Supplemental Material for Survival of the fractional Josephson effect in time-reversal-invariant topological superconductors

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LOCAL MIXING REVIEW

Reference 1 derived that when the operators describing a Majorana Kramers pair depend on some parameters $\boldsymbol{\eta}$, the local mixing angle is given by

$$\theta = \frac{1}{2}\oint d\boldsymbol{\eta}\{\gamma(\boldsymbol{\eta}), \nabla_{\boldsymbol{\eta}}\tilde{\gamma}(\boldsymbol{\eta})\}. \tag{1}$$

We review a simple example of how a non-zero mixing angle can arise microscopically.

Consider a TRITOPS wire modeled by two Kitaev chains. In the dimerized limit, the Hamiltonian is given by

$$H_0 = \frac{\varepsilon}{2}\sum_{j,\sigma} i\gamma_{jb\sigma}\gamma_{j+1a\sigma} \tag{2}$$

where $\sigma \in \{\uparrow, \downarrow\}$ labels the two time-reversed copies, a and b the two Majorana flavors that form the spinful fermion, and j the site. The two Majorana operators corresponding to the same j and σ transform oppositely under $\mathcal{T}$: here we take the signs in Eq. (**??**) to be $s_a = -1$, $s_b = 1$. This model has four MZMs, $\gamma_{1a\sigma}$, $\gamma_{Nb\sigma}$. Let us assume that local perturbations on the left end of the wire take the form of a chemical potential H_μ and s-wave pairing H_Δ:

$$H_\mu(t) = \frac{\beta\cos\alpha(t)}{2}\sum_\sigma (i\gamma_{1a\sigma}\gamma_{1b\sigma} + 1) \tag{3}$$

$$H_\Delta(t) = \frac{\beta\sin\alpha(t)}{2}\left(i\gamma_{1a\uparrow}\gamma_{1b\downarrow} - i\gamma_{1a\downarrow}\gamma_{1b\uparrow}\right), \tag{4}$$

where α parametrizes the ratio of the two terms and is time-dependent. Both Eq. (3) and Eq. (4) commute with $\mathcal{T}$. In the presence of these perturbations, the new zero mode operators become time-dependent as well:

$$\gamma_1(t) = \cos\zeta\gamma_{1a\uparrow} - \sin\zeta\left(\cos\alpha(t)\gamma_{2a\uparrow} + \sin\alpha(t)\gamma_{2a\downarrow}\right) \tag{5}$$

$$\tilde{\gamma}_1(t) = \cos\zeta\gamma_{1a\downarrow} - \sin\zeta\left(\cos\alpha(t)\gamma_{2a\downarrow} - \sin\alpha(t)\gamma_{2a\uparrow}\right), \tag{6}$$

where $\tan\zeta = \beta/\varepsilon$. Solving Eq. (1), we have

$$\theta_1 = -\sin^2\zeta\oint d\alpha = -\sin^2\zeta\int_0^T dt\,\dot{\alpha}(t), \tag{7}$$

where $\alpha(T) = \alpha(0) + 2\pi n$, with $n \in \mathbb{Z}$. Therefore, provided α has non-trivial winding, $\theta_1 \neq 0$ and γ_1, $\tilde{\gamma}_1$ undergo local mixing.

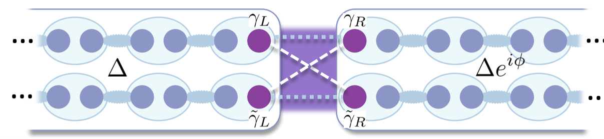

FIG. 1. TRITOPS Josephson junction with each wire modeled by two Kitaev chains in the dimerized limit. Dotted lines indicate hybridization of Majorana Kramers pairs (dark purple) leading to Eq. (**??**). Local mixing arises when Majorana Kramers pairs undergo time-dependent coupling to gapped Majorana modes.

TRITOPS JOSEPHSON JUNCTION

A TRITOPS wire can be thought of as two topological superconductors related by time reversal symmetry. Labeling the two copies with a spin degree of freedom $\sigma \in \{\uparrow, \downarrow\}$, time reversal acts on the fermionic operators of the Jth wire as $\boldsymbol{c}_J = (c_{J,\uparrow}, c_{J,\downarrow})^T$ as [2]

$$\mathcal{T}\boldsymbol{c}_J\mathcal{T}^{-1} = is_J(\phi)e^{i\phi_J}\sigma_y\boldsymbol{c}_J. \tag{8}$$

The sign $s_J(\phi) = \pm 1$ represents a $\mathbb{Z}_2$ gauge-freedom when defining symmetry transformations of superconductors. When multiple TRITOPS are present but disconnected, each satisfies its own time reversal symmetry according to Eq. (8). When two TRITOPS are connected, e.g. by a Josephson junction, the global symmetry transformation must be consistent between the two. Therefore, a TRITOPS Josephson junction is only symmetric under $\mathcal{T}$ when the phase difference between the left and right superconductors is a multiple of π. We label these discrete values the "time-reversal-invariant points" and fix $s_L(\phi_L) = +1$ and $s_R(\phi_R = n\pi + \phi_L) = -(-1)^n$ below.

