

The universal surface bundle over the Torelli space has no sections

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Abstract

For $g > 3$, we give two proofs of the fact that the *Birman exact sequence* for the Torelli group

$$1 \rightarrow \pi_1(S_g) \rightarrow \mathcal{J}_{g,1} \rightarrow \mathcal{J}_g \rightarrow 1$$

does not split. This result was claimed by G. Mess in [Mes90], but his proof has a critical and unreparable error which will be discussed in the introduction. Let $\mathcal{U}\mathcal{J}_{g,n} \xrightarrow{Tu'_{g,n}} \mathcal{B}\mathcal{J}_{g,n}$ (resp. $\mathcal{U}\mathcal{P}\mathcal{J}_{g,n} \xrightarrow{Tu_{g,n}} \mathcal{B}\mathcal{P}\mathcal{J}_{g,n}$) denote the universal surface bundle over the Torelli space fixing n points as a set (resp. pointwise). We also deduce that $Tu'_{g,n}$ has no sections when $n > 1$ and that $Tu_{g,n}$ has precisely n distinct sections for $n \geq 0$ up to homotopy.

1 Introduction

It is a basic problem to understand when bundles have continuous sections, and the corresponding group theory problem as to when short exact sequences have splittings. These are equivalent problems when the fiber, the base and the total space are all $K(\pi, 1)$ -spaces. In this article, we will discuss the “section problems” and the “splitting problems” in the setting of surface bundles. Here by *section* we mean continuous section.

Let $S_{g,n}$ be a closed orientable surface of genus g with n punctures. Let $\text{Mod}_{g,n}$ (resp. $\text{PMod}_{g,n}$) be the *mapping class group* of $S_{g,n}$, i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of S_g fixing n points as a set (resp. pointwise). $\text{Mod}_{g,n}$ and $\text{PMod}_{g,n}$ act on $H^1(S_g; \mathbb{Z})$ leaving invariant the algebraic intersection numbers. Let $\mathcal{J}_{g,n}$ (resp. $\mathcal{P}\mathcal{J}_{g,n}$) be the *Torelli group* (resp. *pure Torelli group*) of $S_{g,n}$, i.e. the subgroup of $\text{Mod}_{g,n}$ (resp. $\text{PMod}_{g,n}$) that acts trivially on $H^1(S_g; \mathbb{Z})$. We omit n when $n = 0$. The following *Birman exact sequence* for the Torelli group provides a relationship between $\mathcal{J}_{g,1}$ and \mathcal{J}_g ; see [FM12, Chapter 4.2].

$$1 \rightarrow \pi_1(S_g) \xrightarrow{\text{point pushing}} \mathcal{J}_{g,1} \xrightarrow{T\pi_{g,1}} \mathcal{J}_g \rightarrow 1. \quad (1.1)$$

The main theorem of this paper is the following:

Theorem 1.1 (Nonsplitting of the Birman exact sequence for the Torelli group). *For $g > 3$, the Birman exact sequence for the Torelli group (1.1) does not split.*

Remark 1.2. Our proof needs the condition $g > 3$. By [Mes92, Proposition 4], \mathcal{J}_2 is a free group. So the Birman exact sequence for \mathcal{J}_2 splits. The case $g = 3$ is open.

Let $\mathcal{B}\mathcal{P}\mathcal{J}_{g,n} := K(\mathcal{P}\mathcal{J}_{g,n}, 1)$ be the *pure universal Torelli space* fixing n punctures pointwise and let

$$S_g \rightarrow \mathcal{U}\mathcal{P}\mathcal{J}_{g,n} \xrightarrow{Tu_{g,n}} \mathcal{B}\mathcal{P}\mathcal{J}_{g,n} \quad (1.2)$$

be the *pure universal Torelli bundle*. Surface bundle (1.2) classifies smooth S_g -bundle equipped with a basis of $H^1(S_g; \mathbb{Z})$ and n ordered points on each fiber. Since $\mathcal{P}\mathcal{J}_{g,n}$ fixes n points, there are n distinct sections $\{Ts_i | 1 \leq i \leq n\}$ of the universal Torelli bundle (1.2). Let $\mathcal{B}\mathcal{J}_{g,n} := K(\mathcal{J}_{g,n}, 1)$ be the *universal Torelli space* fixing n punctures as a set and let

$$S_g \rightarrow \mathcal{U}\mathcal{J}_{g,n} \xrightarrow{Tu'_{g,n}} \mathcal{B}\mathcal{J}_{g,n} \quad (1.3)$$

be the *universal Torelli bundle*. This bundle classifies smooth S_g -bundles equipped with a basis of $H^1(S_g; \mathbb{Z})$ and n unordered points on each fiber. Theorem 1.1 says that $Tu_{g,0}$ has no sections. For $n \geq 0$, we have the following complete answer for sections of (1.3) and (1.2).

Theorem 1.3 (Classification of sections for punctured Torelli spaces). *The following holds:*

(1) For $n \geq 0$ and $g > 3$, every section of the universal Torelli bundle (1.2) is homotopic to Ts_i for some $i \in \{1, 2, \dots, n\}$.

(2) For $n > 1$ and $g > 3$, the universal Torelli bundle (1.3) has no continuous sections.

Let $\mathcal{M}_g := K(\text{Mod}_g, 1)$. As is known, the universal bundle over \mathcal{M}_g :

$$S_g \rightarrow \mathcal{U}\mathcal{M}_g \rightarrow \mathcal{M}_g$$

has no sections. This can be seen from the corresponding algebraic problem of finding splittings of the *Birman exact sequence*

$$1 \rightarrow \pi_1(S_g) \rightarrow \text{Mod}_{g,1} \rightarrow \text{Mod}_g \rightarrow 1.$$

The answer is no because of torsion, e.g. see [FM12, Corollary 5.11]. The key fact is that every finite subgroup of $\text{Mod}_{g,1}$ is cyclic. However, this method does not work for torsion-free subgroups of Mod_g . For any subgroup $G < \text{Mod}_g$, there is an extension Γ_G of G by $\pi_1(S_g)$ as the following short exact sequence.

$$1 \rightarrow \pi_1(S_g) \rightarrow \Gamma_G \rightarrow G \rightarrow 1. \quad (1.4)$$

We call (1.4) the *Birman exact sequence* for G since it is induced from the Birman exact sequence. We pose the following open question:

Problem 1.4 (Virtually splitting of the Birman exact sequence). *Does the Birman exact sequence for a finite index subgroup of $\text{Mod}(S_g)$ always not split?*

Let the *level L congruence subgroup* $\text{Mod}_g[L]$ be the subgroup of Mod_g that acts trivially on $H_1(S_g; \mathbb{Z}/L\mathbb{Z})$ for some integer $L > 1$. Theorem 1.1 implies that a finite index subgroup of Mod_g containing \mathcal{J}_g does not split; in particular, this applies to all the congruence subgroups $\text{Mod}_g[L]$.

