

# Quantum Probability from Decision Theory?

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## Abstract

In a recent paper, Deutsch [1] claims to derive the “probabilistic predictions of quantum theory” from the “non-probabilistic axioms of quantum theory” and the “non-probabilistic part of classical decision theory.” We show that his derivation fails because it includes hidden probabilistic assumptions.

## I. INTRODUCTION

In a recent paper, Deutsch [1] attempts to derive the “probabilistic predictions of quantum theory” from the “non-probabilistic part of quantum theory” and what he views as the “non-probabilistic part of classical decision theory.” For Deutsch this means the following. The nonprobabilistic part of quantum theory is contained in the axioms that associate quantum states with rays in Hilbert space and observables with Hermitian operators; in particular, the eigenvalues of a Hermitian operator are the only possible results of a measurement of the corresponding observable, and if the quantum state is an eigenstate of the Hermitian operator, the eigenvalue is the certain result of a measurement of that observable. The relevant nonprobabilistic part of classical decision theory includes the assumption that a rational decision maker orders all his preferences transitively—that is, if he prefers  $A$  to  $B$  and  $B$  to  $C$ , he must also prefer  $A$  to  $C$ . From these assumptions, Deutsch seeks to derive, first, that quantum mechanics has a probabilistic interpretation and, second, that the quantum probability rule has the standard form of a squared inner product. Deutsch describes his result as follows:

Thus we see that quantum theory permits what philosophy would hitherto have regarded as a formal impossibility, akin to ‘deriving an ought from an is’, namely deriving a probability statement from a factual statement. This could be called deriving a ‘tends to’ from a ‘does’.

We argue in this paper that Deutsch's derivation fails to achieve both its goals. First, as we discuss in Sec. II, the standard nonprobabilistic axioms of classical decision theory, which include the assumption of (complete) transitive preferences, already ensure that the preferences can be ordered in terms of probabilities and utility functions [2,3]. Second, as we detail in Sec. III, Deutsch's derivation of the form of the quantum probability law is flawed because an ambiguity in his notation masks a hidden probabilistic assumption that is essential for the derivation.

Despite the failure of Deutsch's derivation, we are sympathetic to the view that the meaning of probability in quantum mechanics is specified by its role in rational decision making. Indeed, we believe that this view can help illuminate the very nature of quantum theory [4]. We believe, however, that the primary technical machinery underlying this view is already provided by Gleason's theorem [5], an oft-neglected derivation of the quantum probability law. We review the theorem in Sec. IV. Gleason assumes that observables are described by Hermitian operators, supplementing that only by the assumption that the results of measurements cannot always be predicted with certainty and that the uncertainty is described by probabilities that are consistent with the Hilbert-space structure of the observables. From this he is able to derive both that the possible states are density operators and that the quantum probability law is the standard one. Because Gleason's theorem gives both the state space of quantum mechanics and the probability rule, we believe it trumps all other derivations along these lines.

## II. PROBABILITIES AND DECISION THEORY

Classical decision theory, formulated along the lines that Deutsch has in mind, envisions a rational decision maker, or agent, who is confronted with a choice among various games [2,3]. Each game is described by a set of events labeled by an index  $j$ , which the agent believes will occur with probability  $p_j$ . The value the agent attaches to an event within a given game is quantified by its utility  $x_j$ . Decision theory seeks to capture the notion of rational decision making by positing that the agent decides among the games by choosing the one that has the largest expected utility,

$$\sum_j p_j x_j . \tag{1}$$

A simple consequence of this framework is that an agent can give his preferences among games a complete transitive ordering.

Deutsch extracts what he sees as the nonprobabilistic part of decision theory and applies it to quantum mechanics in the following way. A game is again described by events, now interpreted as the outcomes of a measurement of a Hermitian operator that has eigenstates  $|\phi_j\rangle$ . The  $j$ th outcome has utility  $x_j$ . In place of the probabilities of classical decision theory, Deutsch substitutes the normalized quantum state of the system in question,

$$|\psi\rangle = \sum_j \lambda_j |\phi_j\rangle . \tag{2}$$

Thus a quantum game, in Deutsch's formulation, is characterized by a quantum state and utilities that depend on the outcome of a measurement performed on that state. As the

final part of his formulation, Deutsch defines the *value of a game*—the central notion in his argument—as “the utility of a hypothetical payoff such that the player is indifferent between playing the game and receiving that payoff unconditionally.” Deutsch does not assume that the value of a game is an expected utility, for that is precisely the probabilistic aspect of classical decision theory he wants to exclude from his formulation. He does assume that the values are transitively ordered and that a rational decision maker decides among games by choosing the game with the largest value.

