

MONOTONE MATRIX FUNCTIONS OF SUCCESSIVE ORDERS

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ABSTRACT. This paper extends a result obtained by Wigner and von Neumann. We prove that a non-constant real-valued function, $f(x)$, in $C^3(I)$ where I is an interval of the real line, is a monotone matrix function of order $n + 1$ on I if and only if a related, modified function $g_{x_0}(x)$ is a monotone matrix function of order n for every value of x_0 in I , assuming that f' is strictly positive on I .

A real-valued function, f , is matrix monotone of order n over an interval, I , if, for two symmetric $n \times n$ matrices, A, B , with eigenvalues in I , $A \leq B$ implies that $f(A) \leq f(B)$. Call the class of all such functions, $P_n(I)$. Many results on monotone matrix functions may be found in the monograph by Donoghue [2]. Of particular interest is the characterization of functions that are operator monotone, i.e., monotone of all orders n . We denote this class of functions by $P_\infty(I)$. Löwner [4] showed that functions that are operator monotone on I are exactly those functions that are analytic with positive imaginary part in the upper half-plane (Pick functions) and that have an analytic continuation to the lower half-plane across I , where the continuation is by reflection. This is the content of Löwner's Theorem. Many different proofs of this theorem are known (see, e.g., Donoghue [2] for a survey of a number of them). Perhaps the least well-known proof of Löwner's Theorem is due to Wigner and von Neumann [6]. We prove an extension of one of the results in their work.

Define a modified function, $g_{x_0}(x)$ as follows:

$$g_{x_0}(x) = -\frac{1}{f(x) - f(x_0)} + \frac{1}{f'(x_0)(x - x_0)}.$$

We define $g_{x_0}(x_0)$ to be the limit as $x \rightarrow x_0$ of $g_{x_0}(x)$, and this limit exists if f is C^2 there, by L'Hôpital's Rule and Taylor's Theorem.

Theorem 1. *Let f be in $C^3(a, b)$ with the additional property that $f'(x) > 0$ for all $x \in (a, b)$. f is a monotone matrix function of order $n + 1$ ($n \geq 1$) on an interval, (a, b) , if and only if g_{x_0} is a monotone matrix function of order n on the same interval for all $x_0 \in (a, b)$.*

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The assumptions in Theorem 1 are not too restrictive. It is known (see [2]) that $f \in P_n(a, b)$ implies $f \in C^{(2n-3)}(a, b)$. Moreover, if f is non-constant and in $P_2(a, b)$, then $f'(x) > 0$ for all $x \in (a, b)$. Since $P_k(a, b) \subset P_j(a, b)$ for all $k > j$, f' is strictly positive if $f \in P_n(a, b)$ with $n \geq 2$.

It should also be noted that Wigner and von Neumann [6] proved one half of Theorem 1, namely that f monotone of order $n + 1$ in (a, b) implies that g_{x_0} is monotone of order n in (a, b) for any x_0 in (a, b) .

Proof of Theorem 1. The divided differences of a function, f , may be defined as follows:

$$[x_i, x_j]_f = \frac{f(x_i) - f(x_j)}{x_i - x_j}, \quad [x_i, x_i]_f = f'(x_i).$$

It is well known that $f \in P_{n+1}(a, b)$ if and only if for any set of $n + 1$ distinct points $x_0, x_1, x_2, \dots, x_n$ of the interval (a, b) , the Pick matrix formed with the divided differences of f , $K_{ij}^{(f)} = [x_i, x_j]_f = a_{ij}$, is a positive semidefinite matrix. We therefore consider the Pick matrices of f and g_{x_0} . It is straightforward to verify that for x_i and x_j different from x_0 ,

$$[x_i, x_j]_{g_{x_0}} = \frac{a_{00}a_{ij} - a_{i0}a_{j0}}{f'(x_0)(f(x_i) - f(x_0))(f(x_j) - f(x_0))}.$$

Also, by L'Hôpital's Rule and Taylor's Theorem, it can be shown that $g_{x_0}(x_0)$ and $g'_{x_0}(x_0)$ exist if $f \in C^3(a, b)$.