Error in G. Mess [Mes90, Proposition 2]

In the unpublished paper of G. Mess [Mes90, Proposition 2], he claimed that there are no splittings of the exact sequence (1.1). But his proof has a fatal error. Here is how the proof goes. Let C be a curve dividing S_g into 2 parts $S(1)$ and $S(2)$ of genus p and q , where $p, q \geq 2$. Let UTS_g be the unit tangent bundle of genus g surface. Then \mathcal{J}_g contains a subgroup A , which satisfies the following exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(UTS_p) \times \pi_1(UTS_q) \rightarrow A \rightarrow 1.$$

Mess' idea is to prove that the Birman exact sequence for A does not lift. However, in Case a) of Mess' proof for [Mes90, Proposition 2], Mess claimed that if the Dehn twist about C lifts to a Dehn twist about

C' on $S_{g,1}$ and C' bounds a genus p surface with a puncture and a genus g surface, then there is a lift from $\pi_1(UTS_p)$ to $\pi_1(UTS_{p,1})$. This is a wrong claim. Actually even A does have a lift. We construct a lift of A as the following. Let $\text{PConf}_2(S_p)$ be the *pure configuration space* of S_p , i.e. the space of 2-tuples of distinct points on S_p . Let

$$\text{PConf}_{1,1}(S_p) = \{(x, y, v) | x \neq y \in S_g \text{ and } v \text{ a unit vector at } x\}.$$

We have the following pullback diagram:

$$\begin{array}{ccc} \pi_1(\text{PConf}_{1,1}(S_p)) & \longrightarrow & \pi_1(\text{PConf}_2(S_p)) \\ \downarrow f & \lrcorner & \downarrow g \\ \pi_1(UTS_p) & \longrightarrow & \pi_1(S_p). \end{array} \tag{1.5}$$

The lift of $\pi_1(UTS_p)$ should lie in $\pi_1(\text{PConf}_{1,1}(S_p))$ instead of $\pi_1(UTS_{p,1})$ as Mess claimed. As long as we can find a lift of f , we will find a section of A to $\mathcal{J}_{g,*}$. By the property of pullback diagrams, a section of g can induce a section of f in diagram (1.5). To negate the argument of [Mes90, Proposition 2], we only need to construct a section of g . We simply need to find a self-map of S_g that has no fixed point. For example, the composition of a retraction of S_p onto a curve c and a rotation of c at any nontrivial angle does not have a fixed point. Therefore Mess' proof is invalid and does not seem to be repairable.

Strategy of the proof of Theorem 1.1

Let T_a be the Dehn twist about a simple closed curve a on S_g . Our strategy is the following: assume that we have a splitting of (1.1). The main result of [Joh83] shows that all *bounding pair map* i.e. $T_a T_b^{-1}$ for a pair of nonseparating curves a, b that bound a subspace generate \mathcal{J}_g . Firstly we need to understand the lift of $T_a T_b^{-1}$. We show that the lift of a bounding pair $T_a T_b^{-1}$ has to be a bounding pair $T_{a'} T_{b'}^{-1}$ for a', b' on $S_{g,1}$. Moreover, the curve a' does not depend on the choice of b , i.e. for any other curve c that forms a bounding pair with curve a , the lift of $T_a T_c^{-1}$ is $T_{a'} T_{c'}^{-1}$ for the same a' . Therefore, we have a lift from the set of isotopy classes of curves on S_g to the set of isotopy classes of curves on $S_{g,1}$. Then we use the lantern relation to derive a contradiction. Our main tool is the canonical reduction system for a mapping class, which in turn uses the Thurston classification of isotopy classes of diffeomorphisms of surfaces. This idea originated from [BLM83].

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2 The proof of Theorem 1.1

Let $g > 3$. We assume that the exact sequence (1.1) has a splitting which is denoted by ϕ such that $F \circ \phi = id$. The goal of this section is to prove Theorem 1.1 by contradiction. In all the figures in this section, $*$ represents the puncture of $S_{g,1}$, the genus g surface with one puncture.

2.1 Background

In this subsection we discuss some properties of canonical reduction systems and the lantern relation. Let $S = S_{g,p}^b$ be a surface with b boundary components and p punctures. Let $\text{Mod}(S)$ (reps. $\text{PMod}(S)$) be the *mapping class group* (resp. *pure mapping class group*) of S , i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of S fixing the boundary components pointwise and the punctures as a set (resp. pointwise). By “simple closed curves”, we often mean isotopy class of simple closed curves, e.g. by “preserve a simple closed curve”, we mean preserve the isotopy class of a curve.

Thurston’s classification of elements of $\text{Mod}(S)$ is a very powerful tool to study mapping class groups. We call a mapping class $f \in \text{Mod}(S)$ *reducible* if a power of f fixes a nonperipheral simple closed curve. Each nontrivial element $f \in \text{Mod}(S)$ is of exactly one of the following types: periodic, reducible, pseudo-Anosov. See [FM12, Chapter 13] and [FLP12] for more details. We now give the definition of canonical reduction system.

Definition 2.1 (Reduction systems). A *reduction system* of a reducible mapping class h in $\text{Mod}(S)$ is a set of disjoint nonperipheral curves that h fixes as a set up to isotopy. A reduction system is *maximal* if it is maximal with respect to inclusion of reduction systems for h . The *canonical reduction system* $\text{CRS}(h)$ is the intersection of all maximal reduction systems of h .

For a reducible element f , there exists n such that f^n fixes each element in $\text{CRS}(f)$ and after cutting out $\text{CRS}(f)$, the restriction of f^n on each component is either periodic or pseudo-Anosov. See [FM12, Corollary 13.3]. Now we mention three properties of the canonical reduction systems that will be used later.

Proposition 2.2. $\text{CRS}(h^n) = \text{CRS}(h)$ for any n .

Proof. This is classical; see [FM12, Chapter 13]. □

For a curve a on a surface S , denote by T_a the Dehn twist about a . For two curves a, b on a surface S , let $i(a, b)$ be the geometric intersection number of a and b . For two sets of curves P and T , we say that S and T *intersect* if there exist $a \in P$ and $b \in T$ such that $i(a, b) \neq 0$. Notice that two sets of curves intersecting does not mean that they have a common element.

Proposition 2.3. Let h be a reducible mapping class in $\text{Mod}(S)$. If $\{\gamma\}$ and $\text{CRS}(h)$ intersect, then no power of h fixes γ .

Proof. Suppose that h^n fixes γ . Therefore γ belongs to a maximal reduction system M . By definition, $\text{CRS}(h) \subset M$. However γ intersects some curve in $\text{CRS}(h)$; this contradicts the fact that M is a set of disjoint curves. □

Proposition 2.4. Suppose that $h, f \in \text{Mod}(S)$ and $fh = hf$. Then $\text{CRS}(h)$ and $\text{CRS}(f)$ do not intersect.

Proof. By conjugation, we have that $\text{CRS}(hfh^{-1}) = h(\text{CRS}(f))$. Since $hfh^{-1} = f$, we get that $\text{CRS}(f) = h(\text{CRS}(f))$. Therefore h fixes the whole set $\text{CRS}(f)$. A power of h fixes all curves in $\text{CRS}(f)$. By Proposition 2.3, curves in $\text{CRS}(h)$ do not intersect curves in $\text{CRS}(f)$. □

We denote the symmetric difference of two sets A, B by $A \triangle B$.

Lemma 2.5. Let $h, f \in \text{Mod}(S)$ be two reduced mapping classes such that $hf = fh$. Then $\text{CRS}(h) \triangle \text{CRS}(f) \subset \text{CRS}(hf)$.

Proof. Suppose that $\gamma \in \text{CRS}(h)$ and $\gamma \notin \text{CRS}(f)$. By Corollary 2.4, γ does not intersect $\text{CRS}(f)$. The canonical form of f has a component C that contains γ . From $fhf^{-1} = h$ we know that f permutes $\text{CRS}(h)$, e.g. a power of f fixes γ . Since a pseudo-Anosov element does not fix any curve, a power of f is the identity on C .

Since $hfh^{-1} = f$, we know that h permutes the components in the canonical form of f , e.g. $h(C)$ is another component in the canonical form of f . Since f permutes $\text{CRS}(h)$, a power of f fixes γ . This shows that C and $h(C)$ intersect, therefore we have that $h(C) = C$.

