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# FACTORIZATION PROPERTIES OF THE DUAL RESONANCE MODEL: A GENERAL TREATMENT OF LINEAR DEPENDENCES*

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In the dual resonance model, introducing a generalized gauge-transformation operator which leaves the vertex functions invariant, we present a general treatment of linear dependences among the vertex functions. We also prove that the two propagators recently discussed in connection with the twisting operator are equal up to our gauge transformation.

Soon after the proposal by Veneziano[1] of a four-point function satisfying duality as well as Regge behavior, a generalization to the $N$-point function was obtained.[2,3] More recently, it has been shown[4,5] that the dual resonance model proposed in Ref. 3 obeys factorization, in the sense that residues of the poles can be expressed in the form

$$\sum_i f_i^*(\overline{q}, N_{\overline{q}}) f_i(p, N_p), \tag{1}$$

where $p$ ($\overline{q}$) stands for the incoming (outgoing) monenta, $N_p$ ($N_{\overline{q}}$) is the number of incoming (outgoing) lines, and the summation runs over a finite number of terms. If these functions $f_i$ were linearly independent, then the number of terms appearing in the sum would specify the degree of degeneracies at the resonance. However, there exist linear relations among these functions, as already shown in Refs. 4 and 5. In this Letter we shall introduce a generalized gauge-transformation operator $S$ and present a general treatment of the problem of linear dependences using this operator, which will enable us to obtain all the linear relations in a simple closed form.

It has proved very convenient to study the dual $N$-point function by employing the operator formalism recently proposed by Fubini, Gordon, and Veneziano.[6] According to these authors, one introduces a set of "harmonic-oscillator" operators $a_\mu^{(i)}$ obeying the usual commutation relations

$$[a_\mu^{(i)}, a_\nu^{\dagger(j)}] = \delta_{ij} g_{\mu\nu}, \tag{2}$$

and a Hilbert space generated by the repeated application of the operators $a_\mu^{\dagger(i)}$ on a "vacuum" vector $|\varphi\rangle$. Then the $N$-point function can be written as

$$A_{r+1,\,s+1} = \int dz\, z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}\langle\overline{q}|z^R|p\rangle, \tag{3}$$

where the momenta $p$ and $\overline{q}$ are defined in Fig. 1(a), the operator $R = \sum_n n a^{\dagger(n)} a^{(n)}$, the vector $|p\rangle$ is given by

$$|p\rangle = \int dx\, \varphi(x,p) \exp\Big(\sum_{n=1}^{\infty}\sum_{l=0}^{r} \rho_l{}^n p_l \frac{a^{\dagger(n)}}{\sqrt{n}}\Big)|\varphi\rangle, \tag{4}$$

and an analogous expression is given for the vector $|\overline{q}\rangle$. In Eq. (4), $\varphi(x,p)$ is the integrand of the ($r$

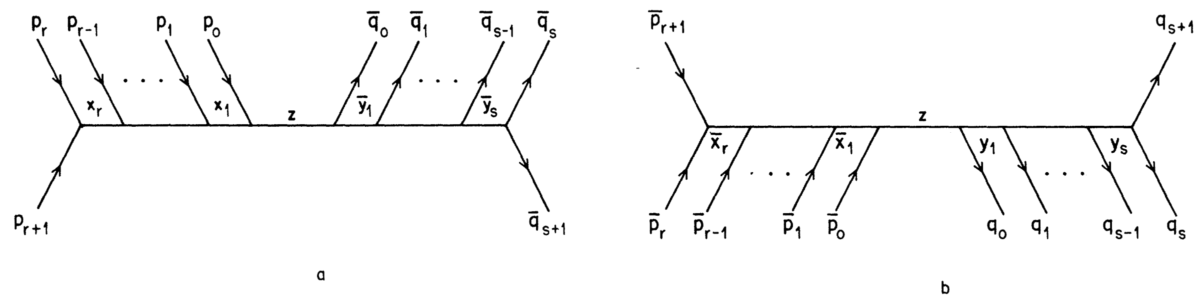


FIG. 1. The two multiperipheral configurations considered in this paper.

