



## Supporting Information

for *Adv. Sci.*, DOI: 10.1002/advs.202001126

### Generalized Solutions of Parrondo's Games

*Jin Ming Koh and Kang Hao Cheong\**

# Supporting Information: Generalized Solutions of Parrondo's Games

Jin Ming Koh<sup>1,2</sup> and Kang Hao Cheong<sup>1,3,\*</sup>

<sup>1</sup>*Singapore University of Technology and Design (SUTD), 8 Somapah Rd, S487372, Singapore*

<sup>2</sup>*California Institute of Technology, Pasadena, CA 91125, USA*

<sup>3</sup>*SUTD-Massachusetts Institute of Technology International Design Centre, Singapore*

## Appendix A: Three-State Game Pair

### A1. Conditions

Let  $f_z$  be the probability that the capital ever reaches zero given a starting amount of  $z \in \mathbb{Z}$  units. It is then consequent of Markov chain theory [1, 2] that either  $f_z = 1 \forall z \geq 0$ , reflecting that the game is fair or losing, or that  $f_z < 1 \forall z > 0$ , in which case it is possible that the capital will grow indefinitely and the game is winning. The set  $\{f_z\}$  is the minimal non-negative solution to the recurrence equation

$$f_{Mk} = p'_1 f_{Mk+1} + r'_1 f_{Mk} + q'_1 f_{Mk-1}, \quad (\text{A1a})$$

$$f_{Mk+l} = p'_2 f_{Mk+l+1} + r'_2 f_{Mk+l} + q'_2 f_{Mk+l-1}, \quad (\text{A1b})$$

where  $k \geq 1$ ,  $l \in \{1, \dots, M-1\}$ , and boundary condition  $f_0 = 1$  applies. Solving Eq. (A1b) yields a general solution  $f_{Mk+l} = a_1 \phi_2^l + a_2$ , where

$$a_1 = \frac{f_{Mk} - f_{M(k+1)}}{1 - \phi_2^M}, \quad (\text{A2a})$$

$$a_2 = \frac{f_{M(k+1)} - f_{Mk} \phi_2^M}{1 - \phi_2^M}. \quad (\text{A2b})$$

Substituting this solution into Eq. (A1a) and solving yields  $f_{Mk} = a_3(\varphi^k - 1) + 1$  with  $\varphi \equiv \phi_1' \phi_2'^{M-1}$ . If  $\varphi \geq 1$ , the minimal non-negative solution occurs when  $a_3 = 0$ , so  $f_{Mk} = 1 \forall k \geq 0$ ; otherwise, the minimal non-negative solution occurs when  $a_3 = 1$ , thus  $f_{Mk} = \varphi^k < 1 \forall k > 0$ . It is hence summarized that  $f_{Mk} = \min\{1, \varphi^k\}$  for stochastically mixed games. This leads to the result in Eq. (1) of the main paper.

### A2. Stationary Distribution

The eigenvalue equation  $\omega = \omega H$  produces the set of equations

$$\omega_1 = \omega_1 r'_1 + \omega_2 q'_2 + \omega_M p'_2, \quad (\text{A3a})$$

$$\omega_2 = \omega_1 p'_1 + \omega_2 r'_2 + \omega_3 q'_2, \quad (\text{A3b})$$

$$\omega_m = \omega_{m-1} p'_2 + \omega_m r'_2 + \omega_{m+1} q'_2, \quad (\text{A3c})$$

$$\omega_M = \omega_1 q'_1 + \omega_{M-1} p'_2 + \omega_M q'_2, \quad (\text{A3d})$$

where  $m \in [3, M-1]$ . Eq. (A3c) is first solved with  $\omega_2$  and  $\omega_3$  as boundary conditions. Invoking Eqs. (A3a) and (A3b) eliminates  $\omega_2$  and  $\omega_3$ , and the normalization constraint  $\sum_{m=1}^M \omega_m = 1$  sets  $\omega_1$ . This yields the solution given in Eq. (3) of the main paper.