Suppose that on the component C , the curve $\gamma \notin \text{CRS}(hf)$. This means that there is a curve $\gamma' \subset C$ such that $(hf)^n(\gamma') = \gamma'$ for some integer n and $i(\gamma, \gamma') \neq 0$. A power of f is the identity on C , therefore f fixes γ' . However no power of h fixes γ' by Proposition 2.3. Therefore, no power of hf fixes γ' . This is a contradiction, which shows that $\gamma \in \text{CRS}(hf|_C)$.

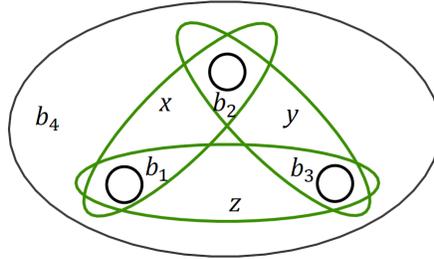
For any element $e \in \text{Mod}(S)$ such that the action on S can be broken into actions on components $\{C_1, \dots, C_k\}$, we have

$$\text{CRS}(e|_{C_1}) \cup \dots \cup \text{CRS}(e|_{C_k}) \subset \text{CRS}(e).$$

Therefore $\gamma \in \text{CRS}(hf|_C) \subset \text{CRS}(hf)$. □

Now, we introduce a remarkable relation for $\text{Mod}(S)$ that will be used in the proof.

Proposition 2.6 (The lantern relation). *There is an orientation-preserving embedding of $S_{0,4} \subset S$ and let $x, y, z, b_1, b_2, b_3, b_4$ be simple closed curves in $S_{0,4}$ that are arranged as the curves shown in the following figure.*



In $\text{Mod}(S)$ we have the relation

$$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}.$$

Proof. This is classical; see [FM12, Chapter 5.1]. □

2.2 Lifts of bounding pair maps

Let $\{a, b\}$ be a *bounding pair* as in the following figure, i.e. a, b are nonseparating curves such that a and b bounds a subsurface. Denote by T_c the Dehn twist about a curve c . In this subsection, we determine $\phi(T_a T_b^{-1})$. For two curves c and d , denote by $i(c, d)$ the geometric intersection number of c and d . For a curve c' on $S_{g,1}$, when we say c' is isotopic to a curve c on S_g , we mean that c' is isotopic to c on S_g .

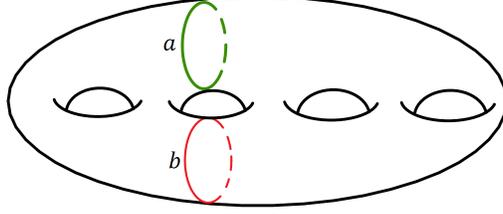


Figure 2.1: A bounding pair a, b

Lemma 2.7. *Let $\{a, b\}$ be a bounding pair as in Figure 2.1. Up to a swap of a and b , $\text{CRS}(\phi(T_a T_b^{-1}))$ can be one of the following two cases. Moreover, either $\phi(T_a T_b^{-1}) = T_{a'} T_{b'}^{-1}$ as in case 1 or there exists an integer n such that $\phi(T_a T_b^{-1}) = T_{a'}^n T_{a''}^{1-n} T_{b'}^{-1}$ as in case 2. For a Dehn twist T_s about a separating curve s , there exists a pair of disjoint curves s' and s'' such that they are all isotopic to s and $\phi(T_s) = T_{s'}^n T_{s''}^{1-n}$ for some integer n .*

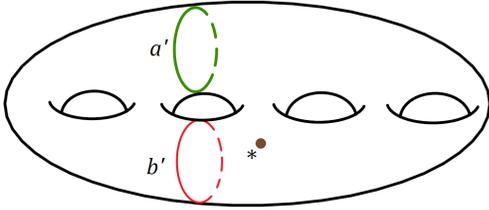


Figure 2.2: Case 1

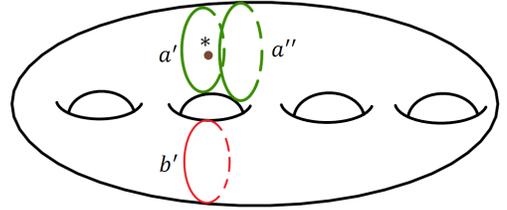


Figure 2.3: Case 2

Proof. Let $T_a T_b^{-1} \in \mathcal{J}_g$ be a bounding pair map. Since the centralizer of $T_a T_b^{-1} \in \mathcal{J}_g$ contains a copy of \mathbb{Z}^{2g-3} as a subgroup of \mathcal{J}_g , the centralizer of $\phi(T_a T_b^{-1}) \in \mathcal{J}_{g,1}$ contains a copy of \mathbb{Z}^{2g-3} as well. However by [McC82, Theorem 1], the centralizer of a pseudo-Anosov element is virtually cyclic group. $g > 3$ implies that $2g - 3 > 3$. Therefore $\phi(T_a T_b^{-1}) \in \mathcal{J}_{g,1}$ is not pseudo-Anosov. For any curve γ' on $S_{g,1}$, denote by γ the same curve on S_g . We decompose the proof into the following three steps.

Claim 2.8 (Step 1). *$\text{CRS}(\phi(T_a T_b^{-1}))$ only contains curves that are isotopic to a or b .*

Proof. Suppose the opposite that there exists $\gamma' \in \text{CRS}(\phi(T_a T_b^{-1}))$ such that γ is not isotopic to a or b . There are two cases.

Case 1: γ intersect a and b . Since a power of $\phi(T_a T_b^{-1})$ fixes γ' , a power of $T_a T_b^{-1}$ fixes γ . By $\text{CRS}(T_a T_b^{-1}) = \{a, b\}$ and Lemma 2.3, we know that $T_a T_b^{-1}$ does not fix γ . This is a contradiction.

Case 2: γ does not intersect a and b . In this case by the change of coordinate principle, we can always find a separating curve c such that $i(a, c) = 0$, $i(b, c) = 0$ and $i(c, \gamma) \neq 0$. Since $T_a T_b^{-1}$ and T_c commute in \mathcal{J}_g , the two mapping classes $\phi(T_a T_b^{-1})$ and $\phi(T_c)$ commute in $\mathcal{J}_{g,1}$. This shows that a power of $\phi(T_c)$ fixes $\text{CRS}(\phi(T_a T_b^{-1}))$; more specifically a power of $\phi(T_c)$ fixes γ' . However by Lemma 2.3, no power of T_c fixes γ . This is a contradiction. \square

Claim 2.9 (Step 2). *$\text{CRS}(\phi(T_a T_b^{-1}))$ must contain curves that are isotopic to a and b .*

Proof. Suppose the opposite that $\text{CRS}(\phi(T_a T_b^{-1}))$ does not contain a curve γ' such that γ is isotopic to a . Then by Step 1, $\text{CRS}(\phi(T_a T_b^{-1}))$ either contains one curve b' isotopic to b or two curves b' and b'' both

isotopic to b . After cutting $\text{CRS}(\phi(T_a T_b^{-1}))$, we have a component C that is not a punctured annulus. C is homeomorphic to the complement of b in S_g .