A simple model of a TRITOPS Josephson junction is

$$H_{\rm JJ} = \tilde{\lambda}c^\dagger_{L\uparrow}c_{R\uparrow} - \tilde{\lambda}^* c^\dagger_{L\downarrow}c_{R\downarrow} + \lambda c^\dagger_{L\uparrow}c_{R\downarrow} + \lambda^* c^\dagger_{L\downarrow}c_{R\uparrow} + h.c. \tag{9}$$

where L/R denote whether the fermion belongs to the wire on the left/right end of the junction and we can generically allow for different tunneling amplitudes between wires with the same and different σ labels.

Each fermionic operator can be written as

$$c_{J\sigma} = \frac{e^{-\frac{i\phi_J}{2}}}{2}\left(\gamma_{Ja\sigma} + i\gamma_{Jb\sigma}\right), \tag{10}$$

where ϕ_J is the superconducting phase of the wire on the Jth side of the junction ($J \in \{L, R\}$), and the operators $\gamma_{c\sigma}$, $c \in \{a, b\}$ satisfy the Majorana anticommutation relation

$$\{\gamma_{c\sigma}, \gamma_{c'\sigma'}\} = 2\delta_{cc'}\delta_{\sigma\sigma'}. \tag{11}$$

Equations (8) and Eq. (10) imply Eq. (**??**).

Each copy of a topological superconductor has a single MZM at its end point. Projecting to the low-energy subspace takes

$$c_{L\sigma} \to \frac{e^{-\frac{i\phi_L}{2}}}{2}\gamma_{La\sigma}, \qquad c_{R\sigma} \to \frac{ie^{-\frac{i\phi_R}{2}}}{2}\gamma_{Rb\sigma}. \tag{12}$$

Fixing $\phi_L = 0$ and $\phi_R = \phi$ and dropping the a/b label of the Majorana operators, Eq. (**??**) becomes

$$H_{\mathrm{JJ}} = \frac{1}{2}\sum_{\sigma=\uparrow/\downarrow}\Big[\cos(\phi/2)\left(\sigma\mathrm{Re}[\tilde{\lambda}]\, i\gamma_{L\sigma}\gamma_{R\sigma} + \mathrm{Re}[\lambda]\, i\gamma_{L\sigma}\gamma_{R\bar{\sigma}}\right) + \sin(\phi/2)\left(\mathrm{Im}[\tilde{\lambda}]\, i\gamma_{L\sigma}\gamma_{R\sigma} + \sigma\mathrm{Im}[\lambda]\, i\gamma_{L\sigma}\gamma_{R\bar{\sigma}}\right)\Big]. \tag{13}$$

In the above, we have written $\bar{\sigma}$ to indicate the opposite choice of σ for the subscripts, and σ as a coefficient to correspond to $\pm$ for $\uparrow$ / $\downarrow$. We recover Eq. (**??**) by setting $\tilde{\lambda} = 0$ for simplicity, denoting the real and imaginary parts of λ with subscripts e/o, and identifying $(\gamma_{L,\uparrow}, \gamma_{L,\downarrow}, \gamma_{R,\uparrow}, \gamma_{R,\downarrow}) \to (\gamma_L, \tilde{\gamma}_L, \tilde{\gamma}_R, \gamma_R)$.

Deriving H_{eff}

We now derive Eq. (**??**) in the main text. The Majorana operators in Eqs. (**??**) and (**??**) can be written in terms of complex fermionic operators as

$$f^\dagger = \frac{1}{2}\left(\gamma'_\varepsilon - i\gamma_\varepsilon\right) \tag{14}$$

$$\tilde{f}^\dagger = \frac{1}{2}\left(\tilde{\gamma}'_\varepsilon - i\tilde{\gamma}_\varepsilon\right) \tag{15}$$

$$c^\dagger = \frac{1}{2}\left(\gamma_L + i\gamma_R\right) \tag{16}$$

$$\tilde{c}^\dagger = \frac{1}{2}\left(\tilde{\gamma}_L + i\tilde{\gamma}_R\right) \tag{17}$$

