by confining to the collisionless regime and taking at some subsequent point $\eta \downarrow 0$.)

The induced current is found from

$$\mathbf{J}(\mathbf{r}, t) = -e\,\mathrm{Tr}[\rho^{(1)}\mathbf{J}_{0p}^{(0)}(\mathbf{r}) + \rho^{(0)}\mathbf{J}_{0p}^{(1)}(\mathbf{r}t)],$$

where

$$\mathbf{J}_{0p}^{(0)}(\mathbf{r}) \equiv \tfrac{1}{2}[\mathbf{p}_e/m\,\delta(\mathbf{r} - \mathbf{r}_e) + \delta(\mathbf{r} - \mathbf{r}_e)\mathbf{p}_e/m]$$

and

$$J_{0p}^{(1)} \equiv \frac{e}{m}\mathbf{A}(\mathbf{r}t)\delta(\mathbf{r} - \mathbf{r}_e).$$

Using Eq. (11), the following expression results:

$$\mathbf{J}(\omega, \beta) = \frac{e^2n_0}{m}\mathbf{A}(\omega, \beta) - \frac{e^2\hbar^2}{m^2V}\sum_{\mathbf{k}}\frac{f_0(\mathbf{k}+\beta) - f_0(\mathbf{k})}{\hbar\omega - (\mathcal{E}_{\mathbf{k}+\beta} - \mathcal{E}_{\mathbf{k}}) - i\eta} \times (\mathbf{k} + \tfrac{1}{2}\beta)[(\mathbf{k} + \tfrac{1}{2}\beta)\cdot\mathbf{A}(\omega, \beta)], \tag{12}$$

where $n_0 = (1/V)\sum_{\mathbf{k}} f_0(\mathbf{k})$ is the free-carrier density.

We finally obtain the susceptibility $\chi_p(\omega, \beta)$, defined by Eq. (2), by substituting $\mathbf{A} = (i/\omega)\mathbf{E}_z$ in Eq. (12) and eliminating the longitudinal-longitudinal term. After some mathematical manipulation one obtains

$$\chi_p(\omega, \beta) = -\frac{e^2}{V\beta^2}\sum_{\mathbf{k}} \frac{f_0(\mathbf{k}+\boldsymbol{\beta}) - f_0(\mathbf{k})}{\hbar\omega - (\mathcal{E}_{\mathbf{k}+\boldsymbol{\beta}} - \mathcal{E}_{\mathbf{k}}) - i\eta}. \quad (13)$$

Expression (13) can now be used in Eq. (9) to solve the coupled-wave dispersion equation. We may readily solve Eq. (9) using first-order expansion[2] of β: $\beta = \beta_1 + \Delta\beta$ (β_1 is the propagation constant of the electromagnetic component with no coupling).

$$\Delta\beta = \tfrac{1}{2}K_1 S\beta_1^2\omega \frac{\chi_p(\omega, \beta_1)}{1 + \chi_p(\omega, \beta_1)/\epsilon}. \quad (14)$$

In particular,

$$\mathrm{Im}\beta = \tfrac{1}{2}K_1 S\beta_1^2\omega \frac{\mathrm{Im}\chi_p}{|1 + \chi_p/\epsilon|^2}. \quad (15)$$

In Eq. (13) the summation over k is replaced by an integration and the limit $\eta \downarrow 0$ is taken, resulting in

$$\mathrm{Re}\chi_p = -\frac{e^2}{(2\pi)^3\beta_1^2}(P)\int d^3k \frac{f_0(\mathbf{k}+\boldsymbol{\beta}_1) - f_0(\mathbf{k})}{\hbar\omega - (\mathcal{E}_{\mathbf{k}+\boldsymbol{\beta}_1} - \mathcal{E}_{\mathbf{k}})}, \quad (16)$$

$$\mathrm{Im}\chi_p = -\frac{\pi e^2}{(2\pi)^3\beta_1^2}\int d^3k\,[f_0(\mathbf{k}+\boldsymbol{\beta}_1) - f_0(\mathbf{k})]\delta[\hbar\omega - (\mathcal{E}_{\mathbf{k}+\boldsymbol{\beta}_1} - \mathcal{E}_{\mathbf{k}})]. \quad (17)$$

These last two equations can now be used in Eq. (15) to find the gain $g = 2\,\mathrm{Im}\beta$ of the electromagnetic mode.