If $\phi(T_a T_b^{-1})$ is pseudo-Anosov on C , then the centralizer of $\phi(T_a T_b^{-1})|_C$ at most contains one copy of \mathbb{Z} by [McC82, Theorem 1]. Combining with $T_{b'}$ and $T_{b''}$, the centralizer of $\phi(T_a T_b^{-1})$ at most contains one copy of \mathbb{Z}^3 as a subgroup. This contradicts the fact that the centralizer of $\phi(T_a T_b^{-1})$ contains a subgroup \mathbb{Z}^{2g-3} as a subgroup because $2g - 3 > 3$. Here we need to use $g \geq 4$. Therefore $\phi(T_a T_b^{-1})$ is identity on C . This contradicts the fact that $T_a T_b^{-1}$ is not identity on C . \square

Claim 2.10 (Step 3). *Either $\phi(T_a T_b^{-1}) = T_{a'} T_{b'}^{-1}$ as in case 1 or there exists an integer n such that $\phi(T_a T_b^{-1}) = T_{a'}^n T_{a''}^{1-n} T_{b'}^{-1}$ as in case 2.*

Proof. Suppose that $\phi(T_a T_b^{-1})$ is pseudo-Anosov on a component C after cutting out $\text{CRS}(\phi(T_a T_b^{-1}))$ from $S_{g,1}$. Since $g(C) \geq 1$, there exists a separating curve s on C such that $\phi(T_s)$ commutes with $\phi(T_a T_b^{-1})$. Therefore $\phi(T_a T_b^{-1})$ fixes $\text{CRS}(\phi(T_s))$, which is either one curve or two curves isotopic to s . Thus a power of $\phi(T_a T_b^{-1})$ fixes curves on C , which means that $\phi(T_a T_b^{-1})$ is not pseudo-Anosov on C . Therefore, $\phi(T_a T_b^{-1})$ is not pseudo-Anosov on each of the components. By the canonical form of a mapping class, a power of $\phi(T_a T_b^{-1})$ is a product of Dehn twists about $\text{CRS}(\phi(T_a T_b^{-1}))$. By the fact that $\phi(T_a T_b^{-1})$ is a lift of $T_a T_b^{-1}$, the lemma holds. \square

The same argument works for T_s the Dehn twist about a separating curve s . \square

When $n = 0$ or $n = 1$, we have that $(T_{a'})^n (T_{a''})^{1-n} T_{b'} = T_{a''} T_{b'}^{-1}$ or $(T_{a'})^n (T_{a''})^{1-n} T_{b'} = T_{a'} T_{b'}^{-1}$. Therefore, we can combine the results to get that $\phi(T_a T_b^{-1}) = (T_{a'})^n (T_{a''})^{1-n} T_{b'}$. In $\phi(T_a T_b^{-1}) = (T_{a'})^n (T_{a''})^{1-n} T_{b'}$, denote $(T_{a'})^n (T_{a''})^{1-n}$ by the *a component* of $\phi(T_a T_b^{-1})$. Notice that by symmetry, the *b component* of $\phi(T_a T_b^{-1})$ could also be a product of Dehn twists. In the following lemma, we will prove that the *a component* of $\phi(T_a T_b^{-1})$ does not depend on the choice of b .

Lemma 2.11. *For two bounding pairs $\{a, b\}$ and $\{a, c\}$, the *a component* of $\phi(T_a T_b^{-1})$ is the same as the *a component* of $\phi(T_a T_c^{-1})$.*

Proof. If b, c are disjoint, $\phi(T_a T_b^{-1})$ and $\phi(T_a T_c^{-1})$ commute. By Lemma 2.5, we have

$$\text{CRS}(\phi(T_a T_b^{-1})) \triangle \text{CRS}(\phi(T_a T_c^{-1})^{-1}) \subset \text{CRS}(\phi(T_a T_b^{-1})) \phi(T_a T_c^{-1})^{-1}.$$

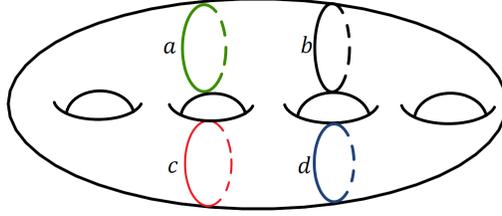
However $\phi(T_a T_b^{-1}) \phi(T_a T_c^{-1})^{-1} = \phi(T_c T_b^{-1})$, we have that $\text{CRS}(\phi(T_a T_b^{-1})) \phi(T_a T_c^{-1})^{-1}$ only contains curves that are isotopic to b or c . So $\text{CRS}(\phi(T_a T_b^{-1})) \triangle \text{CRS}(\phi(T_a T_c^{-1})^{-1})$ does not contain curves isotopic to a . This shows that the *a components* of $\phi(T_a T_b^{-1})$ and $\phi(T_a T_c^{-1})$ are the same so that they can cancel each other through multiplication.

When b, c intersect, there are a series of curves $\{b_1 = b, b_2, \dots, b_n = c\}$ such that $i(b_i, b_{i+1}) = 0$ and $i(a, b_i) = 0$ for all $1 \leq i \leq n - 1$. This fact can be deduced from the connectivity of the complex of homologous curves, e.g. see [Put08]. Therefore, the *a components* of $\phi(T_a T_b^{-1})$ and $\phi(T_a T_c^{-1})$ are the same. \square

We denote by the capital letter A the subset of curves in $\text{CRS}(\phi(T_a T_b^{-1}))$ that are isotopic to a . By Lemma 2.11, A only depends on the curve a . It can be a one-element set or a two-element set.

Lemma 2.12. *If $i(a, b) = 0$, then A is disjoint from B .*

Proof. Suppose that a, b are nonseparating. The case of separating curves are the same. If a, b bound, then by Lemma 2.7, A and B are disjoint. If a, b do not bound, then there are curves c, d such that they form the following configuration.



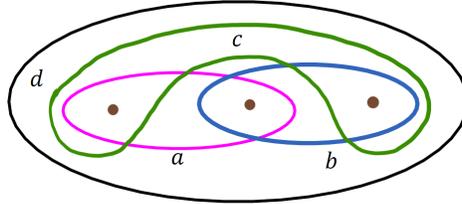
$g \geq 4$ is needed here. Since $\phi(T_a T_c^{-1})$ and $\phi(T_b T_d^{-1})$ commute, their canonical reduction systems do not intersect by Corollary 2.4. Therefore A and B are disjoint. \square

2.3 A nonsplitting lemma for the braid group

Let D_n be an n -punctured 2-disk. The n -strand *pure braid group* is denoted by PB_n , i.e. the pure mapping class group of D_n fixing the n punctures pointwise. In this subsection, we prove a nonsplitting lemma for the braid group that will be used in the proof of Theorem 1.1.

Lemma 2.13. *Let $\mathcal{F} : PB_4 \rightarrow PB_3$ be the forgetful map forgetting the 4th punctures. There is no homomorphism $\mathcal{G} : PB_3 \rightarrow PB_4$ such that Dehn twists map to Dehn twists, the center maps to the center and $\mathcal{F} \circ \mathcal{G} = id$.*

Proof. Suppose the opposite that we have $\mathcal{G} : PB_3 \rightarrow PB_4$ such that Dehn twists map to Dehn twists, the center maps to the center and $\mathcal{F} \circ \mathcal{G} = id$. Let c be a simple closed curve on D_3 and we call c' the lift on D_4 such that $\mathcal{G}(T_c) = T_{c'}$. In the figure below, the lantern relation gives $T_a T_b T_c = T_d \in PB_3$. Therefore we have $T_{a'} T_{b'} T_{c'} = T_{d'} \in PB_4$. Because \mathcal{G} maps the center to the center, d' is the boundary curve of D_n . Since a, b, c do not intersect d , we have that T_a, T_b, T_c commute with T_d .



If $i(a', b') > 2$, then $T_{a'} T_{b'}$ is pseudo-Anosov on some subspace of D_4 by Thurston's construction, e.g. see [Che17a, Proposition 2.13]. Therefore, $T_{a'} T_{b'} = T_{d'} T_{c'}^{-1}$ is not a multitwist. So $i(a', b') = 2$. Every curve in D_n surrounds several points. For example, $a \subset D_3$ surrounds 2 points. There are several cases we need to concern about the number of surrounding points.

