Bäcklund transformations in the Hauser–Ernst formalism for stationary axisymmetric spacetimes [a]

Christopher M. Cosgrove [b]
W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125

(Received 30 March 1981; accepted for publication 26 June 1981)

It is shown that Harrison's Bäcklund transformation for the Ernst equation of general relativity is a two-parameter subset (not subgroup) of the infinite-dimensional Geroch group **K**. We exhibit the specific matrix $u(t)$ appearing in the Hauser–Ernst representation of **K** for vacuum spacetimes which gives the Harrison transformation. Harrison transformations are found to be associated with quadratic branch points of $u(t)$ in the complex t plane. The coalescence of two such branch points to form a simple pole exhibits in a simple way the known factorization of the (null generalized) HKX transformation into two Harrison transformations. We also show how finite (i.e., already exponentiated) transformations in the **B** group and nonnull groups of Kinnersley and Chitre can be constructed out of Harrison and/or HKX transformations. Similar considerations can be applied to electrovac spacetimes to provide hitherto unknown Bäcklund transformations. As an example, we construct a six-parameter transformation which reduces to the double Harrison transformation when restricted to vacuum and which generates Kerr–Newman–NUT space from flat space.

PACS numbers: 04.20.Cv, 04.20.Jb, 04.40.c

1. INTRODUCTION

Since the pioneering work of Geroch,[1,2] it has been known that the partial differential equations governing the metric of the stationary axisymmetric vacuum gravitational field admit an infinite-dimensional internal symmetry group of transformations. This large internal symmetry group (the Geroch group **K**) has encouraged many authors to hope that the complete class of solutions could some day be generated systematically from a particular solution, such as flat space, or from the important subclass of Weyl static solutions. In fact, a number of special transformations, some contained in **K**[3,4] and some Bäcklund transformations known or presumed to be outside **K**,[5–11] are now known which, when iterated, generate asymptotically flat solutions with an arbitrarily large number of parameters. A detailed study of the mathematical interrelationships between these various transformations has been undertaken by the author.[12]

A workable and very fruitful representation in terms of infinitesimal generators for the Geroch group **K** and its electrovac extension **K**′ has been provided by Kinnersley and Chitre[13–16] (KC). Possibly, the most important discoveries arising from their formalism are the **B** group[16] which, among other things, generates the Kerr solution[17] from Schwarzschild, and the HKX transformation.[3] More recently, Hauser and Ernst[18–20] (HE) have deduced (initially from the KC formalism and later by a direct method) a qualitatively different representation which exploits the theory of functions of a complex variable. The HE formalism has the distinct advantage that elements of **K** and **K**′ appear already exponentiated and each may be specified unambiguously by a matrix function $u(t)$ of a complex variable t satisfying certain conditions, a closed contour L in the complex t plane, and a choice of gauge for the $F(t)$ matrix potential[21] of the solution to be transformed. Composition of several transformations is conveniently represented by multiplication of corresponding $u(t)$ matrices. The generation of new solutions from old in their formalism can be accomplished by either solving a linear integral equation[18] or a homogeneous Hilbert problem.[19] A remarkable result of Hauser and Ernst's work is a formula for all $u(t)$ matrices which transform a given initial solution into a given final solution.[20] This provides a quantitative settlement of one form of a well-known conjecture of Geroch.[22]

The present paper addresses certain problems for which the Hauser–Ernst formalism is particularly well suited. First, we study two large classes of $u(t)$ matrices for which the homogeneous Hilbert problem (HHP) can be solved by elementary methods. It is already known[18] that when $u(t)$ has only simple poles in L_+ (the interior of the contour L), the solution of the corresponding HHP is a product of null generalized HKX transformations[3,4,12] and nonsimple poles correspond to confluent forms such as the rank-N transformation.[3]

On the other hand, for certain classes of $u(t)$ matrices for vacuum spacetimes which have quadratic branch points and branch cuts in L_+, we find that the solutions of the HHP's are products of Harrison's Bäcklund transformations.[8] In particular, the Harrison transformation, which was discovered by methods inspired by soliton theory and quite remote from the Geroch group, is here shown to be a two-parameter subset (not a subgroup) of **K**. This result should not be entirely unexpected according to a suggestion of Kinnersley[23] and the fact that arguments given in Ref. 12 to rule out certain other Bäcklund transformations being in **K**[24] were inconclusive for the Harrison transformation.

Conversely, the formula for the Harrison transform of the $F(t)$ potential given by Eqs. (4.44a) and (4.44b) of Ref. 12 can be substituted directly into the HE formula for $u(t)$ in terms of the initial and final solutions[20] [see Eq. (2.25) below]

[a] Supported in part by the National Science Foundation (AST79-22012).
[b] Richard Chace Tolman Research Fellow.

to give, explicitly,

$$u(t) = \left(1 - \frac{s}{t}\right)^{-1/2} \begin{pmatrix} 1 & -cst^{-1} \\ -c^{-1} & 1 \end{pmatrix}, \tag{1.1}$$

s and c real constants. The product of several Harrison transformations is represented by the product of corresponding $u(t)$ matrices [the individual s and c parameters of such a product may be complex such that the product $u(t)$ is real for real t]. In Ref. 12, we proved that the product of two Harrison transformations with same s parameters is a null generalized HKX transformation by directly composing the transformation laws for $F(t)$. In the HE formalism, the proof of this theorem reduces to matrix multiplication.

As yet, finite algebraic transformations have not been written down for the **B** group[16] whose generators are[25]

$$\gamma_{11}^{(k+1)} + \gamma_{22}^{(k-1)}, \quad k = 0, 1, 2, ..., \tag{1.2}$$

and the nonnull KC group, whose generators are[25]

$$q^{XY}\gamma_{XY}^{(k)}, \quad k = 0, 1, 2, ..., \; q^{XY}q_{XY} \neq 0, \tag{1.3}$$

q^{XY} being a symmetric constant SL(2, R) tensor, independent of k, which includes the nonnull HKX transformations.[12] We find that, although we cannot exponentiate the infinitesimal transformations, we can express a large number of finite transformations in these groups as products of Harrison and/or HKX transformations.

Finally, in Sec. 5, we attempt to generalize the Harrison transformation to electrovac spacetimes by considering 3×3 $u(t)$ matrices with *cubic* branch points. Surprisingly, we manage to find a Bäcklund transformation with six parameters which reduces to the *double* Harrison transformation with two complex conjugate s parameters when restricted to vacuum, while the method fails to yield an electrovac version of the *single* Harrison transformation. The new transformation maps flat space to Kerr–Newman–NUT space, which is a satisfying result. However, in the conclusion, we present an argument based on analogy with vacuum that the single Harrison transformation should exist in the electrovac case and have perhaps four parameters. A possible reason for the failure of the HE formalism to account for the electrovac analog of the single Harrison transformation is that the latter is not expected to preserve the reality of the metric and electromagnetic potentials and so an extension to two complex dimensions is indicated.

2. THE HOMOGENEOUS HILBERT PROBLEM OF HAUSER AND ERNST

In this section, we include enough details on the $F(t)$ potential, the matrix representation $u(t)$ for **K**, and the homogeneous Hilbert problem (HHP) for later use. We wish to follow the SL(2) tensor notation of Kinnersley[13] (see also Appendix A of Ref. 12) in which we would identify $F(t) = F_{AB}(t)$, $u(t) = u^A{}_B(t)$, $A, B = 1, 2$.[26] The electrovac case will be postponed till Sec. 5.

The metric of stationary axisymmetric spacetime can be written

$$ds^2 = f_{AB}dx^A dx^B - f^{-1}e^{2\gamma}(d\rho^2 + dz^2), \tag{2.1}$$

where f_{AB}, $f = f_{11}$, and γ are functions of cylindrical coordinates ρ and z only; x^1 is time, x^2 is azimuthal angle. We use the parametrization

$$f_{11} = f, \quad f_{12} = f_{21} = -f\omega, \quad f_{22} = f\omega^2 - \rho^2 f^{-1}. \tag{2.2}$$

As is well known, the vacuum field equations are the integrability conditions for further potentials. The Ernst potential $\mathscr{E}$ and its tensor generalization H_{AB} are defined by[27]

$$\mathscr{E} = H_{11} = f + i\psi, \tag{2.3a}$$

$$\nabla\psi = -\rho^{-1}f^2\tilde{\nabla}\omega, \tag{2.3b}$$

$$H_{AB} = f_{AB} + i\psi_{AB}, \tag{2.4a}$$

$$\nabla\psi_{AB} = -\rho^{-1}f_A{}^X\tilde{\nabla}f_{XB}, \tag{2.4b}$$

and satisfy

$$f\nabla_3{}^2\mathscr{E} = \nabla\mathscr{E}\cdot\nabla\mathscr{E}, \tag{2.5a}$$

$$\nabla H_{AB} = -i\rho^{-1}f_A{}^X\tilde{\nabla}H_{XB}, \tag{2.5b}$$

$$H_{AB} - H_{BA} = 2iz\epsilon_{AB}, \tag{2.6a}$$

$$f_{XA}f^X{}_B = -\rho^2\epsilon_{AB}, \tag{2.6b}$$

where $\nabla = (\partial/\partial\rho, \partial/\partial z)$, $\tilde{\nabla} = (\partial/\partial z, -\partial/\partial\rho)$, $\nabla_3{}^2 = \partial^2/\partial\rho^2 + \partial^2/\partial z^2 + \rho^{-1}\partial/\partial\rho$. An asterisk (*) will denote complex conjugation, e.g., $H^*{}_{AB} = f_{AB} - i\psi_{AB}$.

When H_{AB} is known, a potential $F_{AB}(t)$ $[= F_{AB}(\rho, z, t)]$ which is a function of a complex variable t, as well as ρ and z, can be constructed from the linear differential equation,[28]

$$\nabla F_{AB}(t) = itS^{-2}(t)[(1-2tz)\nabla H_{AX} - 2t\rho\tilde{\nabla}H_{AX}]F^X{}_B(t), \tag{2.7}$$

where

$$S(t) = [(1-2tz)^2 + 4t^2\rho^2]^{1/2}, \quad S(0) = 1, \tag{2.8}$$

subject to

$$F_{AB}(0) = i\epsilon_{AB}, \tag{2.9a}$$

$$\dot{F}_{AB}(0) = H_{AB}, \tag{2.9b}$$

$\dot{F}(t) = \partial F(t)/\partial t$. Two important first integrals of Eq. (2.7) are

$$F_{XA}(t)F^X{}_B(t) = -S^{-1}(t)\epsilon_{AB} \quad \text{or} \quad \det F(t) = -S^{-1}(t), \tag{2.10}$$

$$S(t)F^*{}_{AB}(t) = 2itf_{AX}F^X{}_B(t) - (1-2tz)F_{AB}(t), \tag{2.11}$$

where $F^*{}_{AB}(t)$ is to be understood as the complex conjugate of $F_{AB}(t^*)$.

The differential equation (2.7) and initial conditions (2.9a) and (2.9b) define $F(t)$ up to a gauge change,

$$F_{AB}(t) \to F_A{}^X(t)g_{XB}(t) \quad \text{or} \quad F(t) \to F(t)g(t), \tag{2.12}$$

where $g(t) = -g^A{}_B(t)$ depends on t only. Equations (2.9)–(2.11) imply

$$g_{AB}(0) = \epsilon_{AB} \;\text{ or }\; g(0) = I, \quad g^*(t) = g(t), \quad \det g(t) = 1, \tag{2.13}$$

I being the unit matrix. This gauge freedom may be used to minimize the singularities of $F(t)$ in the complex t plane.[19,20] In all cases, $F(t)$ is analytic at and in a neighborhood of $t = 0$. Also, it is always possible to choose gauge so that $F(t)$ is analytic at and near $t = \infty$. Hauser and Ernst (HE) have imposed the slightly stronger condition

$$F(t)\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \text{ analytic at } t = \infty \quad (\text{"HE gauge"}), \tag{2.14}$$

which can be brought about by a translation, $\omega \to \omega +$ constant. Later, in SL(2)-covariant applications, we shall wish to permit an arbitrary additive constant in ω, so we relax this condition to

$$F(t) \text{ analytic at } t = \infty \quad (\text{"modified HE gauge"}). \tag{2.14'}$$

Next, HE[19,20] have shown that, in a (ρ, z) domain covering at least one point of the z axis in which $\mathscr{E}$ is analytic[29] and $f \neq 0$, gauge can be chosen so that the only singularities of $F(t)$ in the t plane (including $t = \infty$) are quadratic branch points with index $-\frac{1}{2}$ at

$$t_+ = \frac{z + i\rho}{2r^2} \text{ and } t_- = \frac{z - i\rho}{2r^2} \quad (r^2 = \rho^2 + z^2), \tag{2.15}$$

i.e., the zeros of $S(t)$, and the cut is a finite arc from $t = t_+$ to $t = t_-$ (not through $t = 0$). On the z axis, where $\rho = 0$, the branch points and cut degenerate to a simple pole at $t = (2z)^{-1}$. This very special gauge will be called "special HE gauge" if condition (2.14) is also imposed; otherwise [condition (2.14')] we shall call it "modified special HE gauge."