In general the distribution function $f_0(\mathbf{k})$ is the Fermi distribution which under appropriate conditions can be approximated by the Boltzmann distribution. When a dc field is applied, $f_0(\mathbf{k})$ may become a quite complicated function. An equilibrium distribution, shifted by the amount of the drift momentum $\hbar k_0 = mv_0$ (where v_0 is the carrier drift velocity), is commonly used as a first-order approximation, with some experimental justification.[7] We will assume that Boltzmann statistics apply and use a shifted Maxwellian distribution

$$f_0(\mathbf{k}) = \frac{(2\pi)^3 n_0}{(\sqrt{\pi}k_{\mathrm{th}})^3}\exp\left(-\frac{(\mathbf{k}-\mathbf{k}_0)^2}{k_{\mathrm{th}}^2}\right), \quad (18)$$

where $\hbar^2 k_{\mathrm{th}}^2/(2m) \equiv k_B T$ and T is the carrier's temperature, k_B is the Boltzmann constant, and $\mathbf{k}_0$ is in the z direction (Fig. 1).

Equation (15)—(17) with the distribution function (18) give

$$\mathrm{Im}\beta = -\alpha \times \frac{\mathrm{Im}[G(\zeta_b) - G(\zeta_a)]}{\{1 - (\gamma^3/\beta_1^3)\mathrm{Re}[G(\zeta_b) - G(\zeta_a)]\}^2 + \{\gamma^3/\beta_1^3)\mathrm{Im}[G(\zeta_b) - G(\zeta_a)]\}^2} \quad (19)$$

where $\gamma^3 \equiv \omega_p^2 m^2/\hbar^2 k_{\mathrm{th}}$, $\omega_p^2 \equiv n_0 e^2/m\epsilon$, $\alpha \equiv \frac{1}{2}\epsilon\omega K_1 S\gamma^3/\beta_1$, $\zeta_a \equiv (k_a - k_0)/k_{\mathrm{th}}$, and $\zeta_b \equiv (k_b - k_0)/k_{\mathrm{th}}$. $\mathbf{k}_a$ and $\mathbf{k}_b \equiv \mathbf{k}_a + \boldsymbol{\beta}_1$ are the couple of state vectors between which transitions occur (see Fig. 2) and are connected by the condition of energy conservation which is imposed by the δ function in Eq. (17): $\hbar^2(k_b^2 - k_a^2)/(2m) = \hbar\omega$.

The plasma dispersion function is defined (for real argument ζ) as

$$\mathrm{Re}G(\zeta) = \pi^{-1/2}(P)\int_{-\infty}^{\infty}\exp(-x^2)/(x-\zeta)\,dx, \quad (20)$$

$$\mathrm{Im}G(\zeta) = -\pi^{1/2}\exp(-\zeta^2). \quad (21)$$

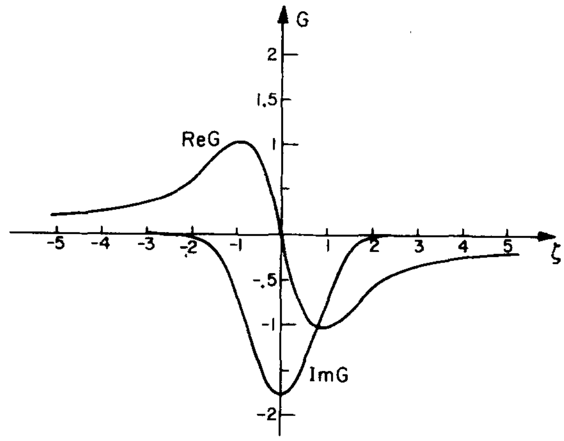

FIG. 3. The plasma dispersion equation for real argument.

It is plotted in Fig. 3 and tabulated in Ref. 8.