Then, defining even-parity basis states so that $|0\rangle$ corresponds to the vacuum state annihilated by $c, \tilde{c}, f, \tilde{f}$ and

$$|1\rangle = c^\dagger\tilde{c}^\dagger|0\rangle \tag{18}$$

$$|2\rangle = f^\dagger c^\dagger|0\rangle \tag{19}$$

$$|3\rangle = f^\dagger\tilde{c}^\dagger|0\rangle \tag{20}$$

$$|4\rangle = \tilde{f}^\dagger c^\dagger|0\rangle \tag{21}$$

$$|5\rangle = \tilde{f}^\dagger\tilde{c}^\dagger|0\rangle \tag{22}$$

$$|6\rangle = f^\dagger\tilde{f}^\dagger|0\rangle \tag{23}$$

$$|7\rangle = f^\dagger\tilde{f}^\dagger c^\dagger\tilde{c}^\dagger|0\rangle \tag{24}$$

so that $f^\dagger f = \frac{1}{2}\left(1 - i\gamma_\varepsilon\gamma'_\varepsilon\right)$ and $c^\dagger c = \frac{1}{2}\left(1 - i\gamma_L\gamma_R\right)$ and similarly for the time-reversed partners. In this basis, the full Hamiltonian can be written in first-quantized form as

$$H = \left(\begin{array}{cc|cccc|cc}
\boldsymbol{\lambda_e} & 0 & \frac{\beta}{2}\cos\alpha & -\frac{\beta}{2}\sin\alpha & \frac{\beta}{2}\sin\alpha & \frac{\beta}{2}\cos\alpha & 0 & 0 \\
0 & -\boldsymbol{\lambda_e} & \frac{\beta}{2}\sin\alpha & \frac{\beta}{2}\cos\alpha & -\frac{\beta}{2}\cos\alpha & \frac{\beta}{2}\sin\alpha & 0 & 0 \\
\hline
\frac{\beta}{2}\cos\alpha & \frac{\beta}{2}\sin\alpha & \varepsilon - \boldsymbol{\lambda_o} & 0 & 0 & 0 & \frac{\beta}{2}\sin\alpha & -\frac{\beta}{2}\cos\alpha \\
-\frac{\beta}{2}\sin\alpha & \frac{\beta}{2}\cos\alpha & 0 & \varepsilon + \boldsymbol{\lambda_o} & 0 & 0 & \frac{\beta}{2}\cos\alpha & \frac{\beta}{2}\sin\alpha \\
\frac{\beta}{2}\sin\alpha & -\frac{\beta}{2}\cos\alpha & 0 & 0 & \varepsilon - \boldsymbol{\lambda_o} & 0 & -\frac{\beta}{2}\cos\alpha & -\frac{\beta}{2}\sin\alpha \\
\frac{\beta}{2}\cos\alpha & \frac{\beta}{2}\sin\alpha & 0 & 0 & 0 & \varepsilon + \boldsymbol{\lambda_o} & \frac{\beta}{2}\sin\alpha & -\frac{\beta}{2}\cos\alpha \\
\hline
0 & 0 & \frac{\beta}{2}\sin\alpha & \frac{\beta}{2}\cos\alpha & -\frac{\beta}{2}\cos\alpha & \frac{\beta}{2}\sin\alpha & 2\varepsilon + \boldsymbol{\lambda_e} & 0 \\
0 & 0 & -\frac{\beta}{2}\cos\alpha & \frac{\beta}{2}\sin\alpha & -\frac{\beta}{2}\sin\alpha & -\frac{\beta}{2}\cos\alpha & 0 & 2\varepsilon - \boldsymbol{\lambda_e}
\end{array}\right) \tag{25}$$

where we have adopted the shorthand $\boldsymbol{\lambda_e} = 2\lambda_e\cos\left(\frac{\phi}{2}\right)$ and $\boldsymbol{\lambda_o} = 2\lambda_o\sin\left(\frac{\phi}{2}\right)$ and suppressed the time-dependence of ϕ and α. Working to order ε^{-2}, the two lowest energies are

$$\epsilon_{1/2}(t) = \pm 2\lambda_e\left(1 - \frac{\beta^2}{2\varepsilon^2}\right)\cos\left(\frac{\phi(t)}{2}\right) \tag{26}$$

with corresponding instantaneous eigenstates

$$|\psi_1(t)\rangle = +|1\rangle - \frac{\beta}{2\varepsilon}\left(\sin\alpha(t)\left[\nu_{-+}(t)|2\rangle + \nu_{--}(t)|5\rangle\right] - \cos\alpha(t)\left[\nu_{--}(t)|3\rangle - \nu_{-+}(t)|4\rangle\right]\right) - \left(\frac{\beta}{2\varepsilon}\right)^2\left(|1\rangle - |6\rangle\right). \quad (27)$$

$$|\psi_2(t)\rangle = -|0\rangle + \frac{\beta}{2\varepsilon}\left(\cos\alpha(t)\left[\nu_{++}(t)|2\rangle + \nu_{+-}(t)|5\rangle\right] - \sin\alpha(t)\left[\nu_{+-}(t)|3\rangle - \nu_{++}(t)|4\rangle\right]\right) + \left(\frac{\beta}{2\varepsilon}\right)^2\left(|0\rangle + |7\rangle\right) \quad (28)$$

We have defined $\nu_{p,p'} = 1 + \left(-p\boldsymbol{\lambda}_e(t) + p'\boldsymbol{\lambda}_o(t)\right)/\varepsilon$ with $p, p' = \pm 1$.