If $F(t)$ is in (modified) special HE gauge, then a change of gauge will automatically introduce (ρ, z)-independent singularities in the finite t plane and/or at $t = \infty$. Furthermore, analytic continuation of $F(t)$ across the cut will reveal, in general, (ρ, z)-independent singularities of various types (including at $t = 0$) on the second Riemann sheet. If $\mathscr{E}$ is not analytic anywhere on the z axis, then special HE gauge may not exist. Nevertheless, in this case, HE[19] have proved that $F(t)$ can be chosen to be analytic in the whole t plane except for four quadratic branch points of index $-\frac{1}{2}$ at $t = t_\pm$ and at $t = t_{0\pm}$, the latter being complex conjugate points independent of ρ and z, each pair being joined by a cut.

In Ref. 20, HE have demonstrated that Eqs. (2.3)–(2.11) are easily solved on the z axis ($\rho = 0$), thereby providing a remarkably simple and convenient characterization of (modified) special HE gauge. First, if $\mathscr{E}$ is analytic and $f \neq 0$ on an open interval $\mathscr{I}$ of the z axis, then $\partial H_{AB}/\partial\rho$, $\partial F_{AB}(t)/\partial\rho$, and all derivatives of odd order with respect to ρ vanish on $\mathscr{I}$. Then, $\omega =$ constant $= \omega_0$, say, on $\mathscr{I}$. In special HE gauge, $\omega = 0$ on $\mathscr{I}$. Hence, integrating Eqs. (2.5b) and (2.7) along $\mathscr{I}$, we find

$$H_{AB}(0, z) = \begin{pmatrix} \mathscr{E}(0, z) & 2iz \\ 0 & 0 \end{pmatrix}, \tag{2.16a}$$

$$F_{AB}(0, z, t) = \begin{pmatrix} \dfrac{t\mathscr{E}(0, z)}{1 - 2tz} & \dfrac{i}{1 - 2tz} \\ -i & 0 \end{pmatrix}, \tag{2.16b}$$

in special HE gauge. In modified special HE gauge, where $\omega = \omega_0$ on $\mathscr{I}$,

$$H_{AB}(0, z) = \begin{pmatrix} \mathscr{E} & 2iz - \omega_0\mathscr{E} \\ -\omega_0\mathscr{E} & -2iz\omega_0 + \omega_0{}^2\mathscr{E} \end{pmatrix}, \tag{2.17a}$$

$$F_{AB}(0, z, t) = \begin{pmatrix} \dfrac{t\mathscr{E}}{1 - 2tz} & \dfrac{i - \omega_0 t\mathscr{E}}{1 - 2tz} \\ -i - \dfrac{\omega_0 t\mathscr{E}}{1 - 2tz} & \dfrac{-2itz\omega_0 + \omega_0{}^2 t\mathscr{E}}{1 - 2tz} \end{pmatrix}, \tag{2.17b}$$

where $\mathscr{E} = \mathscr{E}(0, z)$. Clearly, the only singularity in the t plane is a simple pole where t_+ and t_- coalesce at $t = (2z)^{-1}$.

Let us now turn to the description of the matrix element $u(t) = u^A{}_B(t)$ which represents an element of $\mathbf{K}$ in a manner depending on the choice of gauge for $F(t)$ and a contour L in the complex t plane.[19,20] For convenience of expression, we shall often denote an element of $\mathbf{K}$ by its corresponding $u(t)$ matrix if the gauge and choice of L is clear from the context. First, since $F(t)$ is always analytic in an open region containing $t = 0$, we can draw a simple closed curve L surrounding $t = 0$, symmetric about the real axis, whose interior we denote L_+, exterior L_-, such that $F(t)$ is analytic in L_+ and on L. In particular, the points $t = t_\pm$ and the cut joining them must be in L_-. The interior L_+ can be made as large as desired by putting $F(t)$ in (modified) special HE gauge and considering points (ρ, z) close to (0, 0). The matrix $u(t)$ must be analytic at least in an open annulus containing L. Further, it satisfies the algebraic constraints

$$\det u(t) = 1 \quad \text{or} \quad u_{XA}(t)u^X{}_B(t) = \epsilon_{AB}, \tag{2.18a}$$

$$u^*(t) = u(t). \tag{2.18b}$$

In addition, HE impose a boundary condition at $t = \infty$ [Eq. (2.19) below] which is not merely a restriction on the gauge, but actually excludes a significant portion of the group $\mathbf{K}$. This topic will be discussed in a sequel to the present paper. Here, we shall require that $F(t)$ be put in either HE gauge or modified HE gauge and that $u(t)$ be analytic in a neighborhood of $t = \infty$ and

(i) $u^1{}_1(t)$, $tu^1{}_2(t)$, $t^{-1}u^2{}_1(t)$, $u^2{}_2(t)$ are analytic at $t = \infty$ whenever $F(t)$ is in HE gauge; (2.19)

(ii) $u(t)$ is analytic at $t = \infty$ whenever $F(t)$ in in modified HE gauge (2.19')

[cf. Eqs. (2.14) and (2.14')]. When combining or multiplying transformations in $\mathbf{K}$ [represented by multiplication of corresponding $u(t)$ matrices], it is important not to mix the two types (2.19) and (2.19').

When $F(t)$, $u(t)$, and L are given, the $F(t)$-potential $F'(t)$ of a new solution $\mathscr{E}'$ of the field equations may be found by solving the matrix homogeneous Hilbert problem (HHP),

$$X_-(t) = X_+(t)G(t), \tag{2.20}$$

where

$$X_+(t) = F'(t)F(t)^{-1}, \tag{2.21a}$$

$$G(t) = F(t)u(t)F(t)^{-1}. \tag{2.21b}$$

In Eq. (2.20), $G(t)$ is a given matrix analytic on L, and the unknowns X_- and X_+ are required to satisfy

$$X_-(t) \text{ analytic in } L + L_- \text{ and at } t = \infty, \tag{2.22a}$$

$$X_+(t) \text{ analytic in } L + L_+, \quad X_+(0) = I. \tag{2.22b}$$

The HHP can also be written

$$X_-(t) = F'(t)u(t)F(t)^{-1}, \tag{2.23a}$$

in which X_- and F' are the unknowns, $F'_{AB}(0) = i\epsilon_{AB}$, or, in tensor notation,

$$X_{-A}{}^B(t) = -S(t)F'_{AX}(t)u^X{}_Y(t)F^{BY}(t). \tag{2.23b}$$

With the boundary conditions at $t = 0$ and $t = \infty$, the solution of the HHP is unique if it exists.[19] The new metric tensor

f'_{AB} and Ernst potential $\mathscr{E}'$ follow from

$$H'_{AB} = \dot{F}'_{AB}(0), \quad f'_{AB} = \mathrm{Re}H'_{AB}, \quad \mathscr{E}' = H'_{11}. \tag{2.24}$$

There is no loss of generality in requiring $u(t)$ to be analytic throughout L_- as singularities can be absorbed by gauge changes. First, suppose $F(t)$ is in (modified) special HE gauge. Then Eq. (2.23) shows that $F'(t)$ is in the same special gauge if and only if $u(t)$ is analytic throughout L_-. If $u(t)$ is not analytic in L_-, then perform the factorization $u(t) = u_+(t)u_-(t)$, where u_+ and u_- satisfy the conditions for a $u(t)$ matrix and, in addition, u_+ is analytic in L_+, u_- in L_-, and $u_+(0) = I$ (this is an HHP). It follows that $F'(t)u_+(t)$ is an $F(t)$ potential in (modified) special HE gauge while $u_+(t)$ merely effects a gauge change [cf. Eqs. (2.12) and (2.13)]. Similarly, if $F(t)$ for the original solution is not in (modified) special HE gauge but still satisfies (2.14) or (2.14′), then write it as $F(t) = F_{\rm sp}(t)g(t)$ where $F_{\rm sp}(t)$ is in the special gauge and $g(t)$ satisfies Eqs. (2.13). Then perform the factorization $u(t)g(t)^{-1} = u_+(t)u_-(t)$, as before, and the HHP will take the form

$$X_-(t) = F'_{\rm sp}(t)u_-(t)F_{\rm sp}(t)^{-1},$$

where $F'_{\rm sp}(t) = F'(t)u_+(t)$. Except where otherwise stated, we shall henceforth assume that $u(t)$ is analytic throughout L_-. [Notice that the full group of gauge transformations cannot be handled by the HE formalism because $F'(t)$ is required to be analytic in L_+. In some cases where $F'(t)$ is not analytic in L_+ it may be possible to deform the contour L without crossing singularities of $u_-(t)$, but this is clearly impossible when $F'(t)$ and $u_-(t)$ have coincident singularities in L_+. An example of such a coincidence is the "extended" HKX transformation.[12]]

The Hauser–Ernst formula, mentioned in Sec. 1, which gives all $u(t)$ which map a given initial solution $\mathscr{E}$ to a given final solution $\mathscr{E}'$ is derived as follows. Suppose that $\mathscr{E}$ and $\mathscr{E}'$ are analytic on an open interval $\mathscr{I}$ of the z axis containing $(\rho, z) = (0, 0)$ and put $F(t)$ and $F'(t)$ in special HE gauge. Then, on $\mathscr{I}$, $F(t)$ and $F'(t)$ are given by Eq. (2.16b). Substitute into the HHP Eq. (2.23a) and observe that the left-hand side is analytic in L_- whereas the right-hand side apparently has a simple pole at $t = (2z)^{-1}$ in L_-. Setting the residue to be zero gives

$$tu^1{}_2(t)\mathscr{E}\mathscr{E}' + t^{-1}u^2{}_1(t) - iu^1{}_1(t)\mathscr{E}' + iu^2{}_2(t)\mathscr{E} = 0, \tag{2.25}$$

where $\mathscr{E}$ and $\mathscr{E}'$ are to be evaluated at $\rho = 0, z = (2t)^{-1}$. The real and imaginary parts of Eq. (2.25), together with Eq. (2.18a), provide three equations for the four components of $u^A{}_B(t)$. If, instead, we put $F(t)$ and $F'(t)$ in modified special HE gauge, with $\omega = \omega_0$, $\omega' = \omega'_0$ on $\mathscr{I}$, then

$$u(t) = \begin{pmatrix} 1 & \omega'_0 \\ 0 & 1 \end{pmatrix} u_0(t) \begin{pmatrix} 1 & -\omega_0 \\ 0 & 1 \end{pmatrix}, \tag{2.26}$$

where $u_0(t)$ satisfies Eq. (2.25). This matrix product expresses the composition of three transformations: (i) $\omega \to \omega - \omega_0$, (ii) $\mathscr{E} \to \mathscr{E}'$ preserving special HE gauge, and (iii) $\omega' \to \omega' + \omega'_0$.

The Ehlers group **P** is given by

$$\mathscr{E}' = (P)_\alpha \mathscr{E} = \frac{\alpha_4\mathscr{E} + i\alpha_3}{\alpha_1 - i\alpha_2\mathscr{E}}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \tag{2.27}$$

$\det\boldsymbol{\alpha} = 1$, $(P)_\alpha \in \mathbf{P}$. The Matzner–Misner group **L** (rotation of Killing vectors) is given by

$$f'_{AB} = (L)_\beta f_{AB} = b_A{}^C b_B{}^D f_{CD}, \quad \boldsymbol{\beta} = b_A{}^B = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}, \tag{2.28}$$

$\det\boldsymbol{\beta} = 1$, $(L)_\beta \in \mathbf{L}$. Both groups are one-to-two homomorphic to SL(2, R). The $(L)_\beta$ transformation is represented by

$$u^A{}_B(t) = -b^A{}_B = (\boldsymbol{\beta}^T)^{-1} = \begin{pmatrix} \beta_4 & -\beta_3 \\ -\beta_2 & \beta_1 \end{pmatrix}, \tag{2.29}$$

^{T} denoting transpose, and the solution of the HHP (trivial in this case) is

$$F'_{AB}(t) = b_A{}^C b_B{}^D F_{CD}(t) \quad \text{or} \quad F'(t) = \boldsymbol{\beta}F(t)\boldsymbol{\beta}^T. \tag{2.30}$$

Note that $u(t)$ obeys condition (2.19′) and preserves modified special HE gauge. The $(P)_\alpha$ transformation can be substituted into the HE formula (2.25) to yield

$$u(t) = \begin{pmatrix} \alpha_1 & -\alpha_2 t^{-1} \\ -\alpha_3 t & \alpha_4 \end{pmatrix}, \tag{2.31}$$

obeying condition (2.19). The solution of the HHP in this case can be obtained by a straightforward application of methods outlined in Secs. 3 and 5 below. The result is

$$F'(t) = (A + t^{-1}B)F(t)u(t)^{-1}, \tag{2.32}$$

where

$$A = \begin{pmatrix} \alpha_4 + i\alpha_2\mathscr{E}' & 0 \\ i\alpha_2(H_{12} + H'_{12} - 2iz) & \alpha_1 - i\alpha_2\mathscr{E} \end{pmatrix}, \tag{2.33a}$$

$$B = \begin{pmatrix} 0 & 0 \\ \alpha_2 & 0 \end{pmatrix}. \tag{2.33b}$$

Here, $\mathscr{E}'$ is given by Eq. (2.27) and H'_{12} by

$$H'_{12} = \frac{\alpha_1 H_{12} + \alpha_2 H^{(2)}_{11}}{\alpha_1 - i\alpha_2\mathscr{E}}, \tag{2.34}$$

where $H^{(2)}_{AB} = \frac{1}{2}\ddot{F}_{AB}(0)$. Eq. (2.32) preserves special HE gauge.

