

Lower bounds for multicolor Ramsey numbers

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Abstract

We give an exponential improvement to the lower bound on diagonal Ramsey numbers for any fixed number of colors greater than two.

1 Introduction

The Ramsey number $r(t; \ell)$ is the smallest natural number n such that every ℓ -coloring of the edges of the complete graph K_n contains a monochromatic K_t . For $\ell = 2$, the problem of determining $r(t) := r(t; 2)$ is arguably one of the most famous in combinatorics. The bounds

$$\sqrt{2}^t < r(t) < 4^t$$

have been known since the 1940s, but, despite considerable interest, only lower-order improvements [2, 7, 8] have been made to either bound. In particular, the lower bound $r(t) > (1 + o(1)) \frac{t}{\sqrt{2e}} \sqrt{2}^t$, proved by Erdős [3] as one of the earliest applications of the probabilistic method, has only been improved [8] by a factor of 2 in the intervening 70 years.

If we ignore lower-order terms, the best known upper bound for $\ell \geq 3$ is $r(t; \ell) < \ell^{t\ell}$, proved through a simple modification of the Erdős–Szekeres neighborhood-chasing argument [4] that yields $r(t) < 4^t$. For $\ell = 3$, the best lower bound, $r(t; 3) > \sqrt{3}^t$, again comes from the probabilistic method. For higher ℓ , the best lower bounds come from the simple observation of Lefmann [5] that

$$r(t; \ell_1 + \ell_2) - 1 \geq (r(t; \ell_1) - 1)(r(t; \ell_2) - 1).$$

To see this, we blow up an ℓ_1 -coloring of $K_{r(t; \ell_1)-1}$ with no monochromatic K_t so that each vertex set has order $r(t; \ell_2) - 1$ and then color each of these copies of $K_{r(t; \ell_2)-1}$ separately with the remaining ℓ_2 colors so that there is again no monochromatic K_t . By using the bounds $r(t; 2) - 1 \geq 2^{t/2}$ and $r(t; 3) - 1 \geq 3^{t/2}$, we can repeatedly apply this observation to conclude that

$$r(t; 3k) > 3^{kt/2}, \quad r(t; 3k + 1) > 2^t 3^{(k-1)t/2}, \quad r(t; 3k + 2) > 2^{t/2} 3^{kt/2}.$$

Our main result is an exponential improvement to all these lower bounds for three or more colors.

Our principal contribution is the following theorem, proved via a construction which is partly deterministic and partly random. The deterministic part shares some characteristics with a construction of Alon and Krivelevich [1], in that we consider a graph whose vertices are vectors over a finite field where adjacency is determined by the value of their scalar product, while randomness comes in through both random coloring and random sampling.

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Theorem 1. For any prime q , $r(t; q + 1) > 2^{t/2} q^{3t/8 + o(t)}$.

In particular, the cases $q = 2$ and $q = 3$ yield exponential improvements over the previous bounds for $r(t; 3)$ and $r(t; 4)$, both of which came from the probabilistic method (in fact, Lefmann's observation gives an additional polynomial factor in the four-color case, but this is of lower order than the exponential improvements that are our concern).

Corollary 2. $r(t; 3) > 2^{7t/8 + o(t)}$ and $r(t; 4) > 2^{t/2} 3^{3t/8 + o(t)}$.

For the sake of comparison, we note that the improvement for three colors is from 1.732^t to 1.834^t , while, for four colors, it is from 2^t to 2.135^t . Improvements for all $\ell \geq 5$ now follow from repeated applications of Lefmann's observation, yielding

$$r(t; 3k) > 2^{7kt/8 + o(t)}, \quad r(t; 3k + 1) > 2^{7(k-1)t/8 + t/2} 3^{3t/8 + o(t)}, \quad r(t; 3k + 2) > 2^{7kt/8 + t/2 + o(t)},$$

where we used, for instance,

$$r(t; 3k + 1) - 1 \geq (r(t; 3(k - 1)) - 1)(r(t; 4) - 1) \geq (r(t; 3) - 1)^{k-1}(r(t; 4) - 1).$$

2 Proof of Theorem 1

Let q be a prime. Suppose $t \not\equiv 0 \pmod q$ and let $V \subseteq \mathbb{F}_q^t$ be the set consisting of all vectors $v \in \mathbb{F}_q^t$ for which $\sum_{i=1}^t v_i^2 = 0 \pmod q$, noting that $q^{t-2} \leq |V| \leq q^t$. Here the lower bound follows from observing that we may pick v_1, \dots, v_{t-2} arbitrarily and, since every element in \mathbb{F}_q can be written as the sum of two squares, there must then exist at least one choice of v_{t-1} and v_t such that $v_{t-1}^2 + v_t^2 = -\sum_{i=1}^{t-2} v_i^2$.

We will first color all the pairs $\binom{V}{2}$ and then define a coloring of $E(K_n)$ by restricting our attention to a random sample of n vertices in V . Formally:

Coloring all pairs in $\binom{V}{2}$. For every pair $uv \in \binom{V}{2}$, we define its color $\chi(uv)$ according to the following rules:

- If $u \cdot v = i \pmod q$ and $i \neq 0$, then set $\chi(uv) = i$.
- Otherwise, choose $\chi(uv) \in \{q, q + 1\}$ uniformly at random, independently of all other pairs.

Mapping $[n]$ into V . Take a random injective map $f : [n] \rightarrow V$ and define the color of every edge ij as $\chi(f(i)f(j))$.

Our goal is to upper bound the orders of the cliques in each color class.