As described in the main text, when $\beta\dot{\alpha}(t) \ll \varepsilon^2$, transitions between the low and high-energy states are negligible. Solutions to the Schrödinger equation for a state initialized in the low-energy subspace take the form

$$|\Phi(t)\rangle = v_1(t)|\psi_1(t)\rangle + v_2(t)|\psi_2(t)\rangle. \quad (29)$$

The coefficients satisfy the equation of motion

$$i\partial_t \boldsymbol{v} = \left[H_{\text{inst}}(t) + H_{\text{B}}(t)\right]\boldsymbol{v} = H_{\text{eff}}(t)\boldsymbol{v} \quad (30)$$

where

$$H_{\text{inst}}(t) = 2\lambda_e\left(1 - \frac{\beta^2}{2\varepsilon^2}\right)\cos\left(\frac{\phi(t)}{2}\right)\sigma_z \quad (31)$$

$$H_{\text{B}}(t) = \langle\psi_1(t)|\partial_t|\psi_2(t)\rangle\sigma_y = \frac{\dot{\alpha}}{2}\frac{\beta^2}{\varepsilon^2}\sigma_y, \quad (32)$$

recovering Eq. (**??**):

$$H_{\text{eff}}(t) = 2\lambda_e'\cos\left[\phi(t)/2\right]\sigma_z + \dot{\alpha}(t)\frac{\beta^2}{2\varepsilon^2}\sigma_y. \quad (33)$$

Note that all λ_o dependence drops out at order ε^{-2}.

Time evolution according to $H_{\text{eff}}(t)$

Equation (**??**) has the general form $H(t) = a(t)\sigma_z + b(t)\sigma_y$ with instantaneous eigenvalues and eigenstates

$$\epsilon_\pm(t) = \pm\Omega(t) = \pm\sqrt{a(t)^2 + b(t)^2} \quad (34)$$

$$|\pm(t)\rangle = \mp i\beta_\pm|0\rangle + \beta_\mp|1\rangle \quad (35)$$

where $|0\rangle$, $|1\rangle$ are the eigenstates of σ_z corresponding to eigenvalues ± 1, respectively, and we have defined $\beta_\pm(t) = \sqrt{\frac{\Omega(t)\pm a(t)}{2\Omega(t)}}$. Consider the parameter

$$A = \frac{\max_t|\langle +(t)|\dot{H}_{\text{eff}}(t)| - (t)\rangle|}{4\min_{t'}\Omega(t')^2} \quad (36)$$

$$= \frac{\max_t\left[|-\dot{a}(t)b(t) + \dot{b}(t)a(t)|/\Omega(t)\right]}{4\min_{t'}\Omega(t')^2}. \quad (37)$$

The adiabatic theorem asserts that when $A \ll 1$, the system initialized in an energy eigenstate remains in that eigenstate throughout the time evolution [3, 4].

For $H_{\text{eff}}(t)$, A and $\Omega(t)$ evaluate to

$$A = \frac{\lambda_e'|-\cos[\phi(\tilde{t})/2]\ddot{\theta}(\tilde{t}) - \dot{\phi}(\tilde{t})\sin[\phi(\tilde{t})/2]\dot{\theta}(\tilde{t})/2|/\Omega(\tilde{t})}{4\Omega(\bar{t})^2} \quad (38)$$

$$\Omega(t) = \sqrt{4\lambda_e'^2\cos^2[\phi(t)/2] + \dot{\theta}(t)^2/4}, \quad (39)$$

where $\tilde{t}$ is the time that maximizes the numerator and $\bar{t}$ the time that minimizes the denominator. When $\ddot{\theta}(t) = 0$, A reduces to the inverse of the Landau-Zener parameter x:

$$A^{\text{LZ}} = \frac{\lambda_e'\dot{\phi}(t^*)}{\dot{\theta}^2} = \frac{1}{x}. \quad (40)$$

Alternatively, for the quench considered in Fig. **??**, $\dot{\phi} \to 0$ and A becomes

$$A^{\text{quench}} = \frac{\lambda_e'\cos[\phi(t_1)/2]\ddot{\theta}(t_1)/\Omega(t_1)}{4\Omega(\bar{t})^2}, \quad (41)$$

where t_1 is the location of the quench. If $\dot{\theta}_{\text{max}} \ll 2\lambda_e'\cos[\phi(t_1)/2]$, then the denominator reduces to $\dot{\theta}^2_{\text{max}}$ and

$$A^{\text{quench}} \approx \frac{\ddot{\theta}(t_1)}{2\dot{\theta}^2_{\text{max}}} \approx \frac{1}{2\tau\dot{\theta}_{\text{max}}}. \quad (42)$$

Thus the transition probability approaches zero for $\tau\dot{\theta}_{\text{max}} \gg 1/2$. If instead $\dot{\theta}_{\text{max}}/2$ and $2\lambda_e'\cos[\phi(t_1)/2]$ are comparable (as is the case in Fig. **??**), the adiabatic criterion becomes $8\tau\Omega(t_1) \gg 1$.