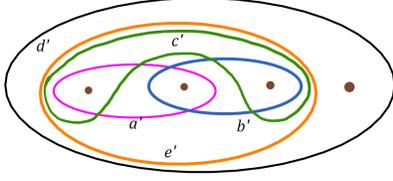


Figure 2.4: Case 1

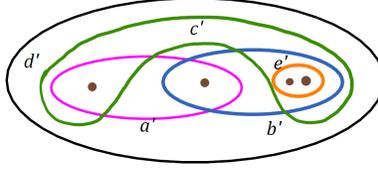


Figure 2.5: Case 2

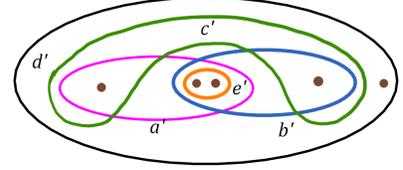


Figure 2.6: Case 3

Case 1: a' bounds 2 points and b' bounds 2 points. Then we have $T_{a'}T_{b'} = T_{e'}T_{c'}^{-1}$ by the lantern relation as is shown in Figure 2.4. We also have the relation $T_{a'}T_{b'} = T_{d'}T_{c'}^{-1}$ from the lift of the relation $T_aT_bT_c = T_d \in PB_3$. However $\text{CRS}(T_{d'}T_{c'}^{-1}) = \{c'\} \neq \text{CRS}(T_{e'}T_{c'}^{-1}) = \{e', c'\}$. This is a contradiction.

Case 2: a' bounds 2 points and b' bounds 3 points. Then we have $T_{a'}T_{b'} = T_{d'}T_{e'}T_{c'}^{-1}$ by the lantern relation as is shown in Figure 2.5. However $\#\text{CRS}(T_{d'}T_{e'}T_{c'}^{-1}) = 2 > 1 = \#\text{CRS}(T_{a'}T_{b'})$. This is a contradiction.

Case 3: a' bounds 3 points and b' bounds 3 points. Then we have $T_{a'}T_{b'} = T_{d'}T_{e'}T_{c'}^{-1}$ by the lantern relation as is shown in Figure 2.6. We have $\#\text{CRS}(T_{d'}T_{e'}T_{c'}^{-1}) = 2 > 1 = \#\text{CRS}(T_{a'}T_{b'})$. This is a contradiction.

□

2.4 Proof of Theorem 1.1

In this proof, we do a case study on the possibilities of $\phi(T_aT_e^{-1})$ for a bounding pair map $T_aT_e^{-1}$. Case 1 is when the a component is not a single Dehn twist. We reach a contradiction by the lantern relation. Case 2 is when the component of every curve is a single Dehn twist, we use Lemma 2.13 to cause contradiction.

Proof of Theorem 1.1. We break our discussion into the following two cases.

Case 1: there is a bounding pair map $T_aT_e^{-1}$ such that

$$\phi(T_aT_e^{-1}) = (T_{a'})^n(T_{a''})^{1-n}T_{e'}^{-1} \text{ where } n \neq 0, 1.$$

There exist curves b, c, d such that a, b, c, d form a 4-boundary disk as Figure 2.7. We need $g > 3$ here.

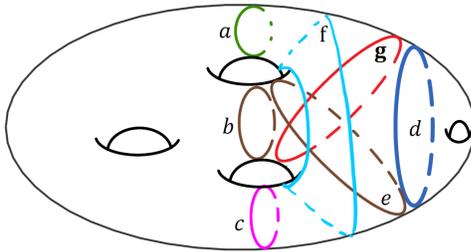


Figure 2.7: On S_g

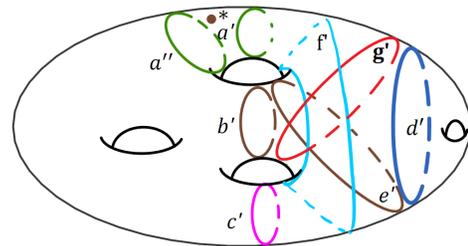


Figure 2.8: On $S_{g,1}$

There are two more curves f, g such that we have the lantern relation $T_a T_e^{-1} T_b T_f^{-1} T_c T_g^{-1} T_d = 1$. The lifts $\{b', c', d', f', g'\}$ of $\{b, c, d, f, g\}$ do not intersect a', a'' as in Figure 2.8. After applying ϕ , we have

$$(T_{a'})^n (T_{a''})^{1-n} T_e^{-1} T_b T_f^{-1} T_c T_g^{-1} T_d = 1.$$

Since $T_{a'} T_e^{-1} T_b T_f^{-1} T_c T_g^{-1} T_d = 1$ by the lantern relation on $S_{g,1}$, we have $(T_{a'})^n (T_{a''})^{1-n} = T_{a'}$. It means that $n = 1$, which contradicts our assumption on n . This proof also works for the Dehn twist T_s about a separating curve s .

Case 2: for any bounding pair map $T_a T_e^{-1}$, we have $\phi(T_a T_e^{-1}) = T_{a'} T_e^{-1}$ and for any Dehn twist T_s about a separating curve s , we have $\phi(T_s) = T_{s'}$

Let $S_{g,p}^b$ be a genus g surface with p punctures and b boundary components. In this case, firstly we want to locate $*$. Let us decompose the surface into pair of pants as the following Figure 2.9.

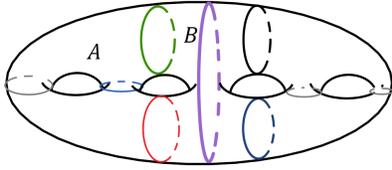


Figure 2.9: A decomposition

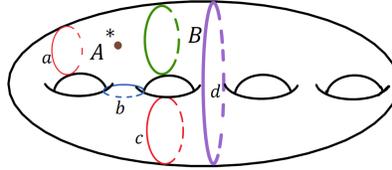


Figure 2.10: possibility 1

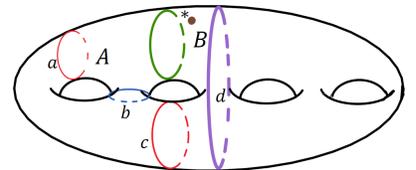


Figure 2.11: possibility 2

The location of $*$ can be either in a pair of pants where all three curves are nonseparating like A or one of them is separating like B . Suppose without loss of generality that $*$ lands on A or B . If $*$ lands on A , we use Figure 2.10 to find four curves a, b, c, d and if $*$ lands on B , we use Figure 2.11 to find four curves a, b, c, d . The curves a, b, c, d that we find satisfy the following properties:

- 1) d is separating and $a \cup b \cup c \cup d$ bounds a 4-boundary sphere $S \approx S_0^4 \subset S_g$.
- 2) The lifts a', b', c', d' are 4 disjoint simple closed curves on $S_{g,1}$ such that d' is separating and $a' \cup b' \cup c' \cup d'$ bounds a 4-boundary sphere with $*$ in $S' \approx S_{0,1}^4 \subset S_{g,1}$. See the following figures.