### A3. Capital Distribution

Suppose that out of  $n$  rounds,  $n_+$  result in wins,  $n_0$  result in draws, and  $n_-$  result in losses. At steady-state, the average outcome probabilities are  $\bar{s} = \omega_1 s'_1 + (1 - \omega_1) s'_2$  for  $s \in \{p, q, r\}$ , where  $\omega_1$  is the stationary distribution of capital state  $S_1$ . The distribution  $\mathcal{P}_n(k)$  representing the probability of having  $k \in \mathbb{Z} \cap [-n, n]$  capital on round  $n$  can thus be written

$$\mathcal{P}_n(k) = \sum \frac{n!}{n_+! n_0! n_-!} \bar{p}^{n_+} \bar{r}^{n_0} \bar{q}^{n_-}, \quad (\text{A4})$$

where the summation occurs over the solution set of simultaneous Diophantine equations  $n_+ + n_0 + n_- = n$  and  $n_+ - n_- = k$ . Such a solution set can be parametrized as  $(n_+, n_0, n_-) = (u + (|k| + k)/2, n - |k| - 2u, u + (|k| - k)/2)$  where  $u \in \mathbb{Z}$  and  $0 \leq u \leq \lfloor (n - k)/2 \rfloor$ . The summation in Eq. (A4) is thus over  $u$ , enabling the closed-form calculation of  $\mathcal{P}_n(k)$ .

The expected capital  $\mu(n)$  can be computed from this explicit capital distribution as

$$\mu(n) = \sum_{k=-n}^n k P_n(k) = (\bar{p} - \bar{q}) n, \quad (\text{A5})$$

which is identical to the result obtained in Eq. (5) of the main paper.

### A4. Fundamental Matrix

We have  $Z = (I - H + \Omega)^{-1}$ , but to simplify calculations, the identity  $Z(I - H) = I - \Omega$  is used. This produces the set of equations

$$\delta_{i1} - \omega_1 = Z_{i1}(1 - r'_1) - Z_{i2} q'_2 - Z_{iM} p'_2, \quad (\text{A6a})$$

$$\delta_{i2} - \omega_2 = -Z_{i1} p'_1 + Z_{i2}(1 - r'_2) - Z_{i3} q'_2, \quad (\text{A6b})$$

$$\delta_{ij} - \omega_j = -Z_{i(j-1)} p'_1 + Z_{ij}(1 - r'_2) - Z_{i(j+1)} q'_2, \quad (\text{A6c})$$

$$\delta_{iM} - \omega_M = -Z_{i1} q'_1 - Z_{i(M-1)} p'_2 + Z_{iM}(1 - r'_2), \quad (\text{A6d})$$

where  $i \in [1, M]$  and  $j \in [3, M-1]$ . Eq. (A6c) is first solved with  $Z_{i2}$  and  $Z_{i3}$  as boundary conditions. Invoking Eqs. (A6a) and (A6b) eliminates  $Z_{i2}$  and  $Z_{i3}$ , and the normalization constraint  $\sum_{j=1}^M Z_{ij} = 1$  sets  $Z_{i1}$ . This yields

\* Corresponding author. E-mail: kanghao\_cheong@sutd.edu.sg

$$Z_{i1} = (p'_2/\rho) \left\{ 2q_2'^M (p'_2 - q'_2)^2 \left\{ q_2' [1 - i - (p'_2 - q'_2)] + \phi_2'^{-M} [Mp'_2\phi_2'^i - q_2' [M + 1 - i - (p'_2 - q'_2)]] \right\} \right. \\ \left. - 2\alpha p_2' \phi_2'^{-M} [M^2 q_2'^M (p'_2 - q'_2)^2 - p_2' q_2' (p_2'^M - q_2'^M) (1 - \phi_2'^M)] \right. \\ \left. + M\beta q_2' (p'_2 - q'_2) \left\{ q_2'^M [M(p'_2 - q'_2) + (p'_2 + q'_2)] + p_2'^M [(M-1)p'_2 - (M+1)q'_2] \right\} \right\}, \quad (\text{A7a})$$

$$Z_{i2} = \left\{ p_2'^M [p_2'\phi_2^i (p'_2 - q'_2) - q_2'\phi_2'^M (p'_2 - q'_2)] + Z_{i1} (1 - \phi_2') \left\{ p_1' p_2'^M q_2'^2 + p_2' q_2'^M [p_2' q_1' - q_2' (p_1' + q_1')] \right\} \right. \\ \left. + \alpha p_2' [Mp_2'^M q_2' - p_2' q_2'^M - (M-1)p_2'^{M+1}] + \beta q_2' \left\{ q_2'^M [M(p'_2 - q'_2) + q_2'] - p_2'^M q_2' \right\} \right\} / \\ q_2'^2 (p'_2 - q'_2) (p_2'^M - q_2'^M), \quad (\text{A7b})$$

$$\rho = 2(p'_2 - q'_2)^2 \left\{ p_2'^M q_2' \left\{ p_2' [(M-1)p_1' + p'_2 + q_1'] - q_2' (Mp_1' + p'_2) \right\} \right. \\ \left. - p_2' q_2'^M \left\{ Mq_1' (p'_2 - q'_2) + q_2' [(p'_2 - q'_2) - (p_1' - q_1')] \right\} \right\}. \quad (\text{A7c})$$