To analyze the time evolution according to Eq. (**??**) more generally, we can consider the Schrödinger equation for a state $|\psi(t)\rangle = \sum_{\sigma=\pm} c_\sigma(t)|\sigma(t)\rangle$. The coefficients $c_\pm(t)$ satisfy

$$i\begin{pmatrix}\dot{c}_+(t)\\ \dot{c}_-(t)\end{pmatrix} = \left(\Omega(t)\sigma_z + v(t)\sigma_y\right)\begin{pmatrix}c_+(t)\\ c_-(t)\end{pmatrix} \quad (43)$$

where

$$v(t) \equiv \langle +(t)|\partial_t| - (t)\rangle = \frac{\dot{a}(t)b(t) - \dot{b}(t)a(t)}{2\Omega(t)^2} \quad (44)$$

$$= \frac{\lambda_e'\left(\dot{\phi}\sin[\phi/2]\dot{\theta}/2 + \cos[\phi/2]\ddot{\theta}\right)}{2\Omega^2}. \quad (45)$$

When $|v(t)| \ll \Omega(t)$, the coefficients evolve according to a diagonal Hamiltonian and the system initialized in the instantaneous ground state will remain in the instantaneous ground state at later times, resulting in the conventional 2π-periodic Josephson effect. (Note that $\max_t |v(t)|/\Omega(t)$ corresponds to A when $\tilde{t} = \bar{t}$.) When $|v(t)| \gg \Omega(t)$, the instantaneous energy states undergo Rabi oscillations, and the current-phase relation will generally be aperiodic.

APERIODICITY FROM LOCAL MIXING

Consider a junction described by Eq. (13). In the even parity sector $i\gamma_L\gamma_R = i\tilde{\gamma}_L\tilde{\gamma}_R$, we can define Pauli matrices

$$\sigma^x = i\gamma_L\gamma_R = i\tilde{\gamma}_L\tilde{\gamma}_R \tag{46}$$

$$\sigma^y = i\gamma_L\tilde{\gamma}_R = -i\tilde{\gamma}_L\gamma_R \tag{47}$$

$$\sigma^z = i\gamma_L\tilde{\gamma}_L = i\gamma_R\tilde{\gamma}_R \tag{48}$$

so that

$$H_{\text{JJ}}^{(e)}(t) = 2\sqrt{\lambda_e^2 + \tilde{\lambda}_e^2}\cos\left(\frac{\phi(t)}{2}\right)\begin{pmatrix} 0 & a - ib \\ a + ib & 0 \end{pmatrix} \tag{49}$$

for $a = \lambda_e/\sqrt{\lambda_e^2 + \tilde{\lambda}_e^2}$, $b = \tilde{\lambda}_e/\sqrt{\lambda_e^2 + \tilde{\lambda}_e^2}$. When ϕ is not equal to an odd multiple of π, the junction eigenstates are

$$|I_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} a - ib \\ \pm 1 \end{pmatrix}. \tag{50}$$

Note that $|I_-\rangle$ is the instantaneous ground state of the junction for $\phi < \pi$, while $|I_+\rangle$ is the instantaneous ground state for $\phi > \pi$. Consider a thought experiment of a phase-biased TRITOPS Josephson junction, undergoing the following protocol. Initialize the system at $\phi = 0$ in the state $|I_+\rangle$, then evolve the phase ϕ such that at the kth time invariant point one of the Majorana Kramers pairs accrues a local mixing angle θ_k. In the absence of any other noise sources, between the $k-1$th and kth time-reversal invariant points, the system is in a superposition of junction eigenstates

$$|\psi_{12}\rangle = \cos\left(\frac{\sum_k \theta_k}{2}\right)|I_+\rangle + i\sin\left(\frac{\sum_k \theta_k}{2}\right)|I_-\rangle \tag{51}$$

with current expectation value

$$\langle I(\phi)\rangle = -\frac{e}{\hbar}\sqrt{\lambda_e^2 + \tilde{\lambda}_e^2}\cos\left(\sum_{j=1}^{k}\theta_j\right)\sin\left(\frac{\phi}{2}\right). \tag{52}$$

When $\theta_k \neq 2\pi$,

$$\cos\left(\sum_{j=1}^{k-1}\theta_j\right) \neq \cos\left(\sum_{j=1}^{k+1}\theta_j\right) \tag{53}$$

thus the current expectation value is not 4π periodic. More generally, $\langle I(\phi)\rangle$ is aperiodic except for fine-tuned choices of the θ_j.