3. QUADRATIC BRANCH POINTS AND THE HARRISON TRANSFORMATION

The homogeneous Hilbert problem can be solved by elementary methods in the case when $u(t)$ has only poles in L_+ and for a large class of cases where $u(t)$ has also quadratic branch points and cuts contained in L_+. The case of N simple poles (at points $t \neq 0$ in L_+) has been treated adequately by Hauser and Ernst[18] using their integral equation. They have identified the transformation for which

$$u(t) = I + \sum_{i=1}^{N} \frac{\alpha_i t}{t - s_i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

α_i, s_i real constants, with the product of N HKX transformations.[3] Allowing all four components of $u(t)$ to have simple poles at $t = s_1, \ldots, s_N \neq 0$ leads to a product of null generalized HKX transformations.[12] HE have also obtained the electrovac counterparts of the latter transformations.[18] The case of nonsimple poles can be treated directly by the same methods or by regarding a pole of multiplicity N as a coalescence of N simple poles. The corresponding limit of the product of N HKX transformations is a combined HKX transformation of ranks 0, 1, ..., $N-1$.[3]

It is instructive to see the null generalized HKX transformation derived from the HHP in an SL(2)-covariant manner. Consider

$$u^A{}_B(t) = -\epsilon^A{}_B + \frac{\alpha t}{t-s} q^A{}_B, \tag{3.2}$$

where s, α, and $q^A{}_B$ are constants and q_{AB} is symmetric and null (recall $-\epsilon^A{}_B = \epsilon_B{}^A = \delta^A_B$). The contour L is drawn to enclose $t=0$ and the pole at $t=s$. The HHP is

$$X_{-A}{}^B(t) = -S(t)F'_{AX}(t)\left[-\epsilon^X{}_Y + \frac{\alpha t}{t-s} q^X{}_Y\right]F^{BY}(t). \tag{3.3}$$

The left-hand side is analytic in $L+L_-$ and at $t=\infty$. The right-hand side is analytic in $L+L_+$ except for a simple pole at $t=s$ and has the value $\epsilon_A{}^B$ at $t=0$. Hence

$$-S(t)F'_{AX}(t)\left[-\epsilon^X{}_Y + \frac{\alpha t}{t-s} q^X{}_Y\right]F^{BY}(t)$$
$$= \epsilon_A{}^B + \frac{t}{t-s} R_A{}^B, \tag{3.4}$$

where $R_A{}^B$ is a constant (in t) tensor to be determined. Expressing $F'_{AB}(t)$ as the subject, we have

$$F'_{AB}(t) = \left[F_{AY}(t) + \frac{t}{t-s} R_A{}^X F_{XY}(t)\right]\left[\epsilon_B{}^Y - \frac{\alpha t}{t-s} q_B{}^Y\right]. \tag{3.5}$$

Comparing residues at $t=s$ of both sides of Eq. (3.4), we find

$$R_A{}^B = -\alpha S(s)F'_{AX}(s)q^X{}_Y F^{BY}(s). \tag{3.6}$$

Since q_{AB} is null and symmetric, it admits the factorization $q_{AB} = q_A q_B$. An expression for the vector $F'_{AX}(s)q^X$ and hence for $R_A{}^B$ can be obtained by multiplying both sides of Eq. (3.5) by q^B and taking the limit as $t \to s$. The results are

$$F'_{AX}(s)q^X = F_{AX}(s)q^X\left[1 + \alpha s S(s) q^{CD} F_{ZC}(s)\dot{F}^Z{}_D(s)\right]^{-1}, \tag{3.7}$$

$$R_A{}^B = -\alpha S(s)F_{AX}(s)q^X{}_Y F^{BY}(s) \times \left[1 + \alpha s S(s) q^{CD} F_{ZC}(s)\dot{F}^Z{}_D(s)\right]^{-1}. \tag{3.8}$$

The expression for the transformed $F(t)$ potential simplifies if we introduce the generating function $G_{AB}(s,t)$ of Kinnersley and Chitre[16] which is given in terms of $F_{AB}(t)$ by[3]

$$G_{AB}(s,t) = \frac{s}{s-t}\epsilon_{AB} + \frac{tS(s)}{s-t} F_{XA}(s)F^X{}_B(t), \tag{3.9}$$

with a suitable limit for $G_{AB}(s,s)$. Then Eqs. (3.5), (3.8), and (3.9) give

$$F'_{AB}(t)$$
$$= \left\{F_A{}^X(t) + \frac{\alpha q^{CD} F_{AC}(s)\left[G_D{}^X(s,t) - s(s-t)^{-1}\epsilon_D{}^X\right]}{1 - \alpha q^{EF} G_{EF}(s,s)}\right\}$$
$$\times\left[\epsilon_{XB} - \frac{\alpha t}{s-t} q_{XB}\right]. \tag{3.10}$$

This is precisely the formula given in Ref. 12 for the generalized HKX transformation, whose infinitesimal form is defined by[25]

$$\sum_{k=0}^{\infty} \alpha s^k q^{XY} \gamma^{(k)}_{XY}, \tag{3.11}$$

for the case of null q^{XY}. It is easy to show that the apparent pole at $t=s$ in the right-hand side of Eq. (3.10) is absent.

The second factor in Eq. (3.10) is a gauge function. The first factor, with the index X lowered by ϵ_{XB}, is the transform of $F_{AB}(t)$ under the "extended" HKX transformation,[12] defined by (3.11) with the sum taken from $k = -\infty$ to $k=\infty$. In Ref. 12, the extended HKX transformation was derived as the limit of the double soliton transformation of Belinsky and Zakharov[9] when the two simple poles of their matrix $\chi(\lambda)$ coalesce to form a double pole.

Transformations corresponding to a simple or nonsimple pole at $t=0$ have a longer history. These can be described as or represented by: (i) products of a finite number of $(P)_\alpha$ and $(L)_\beta$ transformations[1,2]; (ii) a broken null curve in the Geroch representation[2]; (iii) repeated applications of the Lagrangian invariance transformations of Hoenselaers[30]; (iv) transformations $q^{XY}\gamma^{(k)}_{XY}$, q^{XY} null, exponentiated by KC[15] and products thereof. The failure of these early transformations to preserve asymptotic flatness has a rational explanation in terms of the HE representation. Eq. (2.25) shows that $u(t)$ will preserve asymptotic flatness up to a NUT parameter whenever it is asymptotic to the right-hand side of (2.31) as $t \to 0$ (not necessarily with all components of $\boldsymbol{\alpha}$ nonvanishing). Thus asymptotic flatness preservation is not sensitive to the types of singularities of $u(t)$ occurring at points $t \neq 0$, showing in a sense that the "majority" of elements of $\mathbf{K}$ actually preserve asymptotic flatness, but singularities at $t=0$ [except for a simple pole in $u^1{}_2(t)$] are not allowed.

Matrices $u(t)$ which have quadratic branch points arise naturally on account of the determinant condition, $\det u(t) = 1$. Let $v(t)$ be a matrix function of t analytic and possessing an inverse in $L+L_-$, real for real t, and obeying either condition (2.19) or (2.19′). Then

$$u(t) = [\det v(t)]^{-1/2} v(t) \tag{3.12}$$

obeys all the conditions for a $u(t)$ matrix and is analytic in L_-, provided the branch cuts joining the zeros of $\det v(t)$ can be contained in L_+. This can be done if $t=\infty$ is not also a branch point. This situation can be excluded by requiring that $\det v(t) \to 1$ (or any positive real constant) as $t \to \infty$.

If $v(t)$ is a given rational function of t, then the HHP can be solved by essentially the same method as for $u(t)$ rational. The HHP can be rewritten

$$[\det v(t)]^{1/2} X_-(t) = F'(t)v(t)F(t)^{-1}. \tag{3.13}$$

The left-hand side is analytic in L_- and at $t=\infty$ while the right-hand side has only poles in L_+. The branch point singularities have been absorbed in $X_-(t)$. Thus either side of Eq. (3.13) defines a rational function of t and the problem reduces to the determination of the unknown coefficients in the numerator. The transformation represented by such a $u(t)$ will be seen to be a product of Harrison's Bäcklund transformations[8] (branch points at $t \neq 0$), null HKX transformations[3,12] (poles at $t \neq 0$), and null KC transformations[2,15,30] (either pole or branch point at $t=0$, depending on whether number of accompanying Harrison transformations is even or odd, respectively). Further, when two Harrison transformations with same s parameters are multiplied, the corresponding product $u(t)$ matrix has only a pole at $t=s$ and so represents a null HKX transformation.[12]

The simplest SL(2)-covariant example is

$$v^A{}_B(t) = -\epsilon^A{}_B + t^{-1}h^A{}_B, \quad \det v(t) = 1 - t^{-1}h_X{}^X, \tag{3.14}$$

where h_{AB} is null and nonsymmetric. The explicit solution of the HHP is

$$F'_{AB}(t) = \left[s^{-1}h_A{}^C - \frac{t(h_{ZA} + ih_{AX}h_{ZY}H^{XY})F^{CZ}(s)}{(s-t)h_{MN}F^{NM}(s)}\right] \times F_{CD}(t)(\epsilon_B{}^D - t^{-1}h_B{}^D), \tag{3.15}$$

where $s = h_X{}^X = -h^X{}_X$. The more general case where h_{AB} is nonnull, so that $\det v(t) = 1 - st^{-1} + t^{-2}(\det h)$, is easily shown to factorize into two of the above transformations. The proof of Eq. (3.15) involves elegant and pleasing tensor manipulations, which we leave as an exercise for the reader. Instead, we shall give the derivation for the closely related, though non-SL(2)-covariant, Harrison transformation, for which

$$u(t) = \left(1 - \frac{s}{t}\right)^{-1/2}\begin{pmatrix} 1 & -cst^{-1} \\ -c^{-1} & 1 \end{pmatrix}. \tag{3.16}$$

The $u(t)$ matrix given by (3.12) and (3.14) can be factorized into matrices of the forms (3.16), (2.29), and (2.31) in at least three ways.

The transform of $F(t)$ under the Harrison (H) transformation was calculated in Ref. 12 by first deriving the transforms of $F(t)$ under the groups $\mathbf{P}$, $\mathbf{L}$, $\mathbf{Q}$,[5–7] and $\widetilde{\mathbf{Q}}$,[6,7] and then calculating the products,

$$H = (L)_\beta(\widetilde{Q})_{-4s}(Q)_{4s}(P)_\alpha = (P)_\alpha(Q)_{-4s}(\widetilde{Q})_{4s}(L)_\beta, \tag{3.17a}$$

$$\alpha = \begin{pmatrix} 1 & cs \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & c^{-1} \\ 0 & 1 \end{pmatrix}, \tag{3.17b}$$

$(P)_\alpha \in \mathbf{P}$, $(L)_\beta \in \mathbf{L}$, $(Q)_{\pm 4s} \in \mathbf{Q}$ (commutes with $\mathbf{P}$), $(\widetilde{Q})_{\pm 4s} \in \widetilde{\mathbf{Q}}$ (commutes with $\mathbf{L}$). The resulting formula [Eqs. (4.44a) and (4.44b) of Ref. 12] preserves special HE gauge, and so can be substituted directly into Eq. (2.25) here to give the representing matrix (3.16). It is, of course, more instructive to see Eqs. (4.44a) and (4.44b) of Ref. 12 derived from (3.16) and the HHP. The insights gained will allow us to construct new Bäcklund transformations for electrovac spacetimes in Sec. 5.