We have  $Z_{ij} = Z_{i1}$  for  $j = 1$ , and for  $2 \leq j \leq M$ , the general solution is

$$Z_{ij} = \left\{ Z_{i1} p_1' (p'_2 - q'_2) \left( 1 - \phi_2'^{2-j} \right) - Z_{i2} q_2' (p'_2 - q'_2) \left( 1 - \phi_2'^{1-j} \right) + \alpha \left\{ p_2' \phi_2'^{-1} + \phi_2'^{-j} [p_2' (j-2) - q_2' (j-1)] \right\} \right. \\ \left. - \beta \left[ (j-1)p'_2 - (j-2)q'_2 - p_2' \phi_2'^{2-j} \right] + (p'_2 - q'_2) \left( 1 - \phi_2'^{-R(j-i)} \right) \right\} / (p'_2 - q'_2)^2, \quad (\text{A8})$$

where  $R(x)$  is the unit ramp function.

## Appendix B: Three-state $M$ -branch game pair

### B1. Conditions

Every  $M$  consecutive states is termed a tier. Winning a tier necessitates winning across all  $M$  branches; furthermore, an arbitrary number of losses  $l_i$  at each state  $S_i$  is allowed, so long as there is a corresponding number of wins to compensate. The probability of winning and losing a tier, respectively  $\tilde{p}$  and  $\tilde{q}$ , is thus

$$\tilde{s} = \prod_{i=1}^M s'_i \cdot \sum_{l_1, \dots, l_M=0}^{\infty} \left[ \prod_{i=1}^M (q'_i p'_{i-1})^{l_i} \right], \quad (\text{B1})$$

where  $s'_i = \gamma s + (1-\gamma)s_i$  for  $s \in \{p, r, q\}$  and  $i \in [1, M]$  are the mixed transition probabilities. The game is winning, fair, and losing when  $\tilde{p} > \tilde{q}$ ,  $\tilde{p} = \tilde{q}$ , and  $\tilde{p} < \tilde{q}$  respectively. Cancellation of terms yield the simplistic condition in Eq. (8) of the main paper.

### B2. Stationary Distribution

The eigenvector equation  $\omega = \omega H$  produces the set of equations

$$\omega_1 = \omega_1 r'_1 + \omega_2 q'_2 + \omega_M p'_M, \quad (\text{B2a})$$

$$\omega_m = \omega_{m-1} p'_{m-1} + \omega_m r'_m + \omega_{m+1} q'_{m+1}, \quad (\text{B2b})$$

$$\omega_M = \omega_1 q'_1 + \omega_{M-1} p'_{M-1} + \omega_M r'_M, \quad (\text{B2c})$$

where  $m \in [2, M-1]$ . But, as the recurrence in Eq. (B2b) involves non-constant coefficients, the usual method of solving the characteristic polynomial cannot be used. Instead, a tracking method can be used on the recursion tree to arrive at

$$\omega_m = F_m^{[1]} \omega_1 + G_m^{[1]} \omega_2, \quad (\text{B3})$$

where  $F$  and  $G$  are counting functions as written in the main paper. Invoking Eq. (B2a) to eliminate  $\omega_2$  and the normalization constraint  $\sum_{m=1}^M \omega_m = 1$  to set  $\omega_1$  then

yields the solution for  $\omega_m$  as presented in Eq. (10) of the main paper.