It is interesting to note that Eq. (9) reduces to the expression derived in Ref. 2 in the limit $\beta_1 \to 0$, which appears to be the classical limit. Note also that $\mathrm{Im}\beta$ is positive (which corresponds to gain) when $\mathrm{Im}G(\zeta_b) < \mathrm{Im}G(\zeta_a)$, i.e., according to Eqs. (21) and (18) when $f_0(k_b) > f_0(k_a)$. This may be viewed as the population inversion condition for transitions from b to a (see Fig. 2).

To demonstrate the effect let us examine an example. Consider a structure as in Fig. 1 made out of GaAs and some GaAs alloy (say GaAlAs). The dielectric constant can be expanded in a Fourier series $\epsilon(z) \approx \epsilon_0[\epsilon_{r0} + 2\epsilon_1\cos(2\pi/L)z]$. The interaction impedance for this structure is derived in the Appendix and given by Eq. (A2): $K_1 = (1/n)(\mu/\epsilon_0)^{1/2}(1/S\beta_1^2)(\sin^2\phi/\cos\phi)(\epsilon_{r1}/\epsilon_{r0})^2$. We assume $\epsilon_{r0} = 12$, $\epsilon_1/\epsilon_{r0} = 0.05$, $L = 300$ Å, the mode "zig-zag" angle is chosen as $\phi = 65°$, the electron effective mass is $m/m_0 = 0.08$, $n_0 = 8\times10^{16}$ cm$^{-1/3}$, $T = 50$ °K, $k_0 = 1.4\times10^6$ cm^{-1}, $\omega = 3.5\times10^{13}$ sec^{-1}. Equations (19) and (A2) yield a gain $g_1 = 2(\mathrm{Im}\beta) = 1$ cm^{-1}.

It should be recalled at this point that until now, we have examined the electromagnetic-wave interaction through its first-order space harmonic only. In Ref. 3 we proposed an approximation which accounts for the contribution of the other space harmonics in an additive way. Appreciable contribution should be expected from the -1 space harmonic which has propagation parameter $\beta_{-1} \simeq -\beta_1$. Equation (19) and an equation similar to Eq. (A2) apply for this interaction too.[3] For the particular example above, we find that the interaction via the -1 harmonic contributes attenuation $g_{-1} = 0.5$ cm^{-1}. The total gain in this case is therefore $g = 0.5$ cm^{-1}.

This predicted gain will probably be masked by the background infrared absorption in GaAs, which may not be the case with other materials and more optimal parameter combinations. GaAs structure may, though, be most appropriate for experimental observation of the effect (as function of current) because of the developed state of the art of this material.[5,6]

Inspection of Eq. (19) indicates that higher gain may be possible at higher carrier concentration n_0 and at lower temperatures T. However, under these conditions the electron plasma becomes degenerate and Fermi-Dirac statistics should be used instead of the Boltzmann statistics. This is not attempted in the present paper.

PHASE-MATCHED PLASMON-PHOTON COUPLING

Inspection of Eq. (19) and Fig. 3 reveals that exceedingly large gain (or loss) should be expected if the denominator of Eq. (19) is made very small. However, this denominator is exactly the square of the absolute value of ϵ_p^{-1} [Eq. (5)], and the parameter values ω and β_1 at which it tends to vanish $\epsilon_p^{-1}(\omega, \beta_1) \simeq 0$, very nearly satisfy the plasma dispersion relation and thus are appropriate to the free-plasma-wave propagation. So the gain (or loss) of an electromagnetic wave is high when the propagation constant of one of its space harmonics is close to that of the system's plasma wave. The physical meaning of this is that the electromagnetic wave is phase matched to the system's plasma wave which leads to a high rate of power exchange between the two waves.