The HHP takes the form,

$$\left(1 - \frac{s}{t}\right)^{1/2} X_-(t) = F'(t)\begin{pmatrix} 1 & -cst^{-1} \\ -c^{-1} & 1 \end{pmatrix} F(t)^{-1}. \tag{3.18}$$

The left-hand side is analytic in $L + L_-$ and at $t = \infty$ [the branch cut in $(1 - s/t)^{1/2}$ joins $t = 0$ to $t = s$ in L_+]. The right-hand side is analytic in $L + L_+$ except for a pole at $t = 0$. It follows that

$$F'(t)\begin{pmatrix} 1 & -cst^{-1} \\ -c^{-1} & 1 \end{pmatrix} F(t)^{-1} = cst^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 + icsH'_{11} & c^{-1} \\ ics(H'_{21} + H_{12}) & 1 - icsH_{11} \end{pmatrix},$$

where Eqs. (2.9a) and (2.9b) have been used. Hence

$$F'(t) = \frac{t}{t-s}\begin{pmatrix} 1 + icsH'_{11} & c^{-1} \\ cst^{-1} + ics(H'_{21} + H_{12}) & 1 - icsH_{11} \end{pmatrix} \times F(t)\begin{pmatrix} 1 & cst^{-1} \\ c^{-1} & 1 \end{pmatrix}.$$

The pole at $t = s$ is absent if

$$1 + icsH'_{11} = -c^{-1}T, \tag{3.19a}$$

$$ics(H'_{21} + H_{12}) = -c - (1 - icsH_{11})T, \tag{3.19b}$$

where

$$T = \frac{F_{22}(s) + cF_{21}(s)}{F_{12}(s) + cF_{11}(s)}. \tag{3.20}$$

Hence the final transformation formula is

$$F'(t) = \frac{t}{t-s}\begin{pmatrix} -c^{-1}T & c^{-1} \\ c(s-t)t^{-1} - (1 - ics\mathscr{E})T & 1 - ics\mathscr{E} \end{pmatrix} \times F(t)\begin{pmatrix} 1 & cst^{-1} \\ c^{-1} & 1 \end{pmatrix}. \tag{3.21}$$

When F_{21} and F_{22} are eliminated using Eq. (2.11), we get Eqs. (4.44a) and (4.44b) of Ref. 12. [T is a pseudopotential for the Bäcklund transformation and is a fractional linear function of the pseudopotential q used by Harrison,[7] the q used by Cosgrove,[6,12] and the two α's used by Neugebauer.[11] From Eq. (2.7), a total Riccati equation can be written for T.]

The decomposition of the Harrison transformation into factors $\mathbf{PQ}$ and $\mathbf{L}\widetilde{\mathbf{Q}}$ (respectively, I_1 and I_2 of Neugebauer[11]) is not unique and Eq. (3.17b) presents only one possible parametrization. It has the disadvantage that it breaks down for the important cases $c = \infty$ and $c = 0$, which map Weyl solutions to Weyl solutions. These may be accommodated by trivial rescalings, $f \to (\text{const}) \times f$, and translations, either $\omega \to \omega + \text{const}$ or $\psi \to \psi + \text{const}$. In terms of representing matrices the products are, suppressing the factor $(1 - s/t)^{-1/2}$,

$$v(t) = \lim_{c \to \infty}\begin{pmatrix} (cs)^{-1} & 0 \\ t & cs \end{pmatrix}\begin{pmatrix} 1 & -cst^{-1} \\ -c^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & -t^{-1} \\ t-s & 0 \end{pmatrix}; \tag{3.22}$$

$$v(t) = \lim_{c \to 0}\begin{pmatrix} c^{-1} & 1 \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & -cst^{-1} \\ -c^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 - st^{-1} \\ -1 & 0 \end{pmatrix}. \tag{3.23}$$

The solutions of the corresponding HHP's are, respectively,

$$F'(t) = \begin{pmatrix} -sT & s \\ \dfrac{s-t}{st} + i\mathscr{E}T & -i\mathscr{E} \end{pmatrix} F(t)\begin{pmatrix} 0 & \dfrac{1}{t-s} \\ -t & 0 \end{pmatrix}, \quad T = \frac{F_{21}(s)}{F_{11}(s)}; \tag{3.24}$$

$$F'(t) = \begin{pmatrix} -T & 1 \\ \dfrac{s-t}{t} + is\mathscr{E}T & -is\mathscr{E} \end{pmatrix} F(t)\begin{pmatrix} 0 & -1 \\ \dfrac{t}{t-s} & 0 \end{pmatrix}, \quad T = \frac{F_{22}(s)}{F_{12}(s)}. \tag{3.25}$$

The former preserves special HE gauge, as does (3.21). The latter preserves modified special HE gauge. The corresponding decompositions into **PQ** and **L$\tilde{\mathbf{Q}}$** transformations are, respectively,

$$H = (\tilde{Q})_{-4s}(Q)_{4s}(P)_{\alpha_1} = (P)_{\alpha_1}(Q)_{-4s}(\tilde{Q})_{4s}; \qquad (3.26)$$

$$H = (L)_{\alpha_1}(\tilde{Q})_{-4s}(Q)_{4s} = (Q)_{-4s}(\tilde{Q})_{4s}(L)_{\alpha_1}; \qquad (3.27)$$

where, in both cases, $\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The factorization of the null HKX transformation (3.10) into two H transformations with same s parameters is expressed by the matrix product

$$\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 - \dfrac{\alpha t q^{12}}{t-s} & \dfrac{\alpha t q^{11}}{t-s} \\ -\dfrac{\alpha t q^{22}}{t-s} & 1 + \dfrac{\alpha t q^{12}}{t-s} \end{pmatrix} = \frac{t}{t-s} \begin{pmatrix} 1 & -c_2 s t^{-1} \\ -c_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -c_1 s t^{-1} \\ -c_1^{-1} & 1 \end{pmatrix}, \qquad (3.28)$$

where

$$c_1 = q^{11}/q^{12}, \quad c_2 = -(q^{11}/q^{12})(1 + \alpha q^{12}), \qquad (3.29a)$$

$$\lambda = 1 + \alpha q^{12}, \quad \mu = -\alpha q^{11}. \qquad (3.29b)$$

the relations between the parameters here is in agreement with those given in Eqs. (6.56)–(6.58) of Ref. 12.

The $u(t)$ matrix (3.16) for the H transformation has quadratic branch points at $t = 0$ and $t = s$, joined by a cut in L_+. The product of two such matrices with $s = s_1$ and $s = s_2$ has branch points at $t = s_1$ and $t = s_2$, but $t = 0$ is an ordinary point [cf. right-hand side of Eq. (3.28)]. The branch cut can be taken from $t = s_1$ to $t = s_2$ without passing through $t = 0$. This shows incidentally that the double Harrison transformation preserves asymptotic flatness.

4. NONNULL KC TRANSFORMATIONS AND THE B GROUP

In Refs. 3, 12, and 15, methods are given which solve the problem of exponentiating particular infinitesimal *null* KC transformations.[14] None of these methods have been found to work in the nonnull case (except for the $s = \infty$ limit of the nonnull HKX[12]). We wish to study the **B** group,[16] whose infinitesimal generators are[25]

$$\beta^{(k-1)} = \gamma_{11}^{(k+1)} + \gamma_{22}^{(k-1)}, \quad k = 0, 1, 2, ..., \qquad (4.1)$$

and the nonnull KC group, whose generators are

$$q^{XY}\gamma_{XY}^{(k)}, \quad k = 0, 1, 2, ..., \qquad (4.2)$$

$q^{XY}q_{XY} = 2 \det q \neq 0$. We shall not attempt to exponentiate any combinations of the infinitesimal generators (the nonnull HKX transformation[12] being an example), but, nevertheless, we shall show how *finite* transformations can be calculated with Harrison and/or null HKX transformations. There are enough of these finite transformations that iteration of them until closure occurs will yield the full infinite-dimensional groups.

First, the $u(t)$ matrix representing an infinitesimal nonnull HKX transformation (α small) is identical to the null case. From Eq. (3.2), we have

$$u^A{}_B(t) = -\epsilon^A{}_B + \frac{\alpha t}{t-s} q^A{}_B + O(\alpha^2). \qquad (4.3)$$

[When q_{AB} is null, the $O(\alpha^2)$ term is identically zero and the remainder of the equation is exact for finite α.] The $u(t)$ matrix representing the individual infinitesimal $\alpha q^{XY}\gamma_{XY}^{(k)}$ transformation is obtainable from the coefficient of s^k in the Taylor series expansion of the right-hand side of Eq. (4.3). An arbitrary infinitesimal generator of the nonnull KC group is expressed by the sum

$$\sum_{k=0}^{\infty} \alpha^{(k)} q^{XY}\gamma_{XY}^{(k)}, \qquad (4.4)$$

$\alpha^{(k)}$ real constants. Define the generating function

$$\alpha(t) = \sum_{k=0}^{\infty} \alpha^{(k)} t^{-k}, \qquad (4.5)$$

analytic at and near $t = \infty$. We may require that $\alpha(t)$ be analytic in $L + L_-$. Then the representing $u(t)$ matrix is

$$u^A{}_B(t) = -\epsilon^A{}_B + \alpha(t) q^A{}_B + O(\alpha^2). \qquad (4.6)$$

Finally, a straightforward exponentiation of this matrix yields

$$u^A{}_B(t) = -[\cosh(q\alpha(t))]\epsilon^A{}_B + q^{-1}[\sinh(q\alpha(t))]q^A{}_B, \qquad (4.7)$$

where the real or pure-imaginary constant q is defined by

$$q_{XA}q^X{}_B = -q^2\epsilon_{AB} \quad \text{or} \quad q^2 = (q^{12})^2 - q^{11}q^{22}. \qquad (4.8)$$

When $\alpha(t)$ is a rational function, as in the case of the nonnull HKX transformation for which $\alpha(t) = \alpha t/(t-s)$, the $u(t)$ matrix has essential singularities in L_+ at the poles of $\alpha(t)$. The methods of this paper are not strong enough to handle essential singularities in $u(t)$. However, finite transformations can be written down when $\tanh(q\alpha(t))$ is a rational function of t, say

$$\tanh(q\alpha(t)) = R(t) \qquad (4.9)$$

(note that $qR(t)$ is real for real t). Then

$$u^A{}_B(t) = [-\epsilon^A{}_B + q^{-1}R(t)q^A{}_B][1 - R^2(t)]^{-1/2}. \qquad (4.10)$$

This is of the form (3.12) and so represents a product of a finite number of Harrison and $(L)_\beta$ transformations. When q is pure-imaginary, the s parameters of the H transformations form complex conjugate pairs such that the product transformation obeys condition (2.18b) and maps real solutions (as determined by f_{AB}) to real solutions.

Consider the simplest case, where

$$R(t) = \frac{at+b}{ct+d}, \quad c > a > 0, \qquad (4.11)$$

a, b, c, d real constants. A short calculation reveals the factorization,

$$u(t) = \begin{pmatrix} \beta_4 & -\beta_3 \\ -\beta_2 & \beta_1 \end{pmatrix} \left(1 - \frac{s_2}{t}\right)^{-1/2} \left(1 - \frac{s_1}{t}\right)^{-1/2} \times \begin{pmatrix} 1 & -c_2 s_2 t^{-1} \\ -c_2^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -c_1 s_1 t^{-1} \\ -c_1^{-1} & 1 \end{pmatrix} \qquad (4.12)$$

[cf. Eqs. (2.29) and (3.16) for the $(L)_\beta$ and H transformations, respectively], where

$$\beta_4 = \Delta^{-1}(c^2 - a^2)^{1/2}(d - bq^{12}/q), \tag{4.13a}$$

$$\beta_3 = -(c^2 - a^2)^{-1/2} a q^{11}/q, \tag{4.13b}$$

$$\beta_2 = \Delta^{-1}(c^2 - a^2)^{1/2} b q^{22}/q, \tag{4.13c}$$

$$\beta_1 = (c^2 - a^2)^{-1/2}(c + aq^{12}/q); \tag{4.13d}$$

$$s_1 = -\frac{d+b}{c+a}, \quad s_2 = -\frac{d-b}{c-a}; \tag{4.14}$$

$$c_1 = \frac{q^{11}}{q^{12}+q}, \quad c_2 = -\frac{\Delta}{(c+a)(d-b)}\frac{q^{11}}{q^{12}+q}; \tag{4.15}$$

$$\Delta = cd - ab + (ad - bc)q^{12}/q. \tag{4.16}$$

There is a second factorization with same $\boldsymbol{\beta}$ matrix and

$$s_1 = -\frac{d-b}{c-a}, \quad s_2 = -\frac{d+b}{c+a}; \tag{4.14$'$}$$

$$c_1 = \frac{q^{11}}{q^{12}-q}, \quad c_2 = -\frac{\Delta}{(c-a)(d+b)}\frac{q^{11}}{q^{12}-q}. \tag{4.15$'$}$$

A matrix of the form (4.10) will represent a product of null HKX transformations alone when $1 - R^2(t)$ is a perfect square. This is the case whenever

$$R(t) = \frac{m^2(t) - n^2(t)}{m^2(t) + n^2(t)} \tag{4.17a}$$

or

$$R(t) = \frac{2m(t)n(t)}{m^2(t) + n^2(t)}, \tag{4.17b}$$

where $m(t)$ and $n(t)$ are polynomials in t.