### B3. Fundamental Matrix

Again, the identity  $Z(I - H) = I - \Omega$  is used. This produces the set of equations

$$\delta_{i1} - \omega_1 = Z_{i1} (1 - r'_1) - Z_{i2} q'_2 - Z_{iM} p'_M, \quad (\text{B4a})$$

$$\delta_{ij} - \omega_j = -Z_{i(j-1)} p'_{j-1} + Z_{ij} (1 - r'_j) - Z_{i(j+1)} q'_{j+1}, \quad (\text{B4b})$$

$$\delta_{iM} - \omega_M = -Z_{i1} q'_1 - Z_{i(M-1)} p'_{M-1} + Z_{iM} (1 - r'_M), \quad (\text{B4c})$$

where  $i \in [1, M]$  and  $j \in [2, M-1]$ . As the recurrence in Eq. (B4b) is non-constant, the method of characteristic polynomials cannot be applied. The recurrence tree is tracked to give

$$Z_{ij} = F_j^{[1]} Z_{i1} + G_j^{[1]} Z_{i2} + T_j^{[3]} - G_j^{[2]} / q'_{i+1}, \quad (\text{B5})$$

where

$$T_m^{[l]} = \sum_{\substack{k \in K(n_1, n_2) \\ (n_1, n_2) \in \zeta_l(m)}} \left( \frac{\omega_{m - \sigma_{|k|}(k) - 1}}{q'_{m - \sigma_{|k|}(k)}} \right) \pi_m(k), \quad (\text{B6a})$$

$$\zeta_d(m) = \bigcup_{i=d}^m \xi_i(m). \quad (\text{B6b})$$

Invoking Eq. (B4a) to eliminate  $Z_{i2}$  and the normalization constraint  $\sum_{j=1}^M Z_{ij} = 1$  to set  $Z_{i1}$  then yields the solution

$$Z_{ij} = \left( F_j^{[1]} + \Lambda G_j^{[1]} + \delta_{1j} \right) Z_i^* - \varrho_i G_j^{[1]} + T_j^{[3]} - \frac{G_j^{[2]}}{q'_{i+1}}, \\ Z_i^* = \frac{1 + \frac{1}{q'_{i+1}} \sum_{j=i}^M G_j^{[2]} - \sum_{j=2}^M T_j^{[3]} + \varrho_i \sum_{j=2}^M G_j^{[1]}}{1 + \sum_{j=2}^M F_j^{[1]} + \Lambda \sum_{j=2}^M G_j^{[1]}}, \\ \varrho_i = \frac{p_M \left( q'_{i+1} T_M^{[3]} - G_M^{[2]} \right) - q'_{i+1} \omega_1}{\left( p'_M G_M^{[1]} + q'_2 \right) q'_{i+1}}. \quad (\text{B7})$$

### Appendix C: Numerical Simulations

Double-precision numerical Monte Carlo simulations of the presented game structures were written in *Java 11*. The stochastic mixing of games and selection of outcomes at each game round were performed using native random-number generators, provided by the `SplittableRandom` class to facilitate parallelization. All statistical results were averaged over at least  $10^6$  trials where applicable to suppress noise; simulations were run on a 16-core machine. Visualization of the simulation data and comparison against theory were performed in *Mathematica 12*.

### Appendix D: Computational Complexity

A technical motivation for pursuing the analytical capital statistics solutions in the main paper is to bypass the computational cost of Markov-chain calculations in making predictions. We analyze this as follows. The basic approach to compute capital statistics is to propagate the initial capital distribution vector using the full transition matrix. For  $n$  game rounds, since the capital may

be changed by a unit amount each round, the capital distribution vector will be  $\mathcal{O}(n)$  long, and the transition matrix will be  $\mathcal{O}(n \times n)$  large. Therefore arriving at the final capital distribution will be  $\mathcal{O}(n^3)$  expensive, followed by post-processing to compute statistics of interest. This is unwieldy, as  $n$  can be large, say,  $\sim \mathcal{O}(10^6)$  in some practical applications.

As was carried out in this paper, and is common in literature, one may contract the matrix to  $M \times M$  by considering capital states modulo  $M$ , and accordingly the time complexity to compute the final distribution vector may be improved to  $\mathcal{O}(nM^2)$ . It is possible to reduce this further, but at least some matrix operations are required, such as spectral decomposition, inverses, and multiplication, so  $\mathcal{O}(M^3)$  is a lower bound. In this work  $M$  is general, and can be arbitrarily large, so this is not entirely satisfactory. Having explicit analytical, or otherwise closed-form, solutions for capital statistics allows the side-stepping of these calculations to give nearly  $\mathcal{O}(1)$  theoretical evaluations, a huge advancement.

---

[1] G. P. Harmer, D. Abbott, P. G. Taylor, and J. M. R. Parrondo, AIP Conf. Proc **511**, 189 (2000).

[2] J. G. Kemeny and J. L. Snell, *Markov Chains* (Springer-Verlag, New York, 1976).