+3)-point function; $\rho_0 = 1$ and $\rho_i = x_i \rho_{i-1}$.

Let us introduce a complete set of occupation-number states $|\lambda\rangle$, where the operator $R$ is diagonal in this space with $R|\lambda\rangle = R_\lambda|\lambda$. By inserting this complete set in Eq. (3), we can perform the integration explicitly and obtain

$$A_{r+1,s+1} = \sum_\lambda B(R_\lambda - \alpha(s), \alpha(0)) \langle \bar{q}|\lambda\rangle\langle\lambda|p\rangle. \tag{5}$$

The residues of the poles clearly obey factorization; the degeneracy is finite since to any particular residuum, only those vectors $|\lambda\rangle$ contribute for which $R_\lambda \leq \alpha(s)$.

The question of linear dependences arises when we ask whether all the functions $f_\lambda(p) = \langle\lambda|p\rangle$ are linearly independent. That the answer is negative has been shown already by the authors of Refs. 4 and 5. In particular, it is evident that, if a transformation $S$ can be found such that

$$S|p\rangle = |p\rangle, \tag{6}$$

then there will be linear dependences, because from Eq. (6) one immediately arrives at

$$0 = \langle\lambda|(1-S)|p\rangle = \sum_{\lambda'} (\delta_{\lambda\lambda'} - S_{\lambda\lambda'}) f_{\lambda'}(p). \tag{7}$$

We have found a one-parameter class of transformations $S(z)$, which leaves the vectors $|p\rangle$ invariant. In this Letter we shall exhibit the operator $S(z)$, prove that $S(z)|p\rangle = |p\rangle$, and show that $S(z)$ generates all the linear dependences among the vertex functions $f_\lambda(p)$, at least in the context of Refs. 4 and 5. The operator $S(z)$ is given by

$$S(z) = S_s(z) S_a(z) S_\pi(z), \tag{8}$$

where

$$S_s(z) = (1-z)^{\Pi^2/2}, \text{ with } \Pi = \sum_{i=0}^{r+1} p_i, \tag{9}$$

$$S_\pi(z) = \exp\left(\sum_n \frac{\Pi a^{(n)} z^n}{\sqrt{n}}\right), \tag{10}$$

and

$$S_a(z) = :\exp(\sum_{ij} d_{ij}(z) a^{\dagger(j)} a^{(i)}):, \tag{11}$$

with

$$d_{ij}(z) = \left(\frac{i}{j}\right)^{1/2} \binom{i-1}{j-1} z^{i-j} (1-z)^j - \delta_{ij}.$$

The operator $S_a(z)$ can also be characterized by

$$S_a(z) a_\mu^{\dagger(i)} S_a^{-1}(z) = \sum_j \left(\frac{i}{j}\right)^{1/2} a_\mu^{\dagger(j)} \binom{i-1}{j-1} z^{i-j} (1-z)^j. \tag{12}$$

It can be shown that the operator $S$ can be written as

$$S = \exp[y(dS/dz)|_{z=0}], \text{ with } y = -\ln(1-z), \tag{13}$$

and

$$(dS/dz)|_{z=0} = -\tfrac{1}{2}\Pi^2 + \sum_i a^{\dagger(i)}\{[i(i+1)]^{1/2}a^{(i+1)} - ia^{(i)}\} + \Pi a^{(1)}. \tag{14}$$

One readily notes that this form is characteristic of that of a gauge transformation. By following Fubini and Veneziano, who called "generalized Ward identities" the linear relations among the functions $f_\lambda(p)$, we can say that $S$ is the "generalized gauge transformation" under which the theory is invariant.

At this point we could proceed directly to prove that $S(z)|p\rangle = |p\rangle$, but we believe that, before demonstrating this equality, it is pertinent to comment on the meaning of the transformation $S(z)$ and the way through which it is obtained.