The same considerations apply to the **B** group,[16] for which

$$u(t) = \begin{pmatrix} \cos\alpha(t) & t^{-1}\sin\alpha(t) \\ -t\sin\alpha(t) & \cos\alpha(t) \end{pmatrix}, \tag{4.18}$$

$\alpha(t)$ analytic in $L_+ + L_-$ and at $t = \infty$. This is the group which maps flat space to itself in special HE gauge, as can be seen by putting $\mathscr{E} = 1 = \mathscr{E}'$ in Eq. (2.25). It is closely related to the nonnull KC group [see Eqs. (7.30)–(7.33b) of Ref. 12[31]]. Finite transformations in the **B** group can be constructed out of Harrison and $(P)_\alpha$ transformations by putting

$$\tan\alpha(t) = R(t), \tag{4.19}$$

a rational function of t. Then

$$u(t) = [1 + R^2(t)]^{-1/2}\begin{pmatrix} 1 & t^{-1}R(t) \\ -tR(t) & 1 \end{pmatrix}. \tag{4.20}$$

In this case, the s parameters form complex conjugate pairs. Also, with the choice,

$$R(t) = \frac{2m(t)n(t)}{m^2(t) - n^2(t)}, \tag{4.21}$$

where $m(t)$ and $n(t)$ are polynomials in t, the $u(t)$ matrix is itself rational and represents a product of null HKX transformations.

5. BÄCKLUND TRANSFORMATIONS FOR ELECTROVAC FIELDS

Hauser and Ernst have extended their complex-variable techniques to the enlarged Geroch group $\mathbf{K}'$ (due to Kinnersley[13]) for electrovac fields.[18–20] Their representation employs 3×3 matrices H_{ab}, $F_{ab}(t)$, and $u_{ab}(t)$ which correspond to the 2×2 matrices H_{AB}, $F_{AB}(t)$, and $u^A{}_B(t)$, respectively, of vacuum. The SL(2) tensor formalism for the potentials has been developed by Kinnersley and Chitre[13–16] and Jones.[32] However, since each 3×3 matrix equation conveys the information of four tensor equations, we shall use the matrix notation in this section.

With the hindsight gained from the vacuum case, it is natural to study electrovac transformations represented by $u(t)$ matrices which have only poles in the complex t plane and those which have algebraic singularities (cubic branch points in this case) for which the HHP can be solved by the methods of Sec. 3. Hauser and Ernst[18] have already treated the case where $u(t)$ has only poles using their integral equation and have obtained the electrovac version of the null HKX transformation and products thereof. The same results could equally well be obtained from the HHP by following the steps which led to Eq. (3.10) here.

In this section we study $u(t)$ matrices of the form

$$u(t) = [\det v(t)]^{-1/3} v(t) \tag{5.1}$$

[cf. Eq. (3.12)], in an attempt to generalize the Harrison transformation to electrovac spacetimes. We succeed in obtaining an electrovac counterpart to the *double* Harrison transformation which maps flat space to Kerr–Newman–NUT space and presumably an n-fold iteration of which maps flat space to the nonlinear superposition of n Kerr–Newman–NUT particles on the z axis. Nevertheless, in Sec. 6, we present an argument which suggests that this transformation may not be as large as it could be and attempt to explain why we were not able to derive the *single* electrovac Harrison transformation (presumed to exist) from the HE formalism.

The electrovac H potential has components

$$H_{ab} = \begin{pmatrix} H_{AB} & \varphi_A \\ 2iL^{(1,1)}_B & 2iK^{(1,1)} \end{pmatrix}, \quad a, b = 1, 2, 3, \quad A, B = 1, 2. \tag{5.2}$$

The definitions of the potentials on the right-hand side are to be found in Refs. 13 and 14. The metric is recoverable from the relation

$$f_{AB} = \tfrac{1}{2}(H_{AB} + H^*_{BA}) + \varphi_A\varphi^*_B - iz\epsilon_{AB}, \tag{5.3}$$

and we again use the parametrization (2.2). To specify an electrovac solution it is sufficient to know only the components (Ernst potentials)

$$\mathscr{E} = H_{11} = f_{11} - \varphi_1\varphi^*_1 + i\Omega_{11}, \quad \Phi = \varphi_1 = H_{13}, \tag{5.4}$$

which satisfy simple field equations given by Ernst[33] (see also Ref. 13[34]).

The $F(t)$ potential has components

$$F_{ab}(t) = \begin{pmatrix} F_{AB}(t) & D_A(t) \\ 2iS_B(t) & 2iQ(t) \end{pmatrix}, \tag{5.5}$$

where the entries on the right-hand side are generating functions whose definitions and direct methods of calculation are found in Ref. 32. (When comparing with the 3×3 matrix equations of Hauser and Ernst,[18–20] one should interchange the first two rows and first two columns as in Ref. 26.) At

$t=0$,

$$F(0)=i\mathfrak{E}(2), \tag{5.6a}$$
$$\dot F(0)=H, \tag{5.6b}$$

where $\mathfrak{E}(2)$ is the value at $t=2$ of

$$\mathfrak{E}(t)=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\tfrac{1}{2}it \end{pmatrix}. \tag{5.7}$$

The $F(t)$ potential satisfies a differential equation and algebraic relations corresponding to Eqs. (2.7), (2.10), and (2.11) which are given in Refs. 18, 19, and 32. We need write down only the determinant condition

$$\det F(t) = -S^{-1}(t). \tag{5.8}$$

The gauge [see Eq. (5.22) below] chosen by Hauser and Ernst ("HE gauge") requires that $F(t)$ be analytic near $t=\infty$ and also that F_{a1}, tF_{a2}, and F_{a3}, $a=1,2,3$, be analytic at $t=\infty$. As in vacuum, there are special gauges such that the only singularities of $F(t)$, in a suitable (ρ,z) domain containing an open interval of the z axis, are quadratic branch points at the zeros t_+ and t_- of $S(t)$. If, in addition, the additive constants in $\omega=-f_{12}/f_{11}$ and φ_2 are chosen so that these latter potentials vanish on the z axis, then the special gauge is unique ("special HE gauge"). The explicit forms of H and $F(t)$ on the z axis in special HE gauge are given by[20]

$$H=\begin{pmatrix} \mathscr{E} & 2iz & \Phi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.9a}$$

$$F(t)=\begin{pmatrix} \dfrac{t\mathscr{E}}{1-2tz} & \dfrac{i}{1-2tz} & \dfrac{t\Phi}{1-2tz} \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.9b}$$

The homogeneous Hilbert problem is to find $X_+(t)$ analytic in $L+L_+$ (drawn as in Sec. 2) and $X_-(t)$ analytic in $L+L_-$ and at $t=\infty$ such that, on L,

$$X_-(t)=X_+(t)G(t), \tag{5.10}$$

where

$$G(t)=F(t)u(t)F(t)^{-1}, \tag{5.11a}$$
$$X_+(0)=I. \tag{5.11b}$$

The matrix $u(t)$, depending only on t, is chosen subject to the constraints

$$\det u(t)=1, \tag{5.12a}$$
$$u^\dagger(t)\mathfrak{E}(t)u(t)=\mathfrak{E}(t), \tag{5.12b}$$
$$u(t) \text{ analytic in } L+L_-, \tag{5.12c}$$
$$\begin{pmatrix} u_{11} & tu_{12} & u_{13} \\ t^{-1}u_{21} & u_{22} & t^{-1}u_{23} \\ u_{31} & tu_{32} & u_{33} \end{pmatrix} \text{ analytic at } t=\infty. \tag{5.12d}$$

[$u^\dagger(t)$ is the Hermitian conjugate of $u(t^*)$, i.e., $u^\dagger(t)=u^*(t)^T$.] When $X_+(t)$ is known, a new electrovac solution can be constructed from

$$F'(t)=X_+(t)F(t), \tag{5.13a}$$
$$H'=\dot F'(0). \tag{5.13b}$$

Hence an alternative form for the HHP is

$$X_-(t)=F'(t)u(t)F(t)^{-1}, \tag{5.14}$$

where $X_-(t)$ and $u(t)$ are analytic in $L+L_-$, $F'(t)$ and $F(t)$ are analytic in $L+L_+$. As before, a composition of several transformations is represented by the product of corresponding $u(t)$ matrices. Notice that $X_+(t)$ and hence $F'(t)$ is not sensitive to the replacement $u(t)\rightarrow\zeta u(t)$, where ζ is a complex cube root of unity.

A simple application of the HHP is to derive an equation for all $u(t)$ which transform a given initial solution $(\mathscr{E},\Phi)$ into a given final solution $(\mathscr{E}',\Phi')$ in special HE gauge.[20] Substitute Eq. (5.9b) for $F(t)$ and $F'(t)$ into the right-hand side of Eq. (5.14). The condition that the pole at $t=(2z)^{-1}$ be absent is

$$(t\mathscr{E}', i, t\Phi')u(t)\begin{pmatrix} i & 0 \\ -t\mathscr{E} & it\Phi \\ 0 & 1 \end{pmatrix}=0, \tag{5.15}$$

where $\mathscr{E}$, Φ, $\mathscr{E}'$, and Φ' are to be evaluated at $\rho=0$, $z=(2t)^{-1}$.

If $u(t)$ is a rational function of t, then the method of solution of the HHP is exactly the same here as in the vacuum case. Similarly, if $v(t)$ is a rational matrix function of t and if the unimodular matrix

$$u(t)=[\det v(t)]^{-1/3}v(t) \tag{5.16}$$

also satisfies conditions (5.12b)–(5.12d), then the solution of the HHP proceeds along exactly the same lines as for a finite product of Harrison transformations in vacuum. Such transformations with $\det v(t)$ not a perfect cube will be new as none of the finite electrovac transformations discussed previously take the form (5.16).[35]

Despite these obvious similarities with vacuum, the single Harrison transformation with $\det v(t)=1-st^{-1}$ does not automatically generalize to electrovac. The simplest transformation of the form (5.16) that we find has two s parameters, complex conjugates of each other, and so more closely resembles the *double* Harrison transformation. This unexpected complication can be traced to condition (5.12b) being quadratic in $u(t)$. [This condition is needed to guarantee the algebraic relation between $F(t)$ and $F^*(t)$ given in Refs. 18, 19, and 32.] Thus, while we want $\det v(t)$ to be *not* a perfect cube, the product $[\det v(t)]\,[\det v^*(t)]$ must be a perfect cube. The simplest choices subject to these constraints take such (essentially equivalent) forms as

$$\det v(t)=\frac{t-s^*}{t-s} \tag{5.17a}$$

or

$$\det v(t)=\frac{s(t-s^*)}{s^*(t-s)}, \tag{5.17b}$$

or

$$\det v(t)=(1-s/t)^2(1-s^*/t), \tag{5.17c}$$

s being a complex constant. With $\det v(t)$ given by Eq.

(5.17b), the simplest possible choice for $u(t)$ is

$$u(t)=\left(\frac{s^*(t-s)}{s(t-s^*)}\right)^{1/3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \dfrac{s(t-s^*)}{s^*(t-s)}\end{pmatrix}. \tag{5.18}$$

This matrix function of t has a branch cut from $t=s$ to $t=s^*$ inside L_+ and appears to represent a very special case of the electrovac version of the double Harrison transformation with no free c parameters. An instructive way to proceed to a more general transformation with four c parameters is to solve the HHP for this special $u(t)$ first, and then recognize that the solution can be generalized in a nontrivial way by exploiting the gauge freedom in $F(t)$.

The solution of the HHP for the $u(t)$ of Eq. (5.18) is

$$F'(t)=\left(F(t)+\frac{t(s-s^*)}{s^*(t-s)}\frac{1}{\Delta}\begin{pmatrix}F_{13}(s^*)\\ F_{23}(s^*)\\ F_{33}(s^*)\end{pmatrix}(m_1(t),\,m_2(t),\,m_3(t))\right)\times\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \dfrac{s^*(t-s)}{s(t-s^*)}\end{pmatrix}, \tag{5.19}$$

where

$$m_b(t)=\begin{vmatrix}F_{11}(s) & F_{12}(s) & F_{1b}(t)\\ F_{21}(s) & F_{22}(s) & F_{2b}(t)\\ F_{31}(s) & F_{32}(s) & F_{3b}(t)\end{vmatrix}, \tag{5.20a}$$

$$\Delta=m_3(s^*). \tag{5.20b}$$

Details of the method of solution of this HHP will be supplied later for a more general $u(t)$. [*Important note*: To calculate $F^{*\prime}(t)$, replace s, s^*, $F(t)$, $F(s)$, and $F(s^*)$ by s^*, s, $F^*(t)$, $F^*(s^*)$, and $F^*(s)$, respectively, in the right-hand sides of Eqs. (5.19), (5.20a), and (5.20b). In particular, $m^*_b(t)$ and Δ^* are the complex conjugates of $m_b(t^*)$ and Δ, respectively.] The components of the first row of $H'=\dot F'(0)$ are

$$\mathscr{E}'=H'_{11}=H_{11}-\frac{i(s-s^*)}{ss^*\Delta}F_{13}(s^*)\begin{vmatrix}F_{11}(s) & F_{12}(s)\\ F_{31}(s) & F_{32}(s)\end{vmatrix}, \tag{5.21a}$$

$$H'_{12}=H_{12}-\frac{i(s-s^*)}{ss^*\Delta}F_{13}(s^*)\begin{vmatrix}F_{21}(s) & F_{22}(s)\\ F_{31}(s) & F_{32}(s)\end{vmatrix}, \tag{5.21b}$$

$$\Phi'=H'_{13}=H_{13}-\frac{s-s^*}{ss^*\Delta}F_{13}(s^*)\begin{vmatrix}F_{11}(s) & F_{12}(s)\\ F_{21}(s) & F_{22}(s)\end{vmatrix}. \tag{5.21c}$$

If the initial solution is a vacuum solution, for which $F_{a3}(t)=0=F_{3b}(t)$, $a, b=1, 2$, $F_{33}(t)=1$, then the above transformation reduces to the identity. On the other hand, if the solution has a nonvanishing electromagnetic field, then Eqs. (5.21a)–(5.21c) define a nontrivial new transformation.