The phase-biased system is not necessarily the most experimentally accessible, as usually phase would be tuned by a magnetic field, whose presence would break the time reversal symmetry of the junction. A more physically relevant setup is for the junction to be voltage-biased, so that the DC Josephson equation implies a constant phase sweep speed $\dot{\phi} = 2eV/\hbar = \omega_J$. When the system undergoes a 4π periodic fractional Josephson effect, the power spectrum of the current

$$P(\omega) = \lim_{C\to\infty}\int_0^C dt\int_0^C dt'\langle I(t')I(t)\rangle e^{i\omega(t'-t)} \tag{54}$$

exhibits a peak at $\omega = \pm\omega_J/2$. An aperiodic current-phase relation manifests as no peak in the power spectrum.

If the only source of noise is local mixing, then the probability $q_\pm$ of occupying junction eigenstates $|I_\pm\rangle$ only changes after passing through a time reversal invariant point. If $s_k = 1 - p_k$ is the probability of transitioning between junction eigenstates (i.e. p_k is the probability of transitioning between instantaneous energy eigenstates) at the kth such point, and $q_\pm(t_k)$ is the occupation probability of $|I_\pm\rangle$ preceding that point, then

$$\begin{pmatrix} q_+(t_{k+1}) \\ q_-(t_{k+1}) \end{pmatrix} = \begin{pmatrix} 1 - s_k & s_k \\ s_k & 1 - s_k \end{pmatrix}\begin{pmatrix} q_+(t_k) \\ q_-(t_k) \end{pmatrix}. \tag{55}$$

Approximating $s_k = \sin^2(\theta_k/2)$ by its average value, $\bar{s}$

$$\begin{pmatrix} q_+(t_{k+1}) \\ q_-(t_{k+1}) \end{pmatrix} = \frac{1}{2}\left(\left[1 + (1-2\bar{s})^k\right]\mathbb{1} + \left[1 - (1-2\bar{s})^k\right]\sigma_x\right)\begin{pmatrix} q_+(t_0) \\ q_-(t_0) \end{pmatrix}. \tag{56}$$

The matrix in Eq. (56) defines the propagator from t_j to t_{j+k}:

$$U\left(t = \frac{2\pi k}{\omega_J}\right) = \frac{1}{2}\left(\left[1 + (1-2\bar{s})^k\right]\mathbb{1} + \left[1 - (1-2\bar{s})^k\right]\sigma_x\right). \tag{57}$$

Note that in the large k limit the system approaches the maximally mixed state at a rate $\omega_J \ln[1 - 2\bar{s}]/2\pi$.

The current is

$$I_{\pm}(t) = \pm\frac{e}{\hbar}\sqrt{\lambda_e^2+\tilde{\lambda}_e^2}\sin\left(\frac{\omega_J}{2}t\right), \tag{58}$$

corresponding to correlator for $t' > t$ [5]

$$\langle I(t')I(t)\rangle = \sum_{ij=\pm} I_i(t')I_j(t)U_{ij}(t'-t) = 2I_0^2\sin\left(\frac{\omega_J t}{2}\right)\sin\left(\frac{\omega_J t'}{2}\right)(1-2\bar{s})^{\frac{\omega_J(t'-t)}{2\pi}} \tag{59}$$

$$= 2I_0^2\sin\left(\frac{\omega_J t}{2}\right)\sin\left(\frac{\omega_J t'}{2}\right)\left(e^{\frac{\omega_J}{2\pi}\ln[1-2\bar{s}](t'-t)}\Theta\left(1-2\bar{s}\right)+e^{\frac{\omega_J}{2\pi}(i\pi+\ln[2\bar{s}-1])(t'-t)}\Theta\left(2\bar{s}-1\right)\right) \tag{60}$$

for $I_0 = \frac{e}{\hbar}\sqrt{\lambda_e^2+\tilde{\lambda}_e^2}$. Therefore, the power spectrum is

$$P(\omega) = \lim_{C\to\infty}\frac{2I_0^2}{C}\int_0^C dt\int_0^C dt' e^{i\omega(t'-t)}\sin\left(\frac{\omega_J t'}{2}\right)\sin\left(\frac{\omega_J t}{2}\right) \times\left(e^{\frac{\omega_J}{2\pi}\ln[1-2\bar{s}]|t-t'|}\Theta(1-2\bar{s})+e^{i\frac{\omega_J}{2}(t-t')}e^{\frac{\omega_J}{2\pi}\ln[2\bar{s}-1]|t-t'|}\Theta(2\bar{s}-1)\right) \tag{61}$$