Colors $1 \leq i \leq q - 1$. There are no i -monochromatic cliques of order larger than t for any $1 \leq i \leq q - 1$. Indeed, suppose that v_1, \dots, v_s form an i -monochromatic clique. We will try to show that they are linearly independent and, therefore, that there are at most t of them. To this end, suppose that

$$u := \sum_{j=1}^s \alpha_j v_j = \bar{0}$$

and we wish to show that $\alpha_j = 0 \pmod q$ for all j . Observe that since $v_j \cdot v_j = 0 \pmod q$ for all j (our ground set V consists only of such vectors) and $v_k \cdot v_j = i \pmod q$ for each $k \neq j$, by considering all the products $u \cdot v_j$, we obtain that the vector $\bar{\alpha} = (\alpha_1, \dots, \alpha_s)$ is a solution to

$$M\bar{\alpha} = \bar{0}$$

with $M = iJ - iI$, where J is the $s \times s$ all 1 matrix and I is the $s \times s$ identity matrix. In particular, we obtain that the eigenvalues of M (over \mathbb{Z}) are $is - i$ with multiplicity 1 and $-i$ with multiplicity $s - 1$. Therefore, if $s \not\equiv 1 \pmod q$, the matrix is also non-singular over \mathbb{Z}_q , implying that $\bar{\alpha} = 0$, as required. On the other hand, if $s \equiv 1 \pmod q$, we can apply the same argument with v_1, \dots, v_{s-1} to conclude that $s - 1 \leq t$. But, we cannot have $s - 1 = t$, since this would imply that $t = 0 \pmod q$, contradicting our assumption. Therefore, we may also conclude that $s \leq t$ in this case.

Colors q and $q + 1$. We call a subset $X \subseteq V$ a *potential clique* if $|X| = t$ and $u \cdot v = 0 \pmod q$ for all $u, v \in X$. Given a potential clique X , we let M_X be the $t \times t$ matrix whose rows consist of all the vectors in X . Observe that $M_X \cdot M_X^T = 0$, where we use the fact that each vector is self-orthogonal. First we wish to count the number of potential cliques and later we will calculate the expected number of cliques that survive after we color randomly and restrict to a random subset of order n .

Suppose that X is a potential clique and let $r := \text{rank}(X)$ be the rank of the vectors in this clique, noting that $r \leq t/2$, since the dimension of any isotropic subspace of \mathbb{F}_q^t is at most $t/2$. By assuming that the first r elements are linearly independent, the number of ways to build a potential clique X of rank r is upper bounded by

$$\left(\prod_{i=0}^{r-1} q^{t-i} \right) \cdot q^{(t-r)r} = q^{tr - \binom{r}{2} + tr - r^2} = q^{2tr - \frac{3r^2}{2} + \frac{r}{2}}.$$

Indeed, suppose that we have already chosen the vectors $v_1, \dots, v_s \in X$ for some $s < r$. Then, letting M_s be the $s \times t$ matrix with the v_i as its rows, we need to choose v_{s+1} such that $M_s \cdot v_{s+1} = \bar{0}$. Since the rank of M_s is assumed to be s , there are exactly q^{t-s} choices for v_{s+1} in \mathbb{F}_q^t and, therefore, at most that many choices for $v_{s+1} \in V$. If, instead, $s \geq r$, then we need to choose a vector $v_{s+1} \in \text{span}\{v_1, \dots, v_r\}$ and there are at most q^r such choices in V .

Now observe that the function $2tr - \frac{3r^2}{2} + \frac{r}{2}$ appearing in the exponent of the expression above is increasing up to $r = \frac{2t}{3} + \frac{1}{6}$, so the maximum occurs at $t/2$. Therefore, by plugging this into our estimate and summing over all possible ranks, we see that the number N_t of potential cliques in V is upper bounded by $q^{\frac{5t^2}{8} + o(t^2)}$.

The probability that a potential clique becomes monochromatic after the random coloring is $2^{1 - \binom{t}{2}}$. Suppose now that p is such that $p|V| = 2n$ and observe that $p = nq^{-t+O(1)}$. If we choose a random subset of V by picking each $v \in V$ independently with probability p , the expected number of monochromatic potential cliques in this subset is, for $n = 2^{t/2}q^{3t/8+o(t)}$,

$$p^t 2^{1 - \binom{t}{2}} N_t \leq q^{-t^2 + o(t^2)} n^t 2^{-\frac{t^2}{2} + o(t^2)} q^{\frac{5t^2}{8} + o(t^2)} = \left(2^{-\frac{t}{2}} q^{-\frac{3t}{8} + o(t)} n \right)^t < 1/2.$$

Since our random subset will also contain more than n elements with probability at least $1/2$, there exists a choice of coloring and a choice of subset of order n such that there is no monochromatic potential clique in this subset. This completes the proof.

Remark. Our method also gives a construction which matches Erdős' bound $r(t) > \sqrt{2}^t$ up to lower-order terms. To see this, we set $V = \mathbb{F}_2^{2t}$ and color edges red or blue depending on whether $u \cdot v = 0$ or $1 \pmod{2}$. If we then sample $2^{t/2+o(t)}$ vertices of V at random, we can show that w.h.p. the resulting set does not contain a monochromatic clique of order t . We believed this to be new, but, after the first version of this article was made public, we learned that such a construction was already discovered by Pudlák, Rödl and Savický [6] in 1988. It was also pointed out to us by Jacob Fox that one can achieve the same end by starting with any pseudorandom graph on n vertices for which the count of cliques and independent sets of order $2c \log_2 n$ is approximately the same as in $G(n, 1/2)$ and sampling n^c vertices. This can be applied, for instance, with the Paley graph.

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