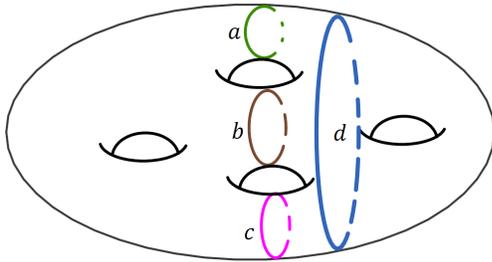


Figure 2.12: On S_g

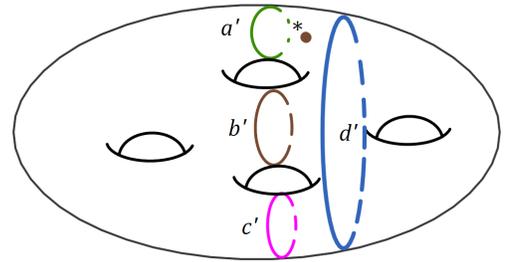


Figure 2.13: On $S_{g,1}$

Claim 2.14. Let W be the subgroup of \mathcal{J}_g generated by bounding pair maps with curves on S . Let W' be the subgroup of $\mathcal{J}_{g,1}$ generated by bounding pair maps with curves on S' such that one of the curves lies in a', b', c' . We have that

$$W \cong PB_3 \text{ and } W' \cong PB_4.$$

Proof. W only acts nontrivially on $S \cong S_0^4$. After gluing punctured disks to the boundaries a, b and c , there is a homomorphism $\mu : W \rightarrow PB_3$. Since every closed curve inside $S_{0,3}^1$ is isotopic to one of the boundary components, every bounding pair map of W maps to a Dehn twist in PB_3 under f . It is clear that ϕ is surjective. If $f \in \ker(\mu)$ as a mapping class on S , then f is either trivial or equal to a product of Dehn twists on a, b, c . However, we claim that a nontrivial product of Dehn twists on a, b, c is not in Torelli group, which shows that μ is injective. Suppose the opposite that $f = T_a^m T_b^n T_c^l \in \mathcal{J}_g$ and $l \neq 0$. Let a', b' be two curves and denote by $I(a', b')$ the *algebraic intersection number* of a' and b' . For $x \in H_1(S_g; \mathbb{Z})$, we have that $T_a^m T_b^n T_c^l(x) = mI(a, x)a + nI(b, x)b + lI(c, x)c + x$. The fact that $T_a^m T_b^n T_c^l$ is in \mathcal{J}_g implies that $mI(a, x)a + nI(b, x)b + lI(c, x)c = 0$ for any x . Since a, b are independent, there exists an element x such that $I(x, a) = 0$ and $I(x, b) = 1$. Since $a \cup b \cup c$ separate implying that $a + b + c = 0$, we have that $I(c, x) = -1$. This contradicts $mI(a, x)a + nI(b, x)b + lI(c, x)c = nb - lc = 0$ because b, c are independent and $l \neq 0$. For the same reason $W' \cong PB_4$. \square

The lifts of elements in W is inside W' and $F : W' \rightarrow W$ is the forgetful map forgetting the puncture $\mathcal{F} : PB_4 \rightarrow PB_3$. To conclude the proof of the theorem we only need to apply Lemma 2.13 that there is no splitting of \mathcal{F} satisfying our assumption. \square

3 Torelli spaces with punctures

In this section, we discuss the “section problem” for the universal Torelli bundle with punctures.

3.1 Translation to a group theoretical problem

We first translate the “section problem” of the universal Torelli surface bundle into a group-theoretic statement. As is discussed in [Che17b, Chapter 2.1], we have the following correspondence when $g > 1$:

$$\left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \text{representations} \\ \rho : \pi_1(B) \rightarrow \text{Mod}_g \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{oriented } S_g\text{-bundles over} \\ B \end{array} \right\}. \quad (3.1)$$

Let $f : E \rightarrow B$ be a surface bundle determined by $\rho : \pi_1(B) \rightarrow \text{Mod}_g$. Let $f_* : \pi_1(E) \rightarrow \pi_1(B)$ be the map on the fundamental groups. By the property of pullback diagrams, finding a splitting of f_* is the same as finding a homomorphism p that makes the following diagram commute, i.e. $\pi_{g,1} \circ p = \rho$.

$$\begin{array}{ccc} \pi_1(E) & \longrightarrow & \text{Mod}_{g,1} \\ \downarrow f_* & \nearrow p & \downarrow \pi_{g,1} \\ \pi_1(B) & \xrightarrow{\rho} & \text{Mod}_g. \end{array} \quad (3.2)$$

We have the following correspondence:

$$\left\{ \begin{array}{c} \text{Homotopy classes of continuous} \\ \text{sections of } S_g \rightarrow E \xrightarrow{f} B \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Homomorphisms } p \text{ satisfying diagram (3.2) up to} \\ \text{conjugacy by an element in } \text{Ker}(\pi_{g,1}) \cong \pi_1(S_g) \end{array} \right\}. \quad (3.3)$$

By the correspondence (3.3), we can translate Theorem 1.3 into the following group-theoretic statement. Let $\mathcal{P}\mathcal{J}_{g,n} \xrightarrow{T\pi_{g,n}} \mathcal{J}_g$ and $\mathcal{J}_{g,n} \xrightarrow{T\pi'_{g,n}} \mathcal{J}_g$ be the forgetful maps forgetting the punctures. Let $\mathcal{J}_{g,n} \xrightarrow{Tp_{g,n,i}} \text{Mod}_{g,1}$

be the forgetful homomorphism forgetting the fixed points $\{x_1, \dots, \hat{x}_i, \dots, x_n\}$. Let $PB_n(S_g)$ (resp. $B_n(S_g)$) be the n -strand surface braid group, i.e. the fundamental group of the space of ordered (resp. unordered) n distinct points on S_g . By the *generalized Birman exact sequence* (see e.g. [FM12, Theorem 9.1]), we have that $\text{Ker}(T\pi_{g,n}) \cong PB_n(S_g)$ and $\text{Ker}(T\pi'_{g,n}) \cong B_n(S_g)$. See [Che17b, Chapter 2.1] for more details. We will prove the following proposition in the next subsection.

Proposition 3.1. *For $g > 1$ and $n \geq 0$. The following holds:*

1) *Every homomorphism p satisfying the following diagram is either conjugate to a forgetful homomorphism $Tp_{g,n,i}$ by an element in $\mathcal{PJ}_{g,n}$ or factors through $T\pi_{g,n}$, i.e. there exists f such that $p = f \circ T\pi_{g,n}$.*

$$\begin{array}{ccccccc} 1 & \rightarrow & PB_n(S_g) & \longrightarrow & \mathcal{PJ}_{g,n} & \xrightarrow{T\pi_{g,n}} & \mathcal{J}_g \longrightarrow 1 \\ & & \downarrow R & & \downarrow p & & \downarrow = \\ 1 & \rightarrow & \pi_1(S_g) & \longrightarrow & \text{Mod}_{g,1} & \xrightarrow{\pi_{g,1}} & \text{Mod}_g \longrightarrow 1. \end{array} \quad (3.4)$$

2) *For $n > 1$, every homomorphism p' satisfying the following diagram factors through $T\pi'_{g,n}$, i.e. there exists f' such that $p' = f' \circ T\pi'_{g,n}$*

$$\begin{array}{ccccccc} 1 & \rightarrow & B_n(S_g) & \longrightarrow & \mathcal{J}_{g,n} & \xrightarrow{T\pi'_{g,n}} & \mathcal{J}_g \longrightarrow 1 \\ & & \downarrow R' & & \downarrow p' & & \downarrow = \\ 1 & \rightarrow & \pi_1(S_g) & \longrightarrow & \text{Mod}_{g,1} & \xrightarrow{\pi_{g,1}} & \text{Mod}_g \longrightarrow 1. \end{array} \quad (3.5)$$

Proof of Theorem 1.3 assuming Proposition 3.1. By Theorem 1.1, the short exact sequence

$$1 \rightarrow \pi_1(S_g) \rightarrow \mathcal{J}_{g,1} \xrightarrow{\pi_{g,1}} \mathcal{J}_g \rightarrow 1$$

has no section. Therefore Proposition 3.1 implies Theorem 1.3. \square

3.2 The proof of Proposition 3.1

The top exact sequence of diagram (3.4) gives us a representation $\rho_T : \mathcal{J}_g \rightarrow \text{Out}(PB_n(S_g))$. The following lemma describes a property of ρ_T . Let $p_i : PB_n(S_g) \rightarrow \pi_1(S_g)$ be the induced map on the fundamental groups of the forgetful map forgetting all points except the i th point.