Deutsch describes this in the following way:

On being offered the opportunity to play such a game at a given price, knowing  $|\psi\rangle$ , our player will respond somehow: he will either accept or refuse. His acceptance or refusal will follow a strategy which, given that he is rational, must be expressible in terms of transitive preferences and therefore in terms of a value  $\mathcal{V}[|\psi\rangle]$  for each possible game.

Notice that Deutsch denotes the value of a game without explicit reference to the utilities and the corresponding eigenstates, which partially define the game. We prefer a more explicit notation. First we define a Hermitian *utility operator*

$$\hat{X} = \sum x_j |\phi_j\rangle \langle \phi_j|. \quad (3)$$

We now can denote the value of a game more explicitly as  $\mathcal{V}(|\psi\rangle; \hat{X})$ , which includes both defining features of a game, the quantum state  $|\psi\rangle$  and the utility operator  $\hat{X}$ . Our notation serves its purpose in Sec. III, where it helps to ferret out a flaw in Deutsch’s derivation.

Before turning to such details of Deutsch’s argument, we consider a more fundamental issue. Deutsch’s attempt to derive the probabilistic interpretation of quantum mechanics from purely nonprobabilistic considerations must fail, because his assumption of complete transitive preferences is tantamount to assuming probabilities at the outset. The conventional understanding of preferences—making decisions in the face of uncertainty—already hints strongly that probabilities will be an essential tool in any decision theory. Indeed, this is the import of a fundamental result of the theory [2,3]: if one assumes complete transitive preferences among games along with standard nonprobabilistic axioms, one can determine simultaneously utility functions and sets of probabilities, such that the agent’s behavior is described as maximizing expected utility. If the preferences among games are quantified by a value function  $\mathcal{V}$ , then for each game there exist probabilities  $p_j$  and transformed utilities  $F(x_j)$ , where  $F$  is a strictly increasing function, such that expected utility gives the same ordering:

$$\mathcal{V}(|\psi\rangle; \hat{X}) = F^{-1} \left( \sum_j p_j F(x_j) \right). \quad (4)$$

The crucial point is that this intimate relationship between preferences and probabilities is purely classical, having nothing to do with quantum mechanics.

Although Deutsch’s argument fails to exclude *a priori* probabilistic considerations, it might nevertheless provide a derivation of the specific form of the quantum probability law. To assess this possibility, we turn now to Deutsch’s specific argument.

### III. EXAMINATION OF DEUTSCH'S DERIVATION

In this section we examine the derivation of what Deutsch terms the “pivotal result” of his argument,

$$\mathcal{V}\left(\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle); \hat{X}\right) = \frac{1}{2}(x_1 + x_2); \quad (5)$$

that is, that the value of a game in which the quantum state is an equal linear combination of two eigenstates of the utility operator is the mean of the utilities. The derivation of the pivotal result is contained in Deutsch's Eqs. (D7)–(D11) and related textual material. (Here and throughout we refer to equations in Deutsch's paper by prefixing a D to the equation number.) We comment briefly on later steps in Deutsch's argument at the end of this section.

Deutsch uses a notation for the value of a game that makes no explicit reference to the utility operator. Furthermore, he employs a notational convention, often used in physics, whereby an eigenvector of an operator—in this case the utility operator—is labeled by its eigenvalue—in this case the utility itself. This labeling can cause confusion when games involving different utility operators are under consideration, as in the argument examined in this section. The resulting ambiguity leads Deutsch to accidentally equate the value of two games whose value cannot be shown to be equal without some additional assumption. Identifying this hidden assumption is the goal of this section.

In deriving the pivotal result, Deutsch posits two properties of the quantum value function. The first, given in Eq. (D8), we call the *displacement property*. Written in our notation, this property becomes

$$\mathcal{V}\left(\sum_j \lambda_j |\phi_j\rangle; \sum_j (x_j + k) |\phi_j\rangle \langle \phi_j|\right) = k + \mathcal{V}\left(\sum_j \lambda_j |\phi_j\rangle; \sum_j x_j |\phi_j\rangle \langle \phi_j|\right). \quad (6)$$

Our notation makes clear that both sides of this equation refer to the same quantum state, but different utility operators. In contrast, Eq. (D8) is ambiguous. The left-hand side of Eq. (D8) refers to the state  $\sum_j \lambda_j |x_j + k\rangle$ , whereas the right-hand side refers to the state  $\sum_j \lambda_j |x_j\rangle$ ; it is unclear whether the two sides refer to different quantum states or to a single state labeled according to two different utility operators. We adopt the latter interpretation as being the one most consistent with Deutsch's discussion. The second property of value functions, Deutsch's *zero-sum property* (D9), becomes in our notation,

$$\mathcal{V}\left(\sum_j \lambda_j |\phi_j\rangle; \sum_j (-x_j) |\phi_j\rangle \langle \phi_j|\right) = -\mathcal{V}\left(\sum_j \lambda_j |\phi_j\rangle; \sum_j x_j |\phi_j\rangle \langle \phi_j|\right). \quad (7)$$