To have $|\epsilon_p^{-1}|$ small, its real and imaginary parts must simultaneously become small,

$$1-(\gamma^3/\beta_1^3)\,\mathrm{Re}[G(\zeta_b)-G(\zeta_a)]=0, \qquad (22)$$

$$(\gamma^3/\beta_1^3)\,\mathrm{Im}[G(\zeta_b)-G(\zeta_a)]\simeq 0. \qquad (23)$$

Since $\zeta_b > \zeta_a$, a necessary condition for satisfying the first condition [Eq. (22)] is $\zeta_a > 1$ or $\zeta_b < -1$, because only then can one have $\mathrm{Re}[G(\zeta_b)-G(\zeta_a)]>0$ (see Fig. 3). In order to satisfy the second condition too [Eq. (23)], it is necessary to have also $|\zeta_a|, |\zeta_b| \gg 1$. Physically this means operation at the "tail" of the carrier distribution function where the Landau damping of the plasmon is small.

Coupling of an electromagnetic wave to a plasma wave can occur together with population inversion ($\zeta_a, \zeta_b < 0$) or without population inversion ($\zeta_a, \zeta_b > 0$). In the first case there will be transfer of energy from the plasma wave to the electromagnetic wave, and vice versa in the second case. Practically it is hard to find physical conditions for plasmon coupling with optical gain, because this requires carrier distribution with thermal spread which is much smaller than the drift velocity. Examples of strong optical attenuation, on the other hand, due to plasmon coupling are much more readily found.

Assume again a GaAs structure as in Fig. 1 with $\epsilon_r = 12$, $\epsilon_1/\epsilon_r = 0.05$, $L = 1040$ Å, $m/m_0 = 0.08$, $n_0 = 5\times10^{17}$ cm^{-3}, $T = 350\,°$K, $k_0 = 0$, $\omega = 5.04\times10^{13}$ sec^{-1}, and $\phi = 20°$. From Eq. (19) one finds that the electromagnetic wave is strongly attenuated in this condition. The net attenuation is $g = -24.5$ cm^{-1}. Furthermore, this effect can be observed as a very sharp absorption peak, since any small change in ζ (for example, by changing k_0) will break the "phase matching" and will turn the denominator of Eq. (19) from a very small number to an appreciable one, thus decreasing the attenuation $|g|$. It must be noted that even higher attenuation with more convenient parameters are possible when we chose ζ_1 larger; however, in this case the effect of collisions should be accounted for, which will not be attempted in the present article.

CONCLUSION

We presented in this article a quantum-mechanical analysis for the interaction of an electromagnetic wave in the infrared regime with a free-carrier plasma in periodic semiconductor structure. We demonstrated by examples the possibility of optical gain and the possibility of phase-matched electromagnetic-wave—plasmon coupling, in periodic structure. These examples were presented just to demonstrate the suggested effects and are not optimized with respect to materials, structural parameters, and physical conditions.

We may mention that appreciable progress has taken place and is still expected to take place in the development of very-short-period artificial periodic structures (superlattice).[5,6] At the present state of the art it seems that periodic structure of the kind suggested in Refs. 2 and 3 (where only the surface is periodically perturbed) may be more readily achievable than the structure of Fig. 1 of this article. Nevertheless, we chose to present the principles of the effects discussed through the example of the structure in Fig. 1, because the lateral variation of the electromagnetic space harmonic in structures with periodically perturbed surface is quite strong and the application of the one-dimensional analysis to that case may have a limited validity. The structure of Fig. 1, though easier to analyze, may be harder to produce. The consistent advancement in the technology of superlattice growth[6] may provide means for its fabrication. A slightly different embodiment of the device (Fig. 4) may be more compatible with the present state of the art of superlattice growth. It may also be a more efficient structure since it is possible to get a larger component of the electric field along the current flow and periodicity direction. Approximative extension of the previous analysis to this structure indicates (for the particular example of attenuation) a fewfold increase in attenuation which can be achieved in this structure.