Four additional parameters can be incorporated into the above transformation by simply changing the gauge of the $F(t)$ potential in the right-hand sides of Eqs. (5.21a)–(5.21c) or by making a similar substitution for $F(s)$ and $F(s^*)$, but not for $F(t)$, in the right-hand side of Eq. (5.19). The differential equations and algebraic relations[18,19,32] which govern $F(t)$ are also satisfied after a substitution

$$F(t)\to F(t)g(t), \tag{5.22}$$

where $g(t)$ is a complex-valued function of t only subject to

$$g(0)=I,\quad \det g(t)=1,\quad g^\dagger(t)\mathfrak{S}(t)g(t)=\mathfrak{S}(t), \tag{5.23}$$

I denoting the 3×3 unit matrix. Since we wish to obtain formulas preserving special HE gauge, we shall make this substitution for $F(s)$ and $F(s^*)$ in Eqs. (5.21a) and (5.21c) and then deduce the new $F'(t)$ from the HHP, rather than Eq. (5.19).

Not all eighteen components of $g(s)$ and $g(s^*)$ will appear in the right-hand sides of Eqs. (5.21a) and (5.21c). The latter depend on only the four independent ratios h_a/h_b and g_a/g_b, $a, b=1, 2, 3$, of the components of the following vectors:

$$\mathbf{h}=\begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix}=g(s^*)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}, \tag{5.24a}$$

$$\mathbf{g}=(g_1, g_2, g_3)=(0, 0, 1)g(s)^{-1}. \tag{5.24b}$$

The algebraic conditions (5.23) imply

$$h^*_1=-\tfrac{1}{2}isg_2,\quad h^*_2=\tfrac{1}{2}isg_1,\quad h^*_3=g_3. \tag{5.25}$$

Since the vectors h_a and g_b may be arbitrarily rescaled, there is no loss of generality in putting, for example, $g_3=1=h_3$. Inspection of later formulas suggests, however, that a more convenient normalization would be to take the quantity

$$\begin{aligned}E&\equiv g_1h_1+g_3h_3+(s/s^*)g_2h_2\\ &=g_3g^*_3+\tfrac{1}{2}is^*g_1g^*_2-\tfrac{1}{2}isg_2g^*_1=E^*\end{aligned} \tag{5.26}$$

to be unity.

When the substitution (5.22) is made for $F(s^*)$ and $F(s)$ in the right-hand sides of Eqs. (5.21a) and (5.21c), the transformed Ernst potentials become

$$\mathscr{E}'=H'_{11}=H_{11}+[i(s-s^*)/ss^*\Delta]h_bF_{1b}(s^*)k_2, \tag{5.27a}$$

$$\Phi'=H'_{13}=H_{13}-[(s-s^*)/ss^*\Delta]h_bF_{1b}(s^*)k_3, \tag{5.27b}$$

where $\mathbf{k}=(k_1, k_2, k_3)$ is the row vector

$$\mathbf{k}=-S(s)\mathbf{g}F(s)^{-1}, \tag{5.28}$$

whose components are

$$k_1=\begin{vmatrix}g_1 & g_2 & g_3\\ F_{21}(s) & F_{22}(s) & F_{23}(s)\\ F_{31}(s) & F_{32}(s) & F_{33}(s)\end{vmatrix}, \tag{5.29a}$$

$$k_2=-\begin{vmatrix}g_1 & g_2 & g_3\\ F_{11}(s) & F_{12}(s) & F_{13}(s)\\ F_{31}(s) & F_{32}(s) & F_{33}(s)\end{vmatrix}, \tag{5.29b}$$

$$k_3=\begin{vmatrix}g_1 & g_2 & g_3\\ F_{11}(s) & F_{12}(s) & F_{13}(s)\\ F_{21}(s) & F_{22}(s) & F_{23}(s)\end{vmatrix}, \tag{5.29c}$$

and

$$\Delta=\mathbf{k}F(s^*)\mathbf{h}=\begin{vmatrix}g_1 & g_2 & g_3 & 0\\ F_{11}(s) & F_{12}(s) & F_{13}(s) & h_bF_{1b}(s^*)\\ F_{21}(s) & F_{22}(s) & F_{23}(s) & h_bF_{2b}(s^*)\\ F_{31}(s) & F_{32}(s) & F_{33}(s) & h_bF_{3b}(s^*)\end{vmatrix}, \tag{5.30}$$

and the summation convention applies to the repeated index b.

Equations (5.27a) and (5.27b) as they stand are sufficient to determine the metric and electromagnetic potential of the transformed solution, but the transformed generating function $F'(t)$ and/or representing matrix $u(t)$ is also needed if we wish to iterate these transformations. If we restrict Eqs. (5.27a) and (5.27b) to the z axis, using Eqs. (5.9a) and (5.9b), and then substitute into Eq. (5.15), we find that $u(t)$ is uniquely determined up to a scalar multiplicative factor. The latter is then uniquely determined (up to a multiplicative cube root of unity) by the unit determinant condition (5.12a). The result of this calculation is

$$u(t) = (s^*(t-s)/s(t-s^*))^{1/3} w(t), \tag{5.31}$$

where

$$w(t) = I + \frac{s(s-s^*)}{s^*E(t-s)}\begin{pmatrix} h_1 \\ h_2 t/s^* \\ h_3 \end{pmatrix}(g_1 t/s, g_2, g_3 t/s), \tag{5.32}$$

E given by Eq. (5.26). The earlier $u(t)$ given by Eq. (5.18) can be recovered by putting $g_1 = g_2 = 0 = h_1 = h_2$. When the parameters satisfy Eq. (5.25), the second condition (5.12b) is satisfied automatically and provides a quick method of calculation of the inverse:

$$w(t)^{-1} = I - \frac{s-s^*}{E(t-s^*)}\begin{pmatrix} h_1 \\ h_2 t/s^* \\ h_3 \end{pmatrix}(g_1 t/s, g_2, g_3 t/s). \tag{5.33}$$

By absorbing the cubic surd in Eq. (5.31) into $X_-(t)$, the HHP implies that $F'(t)w(t)F(t)^{-1}$ is analytic everywhere (including at $t=\infty$) except for a simple pole at $t = s$. Hence, we can write

$$F'(t)w(t)F(t)^{-1} = A(I + [t/(t-s)]B), \tag{5.34}$$

where A and B are constant matrices to be determined. Condition (5.6a) at $t = 0$ gives immediately

$$A = I + \frac{s-s^*}{s^*E}\begin{pmatrix} 0 \\ h_1 \\ ih_3 \end{pmatrix}(g_2, 0, 0). \tag{5.35}$$

Solving Eq. (5.34) for $F'(t)$, we have

$$F'(t) = A\left(I + \frac{t}{t-s}B\right)F(t)w(t)^{-1}. \tag{5.36}$$

The conditions that the poles at $t = s$ and $t = s^*$ on the right-hand side of Eq. (5.36) be absent are, respectively,

$$BF(s)\left\{I - \frac{1}{E}\begin{pmatrix} h_1 \\ h_2 s/s^* \\ h_3 \end{pmatrix}\mathbf{g}\right\} = 0, \tag{5.37a}$$

$$\left(I - \frac{s^*}{s-s^*}B\right)F(s^*)\mathbf{h} = 0. \tag{5.37b}$$

Equations (5.37a) and (5.29) show that B is expressible as the outer product of a column vector and a row vector, the latter being proportional to $\mathbf{k}$. Then Eq. (5.37b) shows that the column vector in question is proportional to $F(s^*)\mathbf{h}$. Hence we find

$$B = [(s-s^*)/s^*\Delta]F(s^*)\mathbf{h}\mathbf{k}, \tag{5.38}$$

which completes the solution of the HHP. An alternative form of the solution (5.36) is

$$F'(t) = A\left[F(t) + \frac{t(s-s^*)}{s^*(t-s)}\frac{1}{\Delta}F(s^*)\mathbf{h}\mathbf{m}(t)\right]w(t)^{-1}, \tag{5.39}$$

with A, Δ, and $w(t)^{-1}$ given by Eqs. (5.35), (5.30), and (5.33), respectively, and the components of $\mathbf{m}(t) = (m_1(t), m_2(t), m_3(t))$ given by

$$m_b(t) = \begin{vmatrix} g_1 & g_2 & g_3 & 0 \\ F_{11}(s) & F_{12}(s) & F_{13}(s) & F_{1b}(t) \\ F_{21}(s) & F_{22}(s) & F_{23}(s) & F_{2b}(t) \\ F_{31}(s) & F_{32}(s) & F_{33}(s) & F_{3b}(t) \end{vmatrix}. \tag{5.40}$$

Observe that

$$\mathbf{m}(0) = i\mathbf{k}\mathscr{E}(2), \tag{5.41a}$$

$$\mathbf{m}(s) = -\mathbf{g}/S(s), \tag{5.41b}$$

$$\mathbf{m}(s^*)\cdot\mathbf{h} = \Delta. \tag{5.41c}$$

From Eqs. (5.6a) and (5.6b), the transformed H potential is found to be

$$H' = A\left[iH\mathscr{E}(2) - \frac{s-s^*}{ss^*\Delta}F(s^*)\mathbf{h}\mathbf{k}\right]A^{-1}i\mathscr{E}(2) + \frac{s-s^*}{ss^*E}\begin{pmatrix} 0 & ig_2h_2 s/s^* & 0 \\ -ig_1h_1 & -ig_2h_1\frac{s}{s^*E}\mathbf{g}\cdot\mathbf{h} & -ig_3h_1 \\ g_1h_3 & g_2h_3\frac{s}{s^*E}\mathbf{g}\cdot\mathbf{h} & g_3h_3 \end{pmatrix}. \tag{5.42}$$

The transformation equations (5.39) and (5.42) preserve special HE gauge and the 11 and 13 components of Eq. (5.42) reduce to Eqs. (5.27a) and (5.27b). The 12 component of the transformed H potential is

$$H'_{12} = H_{12} - i\frac{s-s^*}{ss^*\Delta}h_bF_{1b}(s^*)k_1 + \frac{s-s^*}{s^*E}g_2(h_1H'_{11} + h_3H'_{13}) + \frac{i(s-s^*)}{s^{*2}E}g_2h_2. \tag{5.43}$$

Now, the transformation formula,

$$F'(t) = \left[F(t) + \frac{t(s-s^*)}{s^*(t-s)}\frac{1}{\Delta}F(s^*)\mathbf{h}\mathbf{m}(t)\right] \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{s^*(t-s)}{s(t-s^*)} \end{pmatrix}, \tag{5.44}$$

Δ given by Eq. (5.30), $\mathbf{m}(t)$ by Eq. (5.40), is the result of making the substitution (5.22) for $F(s)$ and $F(s^*)$ but not for $F(t)$ in the formula (5.19). This $F'(t)$ is also a bona fide $F(t)$-potential in HE gauge satisfying all of the defining equations,[18,19,32] but is not in special HE gauge. It gives the same Ernst potentials as Eqs. (5.27a) and (5.27b), and H'_{12} is given by the first two terms on the right-hand side of Eq. (5.43). One should expect to find a simple relationship between the transformations (5.44) and (5.39). Besides the gauge transformations of the form (5.22), we also need the trivial translations, $\omega \to \omega$ + real constant, $\varphi_2 \to \varphi_2$ + complex constant, which are outside the gauge group. These are the $\gamma_{22}^{(0)}$ and $c_2^{(0)}$ transformations of Ref. 14 (or $\gamma_{11}^{(0)}$ and $c_1^{(0)}$ according to our own convention[25]) and are easily exponentiated to give

$$F'(t) = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 2c^* & 0 & 1 \end{pmatrix} F(t) \begin{pmatrix} 1 & -\gamma - icc^*t & ct \\ 0 & 1 & 0 \\ 0 & -2ic^* & 1 \end{pmatrix}, \tag{5.45}$$

γ real, c complex. The transformation law (5.39) preserving special HE gauge can be achieved by appplying first the transformation (5.44), then (5.45) with γ and c chosen so that H'_{2b} and H'_{3b} become constant on the z axis, and finally (5.22). A direct evaluation of γ, c, and $g(t)$ starting from Eq. (5.44) is rather tedious, but it is easy to calculate their values by working backwards from Eq. (5.39). The results are

$$\gamma = -\frac{s-s^*}{s^*E} g_2 h_1, \quad c = \frac{s-s^*}{ss^*E} g_3 h_1, \qquad c^* = \tfrac{1}{2} i \frac{s-s^*}{s^*E} g_2 h_3; \tag{5.46}$$

$g(t)$

$$= \begin{pmatrix} 1 & \gamma - icc^*t & -ct \\ 0 & 1 & 0 \\ 0 & 2ic^* & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{s(t-s^*)}{s^*(t-s)} \end{pmatrix} w(t)^{-1}. \tag{5.47}$$

Note that γ is real and $g(t)$ satisfies the conditions (5.23).