Equation (D9) suffers from the same sort of ambiguity as Eq. (D8): it refers to two states,  $\sum_j \lambda_j |-x_j\rangle$  and  $\sum_j \lambda_j |x_j\rangle$ . In Eq. (7) we again choose the interpretation that there is a single state, but two different utility operators. The zero-sum property is an axiom of Deutsch's nonprobabilistic decision theory, and the displacement property follows from the principle of additivity, another axiom of his analysis.

The derivation of the pivotal result deals with a state  $|\psi\rangle$  that is a superposition of two utility eigenstates:

$$|\psi\rangle = \lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle . \quad (8)$$

When writing the displacement and zero-sum properties for such a state, we can omit the other eigenstates from the utility operator, since as Deutsch shows, the corresponding outcomes do not occur. To shorten the equations, we introduce the abbreviations

$$\hat{\Pi}_i = |\phi_i\rangle\langle\phi_i| , \quad i = 1, 2. \quad (9)$$

We now proceed in our notation through the rest of the argument leading to Eq. (D11). We carry along arbitrary amplitudes  $\lambda_1$  and  $\lambda_2$ , because this helps to illustrate the nature of the hidden assumption. The reasoning begins with the displacement property (D8), specialized to the case  $k = -x_1 - x_2$ :

$$\mathcal{V}(|\psi\rangle; x_1\hat{\Pi}_1 + x_2\hat{\Pi}_2) - x_1 - x_2 = \mathcal{V}(|\psi\rangle; -x_1\hat{\Pi}_2 - x_2\hat{\Pi}_1) . \quad (10)$$

Applying the zero-sum property (7) to the game on the right-hand side yields

$$\mathcal{V}(|\psi\rangle; x_2\hat{\Pi}_1 + x_1\hat{\Pi}_2) + \mathcal{V}(|\psi\rangle; x_1\hat{\Pi}_1 + x_2\hat{\Pi}_2) = x_1 + x_2 . \quad (11)$$

Deutsch uses this result in the case  $\lambda_1 = \lambda_2 = 1/\sqrt{2}$ , where it becomes

$$\mathcal{V}\left(\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle); x_2\hat{\Pi}_1 + x_1\hat{\Pi}_2\right) + \mathcal{V}\left(\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle); x_1\hat{\Pi}_1 + x_2\hat{\Pi}_2\right) = x_1 + x_2 . \quad (12)$$

In Deutsch's notation the values of the two games in this equation are denoted in the same way, so he assumes they are equal,

$$\mathcal{V}\left(\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle); x_2\hat{\Pi}_1 + x_1\hat{\Pi}_2\right) = \mathcal{V}\left(\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle); x_1\hat{\Pi}_1 + x_2\hat{\Pi}_2\right) , \quad (13)$$

which leads immediately to the pivotal result (5).

Equation (13) is the hidden assumption in Deutsch's argument. To see that it is required and that it involves introducing the notion of probabilities, consider the following rule for measurement outcomes: the result associated with eigenstate  $|\phi_1\rangle$  *always* occurs. This deterministic rule is perfectly legitimate at this point in the argument. In Eq. (12) it gives utility  $x_2$  in the first game and utility  $x_1$  in the second, thus satisfying the equation.

Another way to get at the import of Deutsch's hidden assumption is to make a similar assumption for the case of arbitrary expansion coefficients, that is to assume

$$\mathcal{V}(\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle; x_2\hat{\Pi}_1 + x_1\hat{\Pi}_2) = \mathcal{V}(\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle; x_1\hat{\Pi}_1 + x_2\hat{\Pi}_2) . \quad (14)$$

Both this assumption and the more specialized one embodied in Eq. (13) are equally well (or badly) justified at this stage of the argument. The reason is that as yet  $\lambda_1$  and  $\lambda_2$  are just numbers attached to the possible outcomes, having no *a priori* relation to probabilities. Substituting Eq. (14) into Eq. (11) gives

$$\mathcal{V}(\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle; x_1\hat{\Pi}_1 + x_2\hat{\Pi}_2) = \frac{1}{2}(x_1 + x_2) , \quad (15)$$

which generalizes to the rule that the value of a game is the arithmetic mean of the utilities that have nonzero amplitude. This corresponds to the probability rule  $p_j = (\text{number of nonzero amplitudes})^{-1}$ . Notice that this probability rule is contextual in the sense of Gleason's theorem (see discussion in Sec. IV).