Experiments in electromagnetic-wave interaction with semiconductor plasmas in periodic structures may

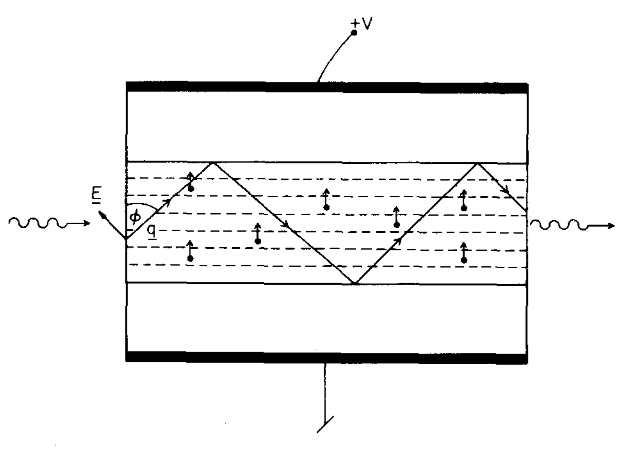

FIG. 4. A superlattice structure for traveling-wave amplification or attenuation.

be interesting for the investigation of solid-state plasmons, drifting carriers distribution, and conduction-band structure. From the device application point of view it may be possible to utilize the optical gain for devices in the infrared regime but achieving net gain (in excess of the free carrier and fundamental lattice absorption) may be rather difficult. The effect of phase-matched plasmon coupling appears attractive as a fast and efficient modulation mechanism in the infrared regime.

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APPENDIX

To calculate the interaction impedance [Eq. (8)], one must know the amplitude of the electromagnetic-wave space harmonic E_{z_1}. The electromagnetic modes which propagate in a dielectric waveguide as in Fig. 1 can be described as two-plane-waves propagating in "zigzag" by reflection from the boundaries. Hence, we may solve E_{z_1} for an infinite structure which is periodic in z and apply it to the structure of Fig. 1.

Assume that the dielectric constant is periodic in the z direction so that

$$\epsilon(z)=\epsilon_0\sum_{m=-\infty}^{\infty}\epsilon_{r_m}\exp(im2\pi z/L).$$

The Floquet wave solution is $\mathbf{E}(\mathbf{r})=\sum_m \mathbf{E}_m \exp i(\omega t-\mathbf{q}_m\mathbf{r})$, where $\mathbf{q}_m=\mathbf{q}_0+m(2\pi/L)\cdot\hat{e}_z$, $\hat{e}_z$ is a unit vector in the z direction, and $|\mathbf{q}_0|\simeq n\omega/c=\sqrt{\epsilon_{r_0}}\,\omega/c$. Then if one keeps only the first-order Fourier expansion of the dielectric constant in the Poisson equation $\nabla\cdot[\epsilon(z)\mathbf{E}(\mathbf{r})]=0$, a very simple set of equations results:

$$\epsilon_{r_0}\mathbf{q}_m\mathbf{E}_m+\epsilon_{r_1}(\mathbf{q}_m\mathbf{E}_{m-1}+\mathbf{q}_m\mathbf{E}_{m+1})=0. \quad \text{(A1)}$$

We substitute $m=1$ in Eq. (A1), and assume $\mathbf{q}_1\cdot\mathbf{E}_2 \ll \mathbf{q}_1\cdot E_0$. Also we assume that $2\pi/L \gg |\mathbf{q}_0|$ so that $\mathbf{q}_1 \simeq \beta_1 \equiv (\mathbf{q}_1\cdot\hat{e}_z)\hat{e}_z$ is in the z direction. It results that $E_{z_1} \simeq -(\epsilon_{r_1}/\epsilon_{r_0})\cdot E_{z_0}$. The electromagnetic-mode power which is carried through cross section S perpendicular to $\hat{e}_z$ is approximated by the power of the zero space harmonic, so that $p=\frac{1}{2}(\epsilon_0/\mu)E_0^2\cos\phi\cdot S$, where $\cos\phi \equiv (\mathbf{e}_z\cdot\mathbf{q})/|\mathbf{q}|$. From Eq. (8) one readily obtains

$$K_1=\frac{1}{n}\left(\frac{\mu}{\epsilon_0}\right)^{1/2}\frac{1}{S\beta_1^2}\frac{\sin^2\phi}{\cos\phi}\left(\frac{\epsilon_{r_1}}{\epsilon_{r_0}}\right)^2. \quad \text{(A2)}$$

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