Generation of Kerr–Newman–NUT space

Let us calculate the effect of the transformation (5.27a) and (5.27b) on flat space, $\mathscr{E} = 1$, $\Phi = 0$. The $F(t)$ potential for flat space is[16]

$$F(t) = \begin{pmatrix} \dfrac{t}{S(t)} & \dfrac{i}{S(t)} & 0 \\ \dfrac{1-2tz+S(t)}{2iS(t)} & \dfrac{1-2tz-S(t)}{2tS(t)} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.48}$$

It is convenient to use oblate spheroidal coordinates $(\bar{x}, y)$ defined by

$$\rho = \bar{\kappa}(\bar{x}^2+1)^{1/2}(1-y^2)^{1/2}, \quad z = \bar{\kappa}\bar{x}y + z_0, \tag{5.49a}$$

$$z_0 - i\bar{\kappa} = (2s)^{-1}, \quad z_0 + i\bar{\kappa} = (2s^*)^{-1}, \quad \operatorname{Im} s > 0. \tag{5.49b}$$

Then

$$S(s) = -\frac{\bar{\kappa}(\bar{x}+iy)}{z_0 - i\bar{\kappa}}, \quad S(s^*) = -\frac{\bar{\kappa}(\bar{x}-iy)}{z_0+i\bar{\kappa}}, \tag{5.50}$$

where the signs have been chosen so that $F(s)$ and $F(s^*)$ reduce to the form (5.9b) on the branch $y = +1$ of the z axis. A straightforward substitution into Eqs. (5.27a) and (5.27b) gives

$$\mathscr{E}' = \frac{g_1h_1(\bar{x}+i) + g^*_1h^*_1(\bar{x}-i) + \frac{1}{2}ig_1g^*_1(1+y) - 2ih_1h^*_1(1-y) + g_3g^*_3(\bar{x}-iy)}{g_1h_1(\bar{x}-i) + g^*_1h^*_1(\bar{x}+i) - \frac{1}{2}ig_1g^*_1(1-y) + 2ih_1h^*_1(1+y) + g_3g^*_3(\bar{x}-iy)}, \tag{5.51a}$$

$$\Phi' = \frac{ig_3(g^*_1 + 2h_1)}{g_1h_1(\bar{x}-i) + g^*_1h^*_1(\bar{x}+i) - \frac{1}{2}ig_1g^*_1(1-y) + 2ih_1h^*_1(1+y) + g_3g^*_3(\bar{x}-iy)}, \tag{5.51b}$$

where we have used Eqs. (5.25) to eliminate g_2, h_2, and h_3.

The solution (5.51a) and (5.51b) is recognizable as the Kerr–Newman–NUT space with magnetic charge. It can be generated from the vacuum Kerr solution by means of the SU(2, 1) Kinnersley group.[36] The appearance of *oblate* spheroidal coordinates indicates that the Kerr solution in question is without horizons, i.e., beyond the extreme ($|a| > m$ in a familiar parametrization[37]). A standard form[38,39] for the vacuum Kerr solution[17] is

$$\xi = px + iqy, \quad \kappa = mp, \quad a = mq, \tag{5.52}$$

where $\xi = (1+\mathscr{E})/(1-\mathscr{E})$,[27] $p^2 + q^2 = 1$, and (x, y) are *prolate* spheroidal coordinates with foci at $z = z_0 \pm \kappa$. This solution can be obtained from flat space by applying two Harrison transformations with real s parameters.[6,40] The beyond the extreme case, $|q| > 1$, where the s parameters form a complex conjugate pair, can be expressed in *oblate* spheroidal coordinates $(\bar{x}, y)$ defined by Eq. (5.49a) and obtainable from the prolate case by the substitutions, $x = i\bar{x}, \kappa = -i\bar{\kappa}, p = -i\bar{p}, \bar{p}^2 = q^2 - 1$. Thus the Ernst potentials are

$$\mathscr{E} = \frac{\bar{p}\,\bar{x} + iqy - 1}{\bar{p}\,\bar{x} + iqy + 1}, \quad \Phi = 0. \tag{5.53}$$

Now apply the Ehlers transformation,[13,36]

$$\mathscr{E}' = \frac{\mathscr{E} + i\lambda}{1 + i\lambda\mathscr{E}}, \quad \Phi' = \frac{(1+i\lambda)\Phi}{1+i\lambda\mathscr{E}}, \tag{5.54}$$

λ real, which introduces a NUT parameter, and the charging transformation of Harrison[41] and Ernst,[33]

$$\begin{aligned} \mathscr{E}' &= \frac{\mathscr{E} + 2b^*\Phi - bb^*}{1 - 2b^*\Phi - bb^*\mathscr{E}}, \\ \Phi' &= \frac{b(\mathscr{E} - 1) + (1 + bb^*)\Phi}{1 - 2b^*\Phi - bb^*\mathscr{E}}, \end{aligned} \tag{5.55}$$

b complex, in either order to the Kerr solution (5.53). The result is[42]

$$\mathscr{E}' = \frac{(1 - i\lambda bb^*)(\bar{p}\,\bar{x} + iqy - 1) + (i\lambda - bb^*)(\bar{p}\,\bar{x} + iqy + 1)}{(1 - i\lambda bb^*)(\bar{p}\,\bar{x} + iqy + 1) + (i\lambda - bb^*)(\bar{p}\,\bar{x} + iqy - 1)}, \tag{5.56a}$$

$$\Phi' = -\frac{2b(1 + i\lambda)}{(1 - i\lambda bb^*)(\bar{p}\,\bar{x} + iqy + 1) + (i\lambda - bb^*)(\bar{p}\,\bar{x} + iqy - 1)}. \tag{5.56b}$$

It is easy now to identify the solution (5.51a) and (5.51b) with (5.56a) and (5.56b) although the relations connecting the parameters are rather messy. Also, having obtained the solution (5.51a) and (5.51b) in oblate spheroidal coordinates ($|q| > 1$), it is a trivial matter to analtyically continue to the prolate case ($|q| < 1$). The six parameters, z_0, κ (or $\bar{\kappa}$), q, λ, Re b, Im b, determine, respectively, the following physical characteristics: position on z axis, mass, angular momentum, NUT parameter, electric charge, and magnetic charge.

It is to be expected that n successive applications of the new transformation of this section to flat space would give the nonlinear superposition of n Kerr–Newman–NUT particles on the z axis. This would generalize the known results for vacuum Kerr–NUT particles (Kramer and Neugebauer[43]) and electrically poised Kerr–Newman–NUT particles (Kobiske and Parker[44]). From Kramer and Neugebauer's work, it is clear that it is not necessary that Coulomb repulsion balance gravitational attraction for the conical stresses to be absent between the particles (assuming NUT singularities have already been removed) as a spin-induced "magnetic-type" gravitational force is also present and is able to balance the more familiar "electric-type" gravity even in the case of uncharged Kerr particles.

At present, the only known examples of two or more *black holes*, for which all spacetime singularities are enclosed by nonsingular even horizons, in equilibrium under mutual gravitational and electromagnetic interactions is the superposition of n static *extreme* Reissner–Nordström holes.[45] The two cases mentioned in the preceding paragraph are beyond the extreme (no horizons) when axial stresses are absent (except for the static limit of the Kobiske–Parker solution which consists of Reissner–Nordstrom holes). Preliminary calculations, however, suggest that it may be possible to balance two nonextreme Kerr–Newman black holes for certain range of the charges, masses, angular momenta, and spatial separation. Further work need to be done to clarify this interesting problem.

6. CONCLUSION

In Secs. 3 and 4 of this paper, we have explored the infinite-dimensional Geroch group **K** of transformations for vacuum spacetimes in the representation of Hauser and Ernst.[18–20] The main result of those sections is that Harrison's Bäcklund transformation[8] is in **K** (though not a subgroup) and that elements of **K** represented by $u(t)$ matrices of the form

$$u(t) = [\det v(t)]^{-1/2} v(t), \tag{6.1}$$

$v(t)$ being a rational matrix function of t, are in fact products of a finite number of Harrison transformations. Each quadratic branch point of $u(t)$ of index $\pm\frac{1}{2}$ in L_+ not at the origin corresponds to an individual Harrison transformation ($t = 0$ is also a branch point when there are an odd number of the latter). This result in a sense generalizes the earlier result of Hauser and Ernst[18] that simple poles of $u(t)$ correspond to null HKX transformations.[3,4] The coalescence of two quadratic branch points to form a simple pole at $t = s$ provides an easy proof of the theorem[12] that the null HKX transformation factorizes into two Harrison transformations with same s parameters.

We have chosen the Hauser–Ernst (HE) representation for **K** instead of the earlier representations of Geroch[2] and Kinnersley and Chitre[13–16] because we needed the transformations to be already exponentiated. Since the Harrison transformations are not subgroups of **K**, it would be rather difficult to give an adequate description of them in terms of infinitesimal generators of **K**. Furthermore, we have found the complex-variable techniques of Hauser and Ernst relatively easy to handle and often lead to significantly simpler computations. In Sec. 4, we exploited the already exponentiated property to exhibit finite transformations in the **B** group and nonnull groups of Kinnersley and Chitre in terms of products of Harrison and/or HKX transformations.

The reader interested in stationary axisymmetric gravitational fields is now confronted with a considerable variety of solution-generating techniques. One impression that the author hopes has been gained from Refs. 6 and 12 is that it is advantageous to be familiar with all of the available methods and their interrelationships as there are situations in which each is best suited. With regard to the HE formalism, for example, one can think of situations where the formalism is: (i) the obvious or only one to consider (e.g., applications in the present paper and Ref. 20); (ii) is more manageable than its competitors (e.g., most applications in Refs. 18 and 19; composition of several transformations in **K**); (iii) is less manageable than its competitors (e.g., SL(2)-covariant manipulations of Belinsky–Zakharov transformations[9] as in Secs. 5 and 6 of Ref. 12); or (iv) is not an applicable method (e.g., applications of the **Q** and $\tilde{\mathbf{Q}}$ groups[5–7,12] or, equivalently, Neugebauer's Bäcklund transformations,[11,40,43] which are not in **K**).

Although the HE formalism does not pretend to represent transformations outside **K**, it can nevertheless be used to provide more satisfying proofs that such transformations are indeed outside **K**. For example, the **Q** group preserves asymptotic flatness and the transform of the $F(t)$ potential given by Eqs. (2.30a)–(2.31b) of Ref. 12 is easily shown to preserve special HE gauge.[46] It follows that there is an infinity of elements of **K** which transform a given solution $\mathscr{E}(\rho, z)$ to $\mathscr{E}'(\rho', z') = (Q)_{4s}\mathscr{E}(\rho, z)$, $(Q)_{4s} \in \mathbf{Q}$. Restricting Eqs. (2.30a)–(2.31b) and (2.2a) and (2.2b) of Ref. 12 to the symmetry axis $(\rho = 0 = \rho')$, we find

$$\mathscr{E}'(0, z') = \mathscr{E}(0, z), \quad z' = z/(1 - 2sz). \tag{6.2}$$

Similarly for $(\widetilde{Q})_{4s} \in \widetilde{\mathbf{Q}}$, we find[12]

$$\mathscr{E}'(0, z') = (1 - 2sz)^{-1}\mathscr{E}(0, z), \quad z' = z/(1 - 2sz). \tag{6.3}$$

When these values are substituted into Eq. (2.25) here, it is clearly impossible to choose a matrix $u(t)$ which is not explicitly dependent on the initial solution. Thus neither $(Q)_{4s}$ nor $(\widetilde{Q})_{4s}$ can be identified with any of the elements of **K** which map $\mathscr{E}$ to $(Q)_{4s}\mathscr{E}$ or to $(\widetilde{Q})_{4s}\mathscr{E}$, respectively. The same comments apply to other subgroups of **Q** and $\widetilde{\mathbf{Q}}$, in particular, the trivial rescaling $(\rho, z) \to (k\rho, kz)$ and translation $(\rho, z) \to (\rho, z - z_0)$. One corollary of the result that the Harrison transformation is in **K** and factorizes into **Q** and $\widetilde{\mathbf{Q}}$ transformations[12] is that any product of elements of **Q**, $\widetilde{\mathbf{Q}}$, and **K** is contained in **K** if and only if the combined transformation leaves (ρ, z) fixed.