We conclude that to derive the pivotal result (5), one must include Eq. (13) as an additional assumption. In our view, including this additional assumption is not just a minor addition to Deutsch's list of assumptions, but rather a major conceptual shift. The assumption is akin to applying Laplace's Principle of Insufficient Reason to a set of indistinguishable alternatives, an application that requires acknowledging *a priori* that amplitudes are related to probabilities. Once this acknowledgement is made, however, the pivotal result (5) is a simple consequence of classical decision theory, as can be seen in the following way. As discussed in Sec. II, the existence of a numerical value  $\mathcal{V}(|\psi\rangle; \hat{X})$  for each game, together with standard nonprobabilistic axioms of decision theory, entails that there exist probabilities  $p_1$  and  $p_2$  such that

$$\mathcal{V}\left(\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle); x_1\hat{\Pi}_1 + x_2\hat{\Pi}_2\right) = F^{-1}(p_1F(x_1) + p_2F(x_2)), \quad (16)$$

where  $F$  is a strictly increasing function. The hidden assumption (13) then takes the form

$$p_1F(x_2) + p_2F(x_1) = p_1F(x_1) + p_2F(x_2). \quad (17)$$

If this is to be true for arbitrary  $x_1$  and  $x_2$  (or for any  $x_1 \neq x_2$ ), it follows that  $p_1 = p_2 = 1/2$ . Thus in the context of classical decision theory, the assumption (13) is equivalent to applying the Principle of Insufficient Reason to the case of equal amplitudes.

It is difficult to assess the validity of Deutsch's argument once one gets past the derivation of the pivotal result. This is mainly because the remainder of the argument repeatedly invokes the principle of substitutability. The difficulty is that this principle—that the value of a game is unchanged when a subgame is replaced by another subgame of equal value—is never given a precise mathematical formulation in the quantum context. In any case, the remainder of the argument can be simplified once one realizes that the Principle of Insufficient Reason is an essential ingredient, for then one gets immediately that for an equal superposition of  $n$  eigenstates, the probability of each outcome is  $1/n$ .

The vagueness of the principle of substitutability has an important consequence. We believe that the probability rule following Eq. (15) satisfies all of Deutsch's assumptions, including a suitably defined principle of substitutability. If it does, then it shows that no amount of cleverness in using Deutsch's assumptions can ever lead uniquely to the standard quantum rule for probabilities. The fly in the ointment is that without a precise formulation of the principle of substitutability, it is not possible to tell whether this rule satisfies it.

#### IV. CONCLUSION

We have seen that if one assumes the nonprobabilistic part of classical decision theory, then one is effectively introducing probabilities at the same time. Indeed, once one realizes that quantum theory deals with uncertain outcomes, one is forced to introduce probabilities, as they provide the *only* language for quantifying uncertainty [2,4,6,7]. From this point

of view, the most powerful and compelling derivation of the quantum probability rule is Gleason’s theorem.

**Gleason’s theorem:** [5,8,9] Assume there is a function  $f$  from the one-dimensional projectors acting on a Hilbert space of dimension greater than 2 to the unit interval, with the property that for each orthonormal basis  $\{|\psi_k\rangle\}$ ,

$$\sum_k f(|\psi_k\rangle\langle\psi_k|) = 1 . \tag{18}$$

Then there exists a density operator  $\hat{\rho}$  such that

$$f(|\psi\rangle\langle\psi|) = \langle\psi|\hat{\rho}|\psi\rangle . \tag{19}$$

It is worthwhile to ponder the meaning of this theorem. It assumes the Hilbert-space structure of observables—that is, that each orthonormal basis corresponds to the mutually exclusive results of a measurement of some observable. It sets as its task to derive the probabilities for the inevitably uncertain measurement outcomes. The only further ingredient required is that the probability for obtaining the result corresponding to a normalized vector  $|\psi\rangle$  depends only on  $|\psi\rangle$  itself, not on the other vectors in the orthonormal basis defining a particular measurement. This important assumption, which might be called the “noncontextuality” of the probabilities, means that the probabilities are consistent with the Hilbert-space structure of the observables. With these assumptions the probabilities for all measurements can be derived from a density operator  $\hat{\rho}$  using the standard quantum probability rule. Remarkably this conclusion does not rely on any assumption about the continuity or differentiability of  $f$ ; the only essential property of  $f$  is that it be bounded.

*By assuming that measurements are described by probabilities that are consistent with the Hilbert-space structure of the observables, Gleason’s theorem derives in one shot the state-space structure of quantum mechanics and the probability rule.* It is hard to imagine a cleaner derivation of the probability rule than this.

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