Lemma 3.2. *Let $h > 1$. For any surjective homomorphism $\phi : PB_n(S_g) \rightarrow F_h$, there exists an element $t \in \mathcal{J}_g$ such that $t(\text{Ker}(\phi)) \neq \text{Ker}(\phi)$.*

Proof. By Theorem [Che17b, Theorem 1.5], any homomorphism $\phi : PB_n(S_g) \rightarrow F_h$ factors through some p_i . Thus we only need to deal with the case $n = 1$. We will prove the lemma by contradiction.

Suppose the opposite that there exists a surjective homomorphism $\phi : \pi_1(S_g) \rightarrow F_h$ such that for any element $e \in \mathcal{J}_g$, we have $e(\text{Ker}(\phi)) = \text{Ker}(\phi)$. Since ϕ is surjective, the induced map on $H_1(_, \mathbb{Z})$ is also surjective. Suppose that $a_1, a_2, \dots, a_h \in \pi_1(S_g)$ such that $\phi(a_1), \dots, \phi(a_h)$ generate F_h . Since the cup product $H^1(F_h, \mathbb{Z}) \otimes H^1(F_h, \mathbb{Z}) \xrightarrow{\text{cup}} H^2(F_h, \mathbb{Z})$ is trivial, the image of $\phi^* : H^1(F_h; \mathbb{Z}) \rightarrow H^1(S_g; \mathbb{Z})$ is an isotropic subspace with dimension at most g . Thus we can find $b \in \pi_1(S_g)$ such that $\phi(b) = 1$ and $[b] \neq 0 \in H_1(S_g; \mathbb{Z})$.

It is clear that $[b]$ and $\{a_1, \dots, a_h\}$ are linearly independent. Let $\pi^0 = \pi_1(S_g)$, and $\pi^{n+1} = [\pi^n, \pi^0]$, we have the following exact sequence.

$$1 \rightarrow \pi^1/\pi^2 \rightarrow \pi^0/\pi^2 \rightarrow \pi^0/\pi^1 \rightarrow 1$$

Let $H := H_1(S_g; \mathbb{Z})$. Let $\omega = \sum_{j=1}^g a_j \wedge b_j$. We know that $\pi^1/\pi^2 \cong \wedge^2 H/\mathbb{Z}\omega$, where the identification is given by $[x, y] \rightarrow x \wedge y$. Notice that \mathcal{J}_g acts trivially on both π^1/π^2 and π^0/π^1 but nontrivially on π^0/π^2 . The action is measured by the Johnson homomorphism $\tau : \mathcal{J}_g \rightarrow \text{Hom}(H, \wedge^2 H/\mathbb{Z}\omega)$; see [Joh80] for more details. Let $t \in \mathcal{J}_g$. For $x \in H$, let $\tilde{x} \in \pi^0$ be a lift of x , i.e. \tilde{x} maps x under the map $\pi^0 \rightarrow H$. The Johnson homomorphism is defined by $\tau(t)(x) = t(\tilde{x})\tilde{x}^{-1} \in \pi^1/\pi^2$. It is standard to check that $\tau(t)$ does not depend on the choice of lift \tilde{x} .

Johnson [Joh80, Theorem 1] proved that the image $\tau(\mathcal{J}_g) = \wedge^3 H/H \subset \text{Hom}(H, \wedge^2 H/\mathbb{Z}\omega)$. Therefore there exists $t \in \mathcal{J}_g$ such that $\tau(t)(b) = a_1 \wedge a_2$. By the definition of the Johnson homomorphism, we have that $t(b)b^{-1} = [a_1, a_2]T$, where $T \in \pi^2$. Since $\phi(b) = 1$, we have that $\phi(t(b)) = 1$ by the assumption that $t(\text{Ker}(\phi)) = \text{Ker}(\phi)$. As a result, $\phi([a_1, a_2])\phi(T) = 1$.

Let $F_h^1 = [F_h, F_h]$ and $F_h^{n+1} = [F_h^n, F_h]$. We have that $\phi(\pi^n) \subset F_h^n$, which implies that $\phi(T) \in F_h^2$. However $\phi([a_1, a_2]) \neq 1 \in F_h^1/F_h^2$. This contradicts the fact that $\phi([a_1, a_2]) = \phi(T)^{-1}$. □

We need the following lemma from [HT85, Lemma 2.2].

Lemma 3.3. *For $g > 1$, a pseudo-Anosov element of $\text{Mod}(S_{g,n})$ does not fix any nonperipheral isotopy class of curves including nonsimple curves.*

Now we have all the ingredients to prove statement 1) in Proposition 3.1.

Proof of 1) in Proposition 3.1. For any $p : \mathcal{P}\mathcal{J}_{g,n} \rightarrow \mathcal{J}_{g,1}$, we have that for $e \in \mathcal{P}\mathcal{J}_{g,n}$ and $x \in PB_n(S_g)$,

$$R(exe^{-1}) = p(e)R(x)p(e)^{-1}.$$

Denote by C_e the conjugation by e in any group. This induces the following diagram:

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{C_e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ \pi_1(S_g) & \xrightarrow{C_{p(e)}} & \pi_1(S_g). \end{array} \tag{3.6}$$

By [Che17b, Theorem 1.5], a homomorphism $R : PB_n(S_g) \rightarrow \pi_1(S_g)$ either factors through a forgetful homomorphism or has cyclic image. We break our discussion into the two cases.

Case 1: Image(R) $\cong \mathbb{Z}$

In this case, the image is generated by $x \in \pi_1(S_g)$. By diagram (3.6), $C_{p(e)}$ preserves Image(R) for any e . It is known that \mathcal{J}_g contains pseudo-Anosov elements; see [FM12, Corollary 14.3]. By Lemma 3.3, a pseudo-Anosov element does not preserve Image(R). Therefore R does not extend to p .

Case 2: R factors through a forgetful homomorphism p_i and does not have cyclic image

In this case, we have a homomorphism $S : \pi_1(S_g) \rightarrow \pi_1(S_g)$ such that $R = S \circ p_i$. If S is a surjection, by the same reason as in the proof of [Che17b, Theorem 2.4], we know that p is conjugate to p_i . If S is not a

surjection, then $\text{Image}(S)$ is a noncyclic free group. By Lemma 3.2 and diagram 3.6, we know R does not extend to p . \square

To prove statement 2) in Proposition 3.1, we need the following lemma.

Lemma 3.4. *For $n > 1$, the image of any homomorphism $B_n(S_g) \rightarrow \pi_1(S_g)$ is a free group.*

Proof. Suppose that there exists a homomorphism $\Phi : B_n(S_g) \rightarrow \pi_1(S_g)$ such that the image is not a free group. Then $\text{Image}(\Phi) \cong \pi_1(S_h)$ where $h \geq g$. After precomposing with the embedding $i : PB_n(S_g) \rightarrow B_n(S_g)$, we have a homomorphism $\Phi' : PB_n(S_g) \rightarrow \pi_1(S_g)$ with image a nontrivial finite index subgroup. By Theorem [Che17b, Theorem 5], the map Φ' factors through some p_i , but there is no surjection from $\pi_1(S_g)$ to a nontrivial finite index subgroup of $\pi_1(S_g)$. This is a contradiction. By the classification of subgroups of $\pi_1(S_g)$, the image of any homomorphism $B_n(S_g) \rightarrow \pi_1(S_g)$ is a free group. \square

Proof of 2) in Proposition 3.1. By Claim 3.4, we know that R' is not a surjection. Therefore, the image of R' is either cyclic or a noncyclic free group. For the cyclic image case, we use the same argument as in the proof of 3.1 to show that R does not extend to p . In the case of noncyclic free group, by Lemma 3.2, we know that R does not extend to p as well. \square

3.3 A nonsplitting statement

In this subsection, we will prove the following corollary using Theorem 1.3.