A reasonable question to ask is whether all known transformations which leave (ρ, z) fixed are members of **K**. The trivial reflection $\mathscr{E} \to \mathscr{E}^*$, which reverses the sense of rotation, is easily shown to be not in **K** by the same argument as in the previous paragraph. The Kramer–Neugebauer mapping[47] [see Ref. 12, Sec. 3, for the transform of $F(t)$] and the Belinsky–Zakharov single-soliton transformation[9,12] create curvature singularities along the whole z axis and generate complex-valued metrics from real-valued and so are necessarily outside the framework of all existing representations of **K**. However, preliminary calculations suggest that these transformations may be obtainable as limits of sequences of bona fide elements of **K** and should therefore be accepted as being in **K** themselves. These results will be included in a separate paper when details are finalized. A simpler example of such a limiting class of group elements is the $s = \infty$ limit of the HKX transformation.[12]

In Sec. 5, we attempted to generalize Harrison's Bäcklund transformation to electrovac spacetimes by studying elements of $\mathbf{K}'$ for which

$$u(t) = [\det v(t)]^{-1/3} v(t) \tag{6.4}$$

in the HE representation, where $v(t)$ is a 3×3 matrix whose entries are rational functions of t. On comparison with Eq. (1.1) for the vacuum Harrison transformation, it is natural to expect that a choice of $v(t)$ such that $u(t)$ has cubic branch points at $t = 0$ and one other point, say $t = s$, in L_+ joined by a cut is a reasonable candidate for the electrovac Harrison transformation. Unfortunately, no such choice is compatible with condition (5.12b) which is quadratic in $u(t)$. The simplest choices for $v(t)$ compatible with conditions (5.12a)–(5.12d) have determinants of the forms (5.17a)–(5.17c) or similar forms with the result that $u(t)$ has cubic branch points in L_+ at $t = s$ and $t = s^*$, s complex, joined by a cut. The Bäcklund transformation deduced therefrom [see Eq. (5.39)] is obviously an electrovac enlargement of the *double* Harrison transformation with complex conjugate s parameters. (When $g_3 = 0 = h_3$, the transformation (5.39) maps vacuum to vacuum and is precisely the double Harrison transformation.) We proved that this transformation maps flat space to the full family of Kerr–Newman–NUT spacetimes with six parameters: mass, angular momentum, NUT parameter, electric charge, magnetic charge, and position on z axis.

The s parameters being complex conjugates in Eq. (5.39) gave us the Kerr–Newman–NUT solution in oblate spheroidal coordinates, i.e., beyond the extreme $(a^2 + e^2 > m^2$,[37] $|q| > 1$[39]). The prolate case $(a^2 + e^2 < m^2$, $|q| < 1)$, which has horizons at $x = \pm 1$, can be obtained by a trivial analytic continuation of the parameters. One is tempted to consider an analytic continuation of the parameters in Eq. (5.39) in order to define a corresponding transformation with two real s parameters (e.g., by formally introducing a second imaginary unit, j say, subject to $i^2 = j^2 = -1$, $i^* = -i$, $j^* = j$), but it is not obvious how this may be achieved.

Of course, every vacuum-to-vacuum transformation in **K** is the restriction to vacuum of an infinite number of elements of $\mathbf{K}'$. Suppose $F_{\text{vac}}(t)$ is a given vacuum $F(t)$ potential and $u_{\text{vac}}(t)$ a given 2×2 matrix obeying conditions (2.18a), (2.18b), and (2.19). Then the 3×3 $F(t)$ potential and corresponding elements of $\mathbf{K}'$ are given by

$$F(t) = \begin{pmatrix} F_{\text{vac}}(t) & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.5a}$$

$$u(t) = \begin{pmatrix} e^{-i\theta(t)} u_{\text{vac}}(t) & 0 \\ 0 & e^{2i\theta(t)} \end{pmatrix}, \tag{6.5b}$$

where $\theta(t) = \theta^*(t)$ is an arbitrary function of t, analytic on L, throughout L_-, and at $t = \infty$. If $u_{\text{vac}}(t)$ is chosen as the representing matrix of the single Harrison transformation [Eq. (1.1)], then it is still not possible to choose $\theta(t)$ such that Eq. (6.5b) takes the form (6.1) with algebraic branch points only at $t = 0$ and $t = s$. On the other hand, if we simply put $\theta(t) = 0$, say, the HHP cannot be solved by the methods of this paper when the given electrovac solution has a nontrivial electromagnetic field.

Perhaps the most positive evidence for the existence of an electrovac counterpart to the *single* Harrison transformation can be gleaned from recent work of Kinnersley and Lemley[48] on the electrovac counterparts of the **Q** and $\widetilde{\mathbf{Q}}$ groups[5–7,12] (or equivalently, I_1 and I_2 Bäcklund transformations,[11] respectively). The arguments which follow are of a speculative nature and are based on analogy with known results in vacuum: the quantitative details should be filled in in the near future. First, note that in Refs. 6 and 12, the vacuum Harrison (H) transformation has been factorized into the forms

$$H = I_2 I_1 = (L)_\beta (\widetilde{Q})_{-4s} (P)_\alpha (Q)_{4s}, \tag{6.6a}$$

$$H = I_1 I_2 = (P)_{\bar{\alpha}} (Q)_{-4s} (L)_{\bar{\beta}} (\widetilde{Q})_{4s}; \tag{6.6b}$$

$(P)_\alpha \in \mathbf{P}$, $(L)_\beta \in \mathbf{L}$ (see Sec. 2); $(Q)_{\pm 4s} \in \mathbf{Q}$, $(\widetilde{Q})_{\pm 4s} \in \widetilde{\mathbf{Q}}$;[46]

$\alpha, \beta, \delta \in SL(2, R)$; $I_1 = (P)_\alpha (Q)_\delta = (Q)_\delta (P)_\alpha$, $I_2 = (L)_\beta (\widetilde{Q})_\delta = (\widetilde{Q})_\delta (L)_\beta$. The four groups, **P**, **L**, **Q**, and $\widetilde{\mathbf{Q}}$, each locally isomorphic to SL(2, R), each contain one nontrivial parameter and two gauge parameters. This fact, together with the algebraic relations between α and β and between $\bar{\alpha}$ and $\bar{\beta}$,[12] limits the vacuum H transformation to two nontrivial parameters.

Let us now try to interpret Eqs. (6.6a) and (6.6b) in the electrovac context. The electrovac counterpart of the **P** group is the well-known SU(2, 1) Kinnersley group[13,36] (denoted **H**′ in Ref. 13) with eight real parameters. Thus α would now be a 3×3 pseudounitary matrix with unit determinant. This group contains three nontrivial parameters [Ehlers transformation (5.54) and Ernst–Harrison charging transformation (5.55)] and five gauge parameters. [If applied to a vacuum solution, the SU(2, 1) group **P** only provides two nontrivial parameters as the phase of b in Eq. (5.55) gives rise to an electromagnetic duality rotation, which is already one of the gauge transformations in **P**.] The full eight-parameter electrovac counterpart of the **L** group is not so well known as it appears that no eight generators in **K**′, three being the $\gamma_{AB}^{(0)}$ which generate a unimodular linear transformation of the Killing vectors,[14] will close to form a representation of the Lie algebra of SU(2, 1). Recently, Kinnersley (private communication) has found an SU(2, 1) group by dropping the requirement that the generators preserve the reality of the metric f_{AB} and the electromagnetic potential $A_A = \frac{1}{2}(\varphi_A + \varphi^*_A)$, and other real potentials. So the **L** group is available if we allow such a complex extension. Next, the electrovac counterparts of **Q** and $\widetilde{\mathbf{Q}}$, presumed to exist, should be locally isomorphic to SL(2, R) as they would necessarily transform the coordinates (ρ, z) exactly according to Eqs. (2.2a) and (2.2b) of Ref. 12 [transformations which leave (ρ, z) invariant are not considered to belong to **Q** or $\widetilde{\mathbf{Q}}$ unless, for convenience, we use the latter symbols for the larger groups, $I_1 = \mathbf{PQ}$, $I_2 = \mathbf{L}\widetilde{\mathbf{Q}}$, respectively]. Since $\widetilde{\mathbf{Q}}$ is presumed to commute with **L**, it would not be surprising if $\widetilde{\mathbf{Q}}$ did not preserve the reality of f_{AB}, A_A, and other real potentials.

The above comments imply that there should exist for electrovac spacetimes two eleven-dimensional groups, $I_1 = \mathbf{PQ}$ and $I_2 = \mathbf{L}\widetilde{\mathbf{Q}}$, each containing four nontrivial parameters and seven gauge parameters. (When applied to a vacuum solution, there would only be three nontrivial parameters, the magnetic charge not being counted.) According to a further analogy with vacuum, the expected algebraic relation between the SU(2, 1) matrices α and β, or between $\bar{\alpha}$ and $\bar{\beta}$, in Eqs. (6.6a) and (6.6b) would restrict the H transformation to at most four parameters (possibly only three). It is certain that this electrovac H transformation, if it exists, would generate complex-valued solutions from real-valued. Thus, it would be necessary to extend the complex-variable formalism of Hauser and Ernst to two complex dimensions (C^2) in order to incorporate the H transformation into their representation for **K**′. Such an analytic continuation to C^2 is straightforward in the Kinnersley–Chitre representation[13–16] (e.g., introduce an imaginary unit j and let the real and imaginary parts of the i-complex potentials be j-complex, $j^* = j$) but will probably require more serious thought for the HE formalism.

If the electrovac version of the single Harrison transformation contains the maximum four parameters then the double transformation must contain eight (seven if applied to a vacuum solution), two more than in our formula (5.39). In that case, it is difficult to imagine what the full transform of flat space would be. The only known solution containing Kerr–Newman–NUT space is the stationary charged C metric[42] with one extra parameter (acceleration), a cosmological constant being inadmissable here. However, this solution must be ruled out because the stationary *uncharged* C metric is the transform of flat space under *three* Harrison transformations, as can be seen by direct substitution into Eq. (2.25).

ACKNOWLEDGMENTS

This research has benefitted from discussions with William Kinnersley and Terry Lemley. I also wish to thank Isidore Hauser and Frederick J. Ernst for making their results available prior to publication.

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[22]Geroch's conjecture (Ref. 2) is that **K** acts transitively on all stationary axisymmetric vacuum solutions. The theorem of Hauser and Ernst (Ref. 20) states that a certain large subgroup of **K** acts multiply transitively on

all solutions which are analytic in a neighborhood of a given point of the z axis. Convincing though nonrigorous arguments for the more restricted class of asymptotically flat solutions have been given by several previous authors.[3,6,11] Crude arbitrary-function-counting arguments suggest that the Geroch conjecture should not be true for the larger class of solutions which have logarithmic singularities along the whole z axis but it may be possible to generate such solutions from flat space by taking suitable limiting transitions of Geroch group elements.

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[24]The closely related Bäcklund transformations of Cosgrove, Maison, and Neugebauer involve the spacetime coordinates (ρ, z). In Sec. 6, we use the HE formalism to demonstrate why transformations with this property cannot be in **K**. The Belinsky–Zakharov $(2n+1)$-soliton transformation and Kramer–Neugebauer mapping were asserted in Ref. 12 to be outside **K** because they create curvature singularities along the whole z axis. However, there is a possibility that these transformations can be incorporated in **K** by taking suitable limits as mentioned in Ref. 22. The Belinsky–Zakharov $2n$-soliton transformation is identifiable with a product of $2n$ Harrison transformations (Ref. 12) and so is covered by the present paper.

[25]The notation $\gamma_{XY}^{(k)}$ is in accordance with Eqs. (2.22) and (2.23) of Ref. 12. The symbol $\gamma_{XY}^{(k)}$ of Eqs. (3.3) of Ref. 14 (restricted to vacuum) has the same meaning as q_{XY} here and in Ref. 12. [Note that, for any SL(2) tensor, $q^{11}=q_{22}, q^{12}=-q_{21}, q^{21}=-q_{12}, q^{22}=q_{11}$.]

[26]When comparing SL(2) tensor equations with matrix equations, it is useful to know that the matrix inverse of R_{AB} is $R^{BA}/(\det R)$, and the inverse of $R_A{}^B$ is $-R^B{}_A/(\det R)$; $\det R=\frac{1}{2}R_{XY}R^{XY}$ and, more generally, $R_{XA}R^X{}_B = R_{AX}R_B{}^X=(\det R)\epsilon_{AB}$. Indices are raised and lowered with $\epsilon_{AB}=\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [e.g., $R_A{}^B=\epsilon^{BX}R_{AX}, R_{AB}=\epsilon_{XA}R^X{}_B$]. The indices 1 and 2 used here correspond to Hauser and Ernst's 4 and 3, respectively. Consequently, to convert one of the matrices in Hauser and Ernst's work to the corresponding matrix here, one must interchange the two rows with each other and similarly the two columns.

[27]The sign of ψ as defined by Eq. (2.3b) is in agreement with Ref. 13, whose conventions we follow. In Refs. 5–7, 12, and 39, the opposite sign was used and so $\mathscr{E}$ there is to be identified with $\mathscr{E}^*$ here.

[28]$F(t)$ can also be calculated by the method of characteristics (Ref. 16) or from the Riccati equations for pseudopotentials (see the last paragraph of Appendix A of Ref. 12).

[29]Analyticity of $\mathscr{E}$ in a real (ρ, z) domain means that it must be possible to define $\mathscr{E}(\rho, z)$, a function of two *complex* variables ρ and z, which is analytic on and near the appropriate parts of the real axes. Taylor series can be used to give an intrinsic definition.

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