$$= \frac{I_0^2}{2\pi\omega_J}\sum_{a=\pm1}\left(\frac{\ln[1-2\bar{s}]}{\left(\frac{\omega}{\omega_J}+\frac{a}{2}\right)^2+\left(\frac{\ln[1-2\bar{s}]}{2}\right)^2}\Theta(1-2\bar{s})+\frac{\ln[2\bar{s}-1]}{\left(\frac{\omega}{\omega_J}+\frac{a-1}{2}\right)^2+\left(\frac{\ln[2\bar{s}-1]}{2}\right)^2}\Theta(2\bar{s}-1)\right). \tag{62}$$

As $\bar{s}\to 0$ ($\bar{p}\to 1$, $r\gg 1$), the power spectrum has two peaks at $\omega = \pm\omega_J/2$, corresponding to a fractional Josephson effect. When $\bar{s}\to 1$ ($\bar{p}\to 0$, $r\ll 1$), the power spectrum peaks at $\omega = 0$, ω_J, corresponding to the standard 2π-periodic Josephson effect. As $\bar{s}\to 1/2$ from either side, $P(\omega)$ flattens- signaling an aperiodic current-phase relation.

$\mathcal{T}^2 = +1$ JOSEPHSON JUNCTIONS

Consider the model for a topological superconductor suggested by Refs. 6 and 7

$$H = \int_x \psi^\dagger(-\frac{\partial_x^2}{2m}-\mu-h\sigma^x-i\alpha\sigma^y\partial_x)\psi+\Delta\psi_\uparrow\psi_\downarrow+H.c. \tag{63}$$

where spin indices have been suppressed, h is a Zeeman term, α is the spin-orbit coupling, and Δ is the superconduting gap. This Hamiltonian is symmetric under $\mathcal{T} = \mathcal{K}$ time-reversal-symmetry [8, 9], which in this model is simply complex conjugation. This symmetry is an artifact of the low-energy Hamiltonian and can be broken by adding higher-order hopping terms or interactions. Nonetheless, such terms are expected to be weak and for low energies the wire satisfies $\mathcal{T}^2 = +1$.

We now derive the Josephson junction Hamiltonian for the setup shown in Fig. **??** when each Majorana nanowire individually satisfies $\mathcal{T}$. Label the fermionic operators by c_{Jj}, $J\in\{L,R\}$ labeling the left/right side of the junction, and $j\in\{1,2\}$ labeling the top or bottom wire. The $c_{J,j}$ transform trivially under $\mathcal{T}$; thus the most general non-interacting Hamiltonian describing the Josephson junction that is even under $\mathcal{T}$ is

$$H_{\text{JJ}}^{(+1)} = \sum_{J,j\neq k}\Lambda_{Jjk}c_{Jj}^\dagger c_{Jk}+\sum_{j,k}\left(2\lambda_{jk}c_{Lj}^\dagger c_{Rk}+h.c\right). \tag{64}$$

where all tunneling amplitudes are real: $\Lambda_{Jjk}, \lambda_{jk}\in\mathbb{R}$.

Time reversal symmetry acts on the complex fermionic operators $c_{Jj} = \frac{e^{-i\phi_J/2}}{2}(\gamma_{Jaj}+i\gamma_{Jbj})$ as $c_{Jj}\to s_J(\phi_J)e^{i\phi_J}c_{Jj}$. Thus, we once again see that $\phi=\phi_R-\phi_L = n\pi$ are the time-reversal invariant points. Fixing $\phi_L = 0$ and $\phi_R = \phi$, the transformation on the Majorana operators is

$$\gamma_{Jaj}\to s_J\gamma_{Jaj}, \quad \gamma_{Jbj}\to -s_J\gamma_{Jbj} \tag{65}$$

with signs $s_L = 1$, $s_R = (-1)^n$ for $\phi = n\pi$.

Projection to the low-energy subspace takes the same form as Eq. (12)

$$c_{Lj}\to\frac{e^{-\frac{i\phi_L}{2}}}{2}\gamma_{Laj} \qquad c_{Rj}\to\frac{ie^{-\frac{i\phi_R}{2}}}{2}\gamma_{Rbj}. \tag{66}$$

From here on, we drop the a/b labels and write the zero mode operators as γ_{Jj}. Under $\mathcal{T}$,

$$i\gamma_{J1}\gamma_{J2}\to -i\gamma_{J1}\gamma_{J2} \tag{67}$$

$$i\gamma_{Lj}\gamma_{Rk}\to s_Ls_Ri\gamma_{Lj}\gamma_{Rk} = (-1)^n i\gamma_{Lj}\gamma_{Rk}. \tag{68}$$

Equation (67) implies $\Lambda_J = 0$ (and is precisely why in the presence of $\mathcal{T}$ the quantum dot-based MZM parity measurement proposed in Ref. 10 does not work). Therefore, we recover Eq. (**??**)

$$H_{\text{JJ}}^{(+1)} = \sum_j i\lambda_{jk} \cos\left(\frac{\phi}{2}\right) \gamma_{Lj}\gamma_{Rk}. \tag{69}$$

The model given in Eq. (25) is purely real and thus also satisfies $\mathcal{T}^2 = +1$ symmetry. As such, the derivation of Eq. (**??**) similarly holds for this system as well.