Corollary 3.5. *For $g > 1$ and $m > n$, the forgetful map $F_{g,m,n} : \mathcal{PJ}_{g,m} \rightarrow \mathcal{PJ}_{g,n}$ forgetting the last $m - n$ points does not have a section.*

Proof. We only need to show that the n sections of the bundle $Tu_{g,n}$ have nontrivial self-intersection. Then we cannot find $n + 1$ disjoint sections on $Tu_{g,n}$. We restrict our attention to the subgroup $PB_n(S_g)$ of $\mathcal{PJ}_{g,n}$. Let $\text{PConf}_n(S_g)$ be the space of n -tuples of distinct points on S_g . Since $\text{PConf}_n(S_g) = K(PB_n(S_g), 1)$, we have that $\text{PConf}_n(S_g)$ is a subspace of $\mathcal{BPJ}_{g,n}$. The bundle on $\text{PConf}_n(S_g)$ is the trivial bundle

$$\text{PConf}_n(S_g) \times S_g \xrightarrow{P_n} \text{PConf}_n(S_g)$$

with n sections $s_i(x_1, \dots, x_n) = (x_1, \dots, x_n, x_i)$. By Poincaré duality, the section is represented by a class in $H^2(\text{PConf}_n(S_g) \times S_g; \mathbb{Z})$. So the self-intersection of a section is a class in $H^4(\text{PConf}_n(S_g) \times S_g; \mathbb{Z})$. Let $p_i(x_1, \dots, x_n) = x_i$ be the projection of $\text{PConf}_n(S_g)$ to S_g . We have the following pullback diagram such that s_i is the pullback of the diagonal section for the trivial bundle P_1 .

$$\begin{array}{ccc} \text{PConf}_n(S_g) \times S_g & \xrightarrow{(p_i, id)} & S_g \times S_g \\ \downarrow P_n & \lrcorner & \downarrow P_1 \\ \text{PConf}_n(S_g) & \longrightarrow & S_g. \end{array} \quad (3.7)$$

Since s_i is the pullback from the trivial bundle P_1 , the self-intersection of s_i is the pullback of the corresponding class γ in $H^4(S_g \times S_g; \mathbb{Z})$. Let $[S_g]$ (resp. $[S_g \times S_g]$) be the fundamental class of S_g (resp. $S_g \times S_g$). It is classical that the class is $\gamma = (2 - 2g)[S_g \times S_g]$. By the Gysin homomorphism,

$$p_i!(p_i, id)^*(2 - 2g)[S_g \times S_g] = (2 - 2g)p_i^*[S_g] \in H^2(\text{PConf}_n(S_g); \mathbb{Z})$$

which is nonzero by the computation in [Che17b, Lemma 3.4]. \square

4 Another proof of Theorem 1.1

We want to point out here that the punctured case can help us with the case of no punctures, i.e. Proposition 3.1 can give us another proof of Theorem 1.1. Notice that the proof of Proposition 3.1 does not depend on Theorem 1.1. Let $\mathcal{J}_{g,p}^b$ be the Torelli group of $S_{g,p}^b$, i.e. the subgroup of $\text{Mod}_{g,p}^b$ that acts trivially on $H^1(S_g; \mathbb{Z})$.

Second proof of Theorem 1.1. Again let $g > 3$. We assume that the exact sequence (1.1) has a splitting which is denoted by ϕ such that $F \circ \phi = id$.

By Lemma 2.7, the image $\phi(T_s)$ of T_s the Dehn twist about a separating curve s is $T_{s'}^n T_{s''}^{1-n}$ where s' and s'' are curves on $S_{g,1}$ that are isotopic to s . Let UTS_g be the unit tangent bundle of genus g surface. Let s be a separating curve that separates S_g into two parts $C_1 \cong S_p^1$ and $C_2 \cong S_q^1$ such that $p, q \geq 2$. The combination of Torelli groups of C_1 and C_2 gives us a subgroup G of \mathcal{J}_g satisfying the following short exact sequence.

$$1 \rightarrow \mathbb{Z} \xrightarrow{(T_s, T_s^{-1})} \mathcal{J}_p^1 \times \mathcal{J}_q^1 \rightarrow G \rightarrow 1$$

The disk pushing subgroup is $\pi_1(UTS_p) \rightarrow \mathcal{J}_p^1$, i.e. see [FM12, Page 118]. The disk pushing subgroups of C_1 and C_2 give us a subgroup A of G satisfying the following short exact sequence.

$$1 \rightarrow \mathbb{Z} \xrightarrow{(T_s, T_s^{-1})} \pi_1(UTS_p) \times \pi_1(UTS_q) \rightarrow A \rightarrow 1 \quad (4.1)$$

Claim 4.1. $\phi(T_s) = T_{s'}$ for a curve s' on $S_{g,1}$ that is isotopic to s .

Proof. We have already proved this result in the proof of Theorem 1.1, Case 1. Here we give another proof using the Euler class. By Lemma 2.7, we have that $\phi(T_s) = T_{s'}^n T_{s''}^{1-n}$. We only need to prove that $\text{CRS}(\phi(T_s))$ only contains one curve.

Suppose the opposite that $\text{CRS}(\phi(T_s))$ contains two curves s' and s'' such that they are isotopic to s . Then $\phi(A)$ is in the centralizer of $\phi(T_s)$. The centralizer $\phi(T_s)$ is the subgroup of $\mathcal{J}_{g,1}$ that fixes s' and s'' . Since $\phi(A)$ also satisfies the fact that it maps to A after forgetting $*$. We know that $\phi(A) \subset \pi_1(UTS_p) \times \pi_1(UTS_q)$. Since by computation,

$$\dim H^2(UTS_p \times UTS_q; \mathbb{Q}) = \dim H^2(S_p \times S_q; \mathbb{Q}) = \dim H^2(A; \mathbb{Q}),$$

we know that the Euler class of (4.1) is nonzero. Therefore (4.1) does not split which proves the claim. \square

Since T_s commutes with each element of G , we have that $T_{s'}$ commutes with each element of $\phi(G)$. Therefore, $\phi(G)$ is a subgroup of the centralizer of $T_{s'}$. The centralizer $C_{\mathcal{J}_{g,1}}(T_{s'})$ of $T_{s'}$ is the subgroup of $\mathcal{J}_{g,1}$ that fixes s' . Since the two components of $S_g - s'$ are not homeomorphic, any element in $C_{\mathcal{J}_{g,1}}(T_{s'})$ has to fix the two components. Therefore $C_{\mathcal{J}_{g,1}}(T_{s'})$ satisfies the following exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \mathcal{J}_p^1 \times \mathcal{J}_{q,1}^1 \rightarrow C_{\mathcal{J}_{g,1}}(T_{s'}) \rightarrow 1.$$

Therefore we have a section of $F : \mathcal{J}_{q,1}^1 \rightarrow \mathcal{J}_q^1$ which maps T_s to $T_{s'}$. This section gives a section of $F_{q,2,1}$ in the following commutative diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{J}_{q,1}^1 & \longrightarrow & \mathcal{P}\mathcal{J}_{q,2} \longrightarrow 1 \\ & & \downarrow & & \downarrow F & & \downarrow F_{q,2,1} \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{J}_q^1 & \longrightarrow & \mathcal{J}_{q,1} \longrightarrow 1. \end{array} \quad (4.2)$$

However, we already prove that $F_{q,2,1}$ does not have a section in Corollary 3.5, this implies that F does not have a section. The statement follows. \square

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