We consider the determinants of the principal submatrices of $K^{(g_{x_0})}$. Consider the determinant of any one such submatrix. The term-by-term expansion of the determinant is a linear combination of fractions with the same denominator. This denominator is a product of terms of the form $(f(x_i) - f(x_0))^2$ (for each row i of $K^{(g_{x_0})}$ in the submatrix) and a positive power of $f'(x_0)$. Therefore this common denominator is positive. Factoring out this common denominator leaves the determinant of the corresponding submatrix of $L^{(g_{x_0})}$, defined by $L_{ij}^{(g_{x_0})} = a_{00}a_{ij} - a_{i0}a_{j0}$, which is just the matrix of the numerators of each entry in $K^{(g_{x_0})}$. Denote the entries of $L^{(g_{x_0})}$ by b_{ij} . Since the determinant of each submatrix of $K^{(g_{x_0})}$ has the same sign as the determinant of the corresponding submatrix of $L^{(g_{x_0})}$, it suffices to consider $L^{(g_{x_0})}$ in what follows.

Suppose f is monotone of order $n + 1$. Let us now consider a $k \times k$ principal submatrix of $L^{(g_{x_0})}$, $L_{sub}^{(g_{x_0})}$, containing diagonal entries $b_{n_i n_i}$. Using Sylvester's Determinant Identity ([5], [1]), we find that $\det L_{sub}^{(g_{x_0})} = a_{00}^{k-1} \det K_{sub}^{(f)}$, where $K_{sub}^{(f)}$ is the $(k + 1) \times (k + 1)$ principal submatrix of $K^{(f)}$ having diagonal entries a_{00} and $a_{n_i n_i}$. Since f is monotone of order $n + 1$, $\det L_{sub}^{(g_{x_0})}$ is nonnegative, and therefore any Pick matrix of g_{x_0} formed by a subset of the points (x_1, x_2, \dots, x_n) (with $x_i \neq x_0$) is a positive semidefinite matrix. If we take a limit of the positive semidefinite Pick matrices as $x_i \rightarrow x_0$, we get a positive semidefinite matrix. This is the first half of the theorem, and was the result used by Wigner and von Neumann in their continued fraction proof of Löwner's Theorem.

The converse is easily proved also. Suppose $g_{x_0} \in M_n(a, b) \quad \forall x_0 \in (a, b)$. Consider any principal submatrix of $K^{(f)}$. If the submatrix is 1×1 , then it is one of the diagonal elements, which are nonnegative since we know that f' is positive on (a, b) . Now consider a $k \times k$ principal submatrix of $K^{(f)}$, which is the Pick matrix associated to f of the distinct points $x_{n_1}, x_{n_2}, \dots, x_{n_k}$ ($n_i \in \{1, 2, \dots, n\}$).

By another application of Sylvester's Determinant Identity, the determinant of this submatrix has the same sign as the determinant of the $(k-1) \times (k-1)$ submatrix of the Pick matrix associated to $g_{x_{n_1}}$ of the points $x_{n_2}, x_{n_3}, \dots, x_{n_k}$. But these latter determinants are nonnegative, which means that f is monotone of order $n+1$. \square

The question of gaps between classes of monotone matrix functions has been addressed in Donoghue [2], and, more recently, in an article by Hansen et al. [3]. The corollary below, which concerns such gaps, follows from Theorem 1.

Corollary 2. *If $n \geq 2$, then $P_n(a, b) = P_{n+1}(a, b)$ implies $P_n(a, b) = P_\infty(a, b)$.*

Proof. Suppose, for some $n \geq 2$, $P_n(a, b) = P_{n+1}(a, b)$. Take $f \in P_{n+1}(a, b)$. Then, $g_{x_0} \in P_n(a, b)$ for all $x_0 \in (a, b)$, and since $P_n = P_{n+1}$, we have that $g_{x_0} \in P_{n+1}(a, b)$ for all $x_0 \in (a, b)$, and, therefore, by our theorem, $f \in P_{n+2}(a, b)$. The corollary then follows by induction. \square

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