Multiwire topological Josephson junctions

We investigate the effect of local mixing on a Josephson junction between two sets of m Majorana wires. Above we argued that $m = 2$ reproduces the aperiodic behavior of a TRITOPS junction. We now demonstrate that interactions restore 4π periodicity for $m = 3$, and 2π periodicity for $m = 4$. Such Josephson junctions offer a testbed for probing the $\mathbb{Z}_8$ classification of Majorana nanowires theorized by Ref. 8.

We consider the low-energy Hamiltonian

$$H = \sum_j E_j \cos\left(\frac{\phi}{2}\right) i\gamma_{Lj}\gamma_{Rj}, \tag{70}$$

where j runs over each of the m wires and L and R signify the wires to the left and right of the junction. After one evolution $\gamma_{Rj} \to -\gamma_{Rj}$. We can combine Majorana fermions into Dirac fermions as $c_j = \gamma_{Lj} + i\gamma_{Rj}$. After one evolution the occupation of this bound state switches. Notice that for m wires we track 2^m bound states, which for free fermions all intersect at $\phi = \pi$ (where all energies are 0).

- $m = 1$. The standard fractional Josephson is immune to local mixing, as the two bound states differ by local fermion parity. No local mixing terms are allowed that mix the states at $\phi = \pi$.

- $m = 2$. The model posited in previous Appendices still respects $\mathcal{T}^2 = +1$ symmetry. The four states in question split into even and odd parity states. Unlike the $m = 1$ wire, however, fermion parity in the junction remains the same after a 2π evolution (as both bound states switch occupation) and so we can restrict ourselves to the even parity sector. The crossing at $\phi = \pi$ is protected by our symmetry, but that does not prevent local mixing.

  Interactions do not play an important role for $m = 2$. The only acceptable interaction at $\phi = \pi$ reads

  $$H_{\text{int}} = w_1(i\gamma_{L1}\gamma_{R1})(i\gamma_{L2}\gamma_{R2}), \tag{71}$$

  which only splits the even and odd parity sectors and does not affect the Josephson periodicity. We recover local mixing, implying (for certain parameter regimes) the loss of 4π periodicity.

- $m = 3$. While it may seem that $m = 3$ wires will suffer from local mixing as well, interactions conspire to restore 4π periodicity (in much the same way that interactions stabilize an 8π-periodic fractional Josephson effect in the absence of local mixing for a junction of proximitized quantum spin Hall edges [11]). Notice that after a 2π evolution, the local fermion parity in the junction changes. We track 8 states, 4 with even parity and 4 with odd parity, and these states all intersect at $\phi = \pi$.

  However, adding interactions

  $$\begin{aligned} H_{\text{int}} = {} & w_1(i\gamma_{L1}\gamma_{R1})(i\gamma_{L2}\gamma_{R2}) \\ & + w_2(i\gamma_{L1}\gamma_{R1})(i\gamma_{L3}\gamma_{R3}) \end{aligned} \tag{72}$$

  will shift the different bands up or down. Instead of crossing at π, many crossings are now shifted away, and so symmetry-breaking perturbations may be added that open up avoided crossings. Not all crossings are avoided; recall that even parity states get mapped to odd parity states and vice versa. Crossings between these states are protected by fermion parity; we recover the 4π periodic Josephson effect.

- $m = 4$. As predicted by Ref. 8, adding interactions to a system with 8 Majoranas makes the system trivial; the term

  $$\begin{aligned} H_{\text{int}} = {} & w_1(i\gamma_{L1}\gamma_{R1})(i\gamma_{L2}\gamma_{R2}) \\ & + w_2(i\gamma_{L1}\gamma_{R1})(i\gamma_{L3}\gamma_{R3}) \\ & + w_3(i\gamma_{L1}\gamma_{R1})(i\gamma_{L4}\gamma_{R4}) \\ & + w_4(i\gamma_{L1}\gamma_{L2})(i\gamma_{L3}\gamma_{L4}) \end{aligned}$$

  completely removes any degeneracy at the crossing while respecting time reversal. The Josephson effect is 2π periodic.

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