The $N$-point function, for the configuration of Fig. 1(a), can be written as

$$A_{r+1,s+1} = \int dz\, z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}\int dx d\bar{y}\, \varphi(x,p)\varphi(\bar{y},\bar{q})F(x,p,\bar{y},\bar{q},z). \tag{15}$$

An expansion of $F(x,p,\bar{y},\bar{q},z)$ in powers of $z$ allows one to extract the residues of the poles. Alternatively, if we had started from the conjugate configuration, i.e., of Fig. 1(b), we would have written

$$A_{r+1,s+1} = \int dz\, z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}\int d\bar{x}dy\, \varphi(\bar{x},\bar{p})\varphi(y,q)F(\bar{x},\bar{p},y,q,z). \tag{16}$$

It can be shown that the transformation $T$, defined by

$$\rho_i \xrightarrow{T} \bar{\rho}_i = 1-\rho_{r+1-i}, \tag{17}$$

transforms $dx\,\varphi(x,p)$ into $d\bar{x}\,\varphi(\bar{x},\bar{p})$. In Eq. (15), by changing integration variables according to $T$ for the $x$'s and the analogous transformation for the $\bar{y}$'s, we obtain

$$A_{r+1,s+1} = \int dz\, z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}\int d\bar{x}dy\, \varphi(\bar{x},\bar{p})\varphi(y,q)G(\bar{x},\bar{p},y,q,z). \tag{18}$$

The function $G(\bar{x},\bar{p},y,q,z) = F(x(\bar{x}),p(\bar{p}),\bar{y}(y),\bar{q}(q),z)$ is not equal to $F(\bar{x},\bar{p},y,q,z)$. The expressions in the right-hand side of Eqs. (15) and (16), and therefore also those of Eqs. (18) and (16), give the same amplitude $A_{r+1,s+1}$. There must be relations among the coefficients of the expansions of $\int d\bar{x}dy\,\varphi(\bar{x},p) \times\varphi(y,q)F(\bar{x},\bar{p},y,q,z)$ and those of $\int d\bar{x}dy\,\varphi(\bar{x},p)\varphi(y,q)G(\bar{x},\bar{p},y,q,z)$ in powers of $z$.

To obtain these relations, let us first examine how we prove that Eqs. (15) and (16) give origin to the same $N$-point function $A_{r+1,s+1}$. It can be easily shown that the transformation $U$, defined by

$$\rho_i \xrightarrow{U} \bar{\rho}_i = (1-\rho_{r+1-i})/(1-z\rho_{r+1-i}), \tag{19}$$

plus the analogous transformation operating on $\bar{y}$'s, transforms the integrand of Eq. (15) into the integrand of Eq. (16), i.e.,

$$dxd\bar{y}\varphi(x(\bar{x}),p(\bar{p}))\varphi(\bar{y}(y),\bar{q}(q))F(x(\bar{x}),p(\bar{p}),\bar{y}(y),\bar{q}(q),z) = d\bar{x}dy\varphi(\bar{x},\bar{p})\varphi(y,q)F(\bar{x},\bar{p},y,q,z). \tag{20}$$

This proves the equality of the two integrated expressions.

At this point, one's interest is immediately drawn to the transformation $S=U^{-1}T$, which has the advantage of sending variables pertaining to an initial configuration into themselves, and not into the conjugate ones. Under $U^{-1}T$,

$$\rho \to \rho' = (1-z)\rho/(1-z\rho). \tag{21}$$

Let us relabel the integration variables in Eq. (4):

$$|p\rangle = \int dx'\,\varphi(x',p)\exp\Big(\sum_{n=1}^{\infty}\sum_{i=0}^{r}\rho_i'^{\,n}p_i\frac{a^{\dagger(n)}}{\sqrt{n}}\Big)|\varphi\rangle,$$

and change integration variables according to Eq. (21). We get

$$|p\rangle = \int dx\,\varphi(x'(x),p)\frac{\partial x'}{\partial x}\exp\Big(\sum_{n=1}^{\infty}\sum_{i=0}^{r}[\rho_i'(\rho)]^n p_i\frac{a^{\dagger(n)}}{\sqrt{n}}\Big)|\varphi\rangle. \tag{22}$$

It can be easily checked that

$$S(z)dx\,\varphi(x,p)\exp\Big(\sum_{n=1}^{\infty}\sum_{i=0}^{r}\rho_i^{\,n}p_i\frac{a^{\dagger(n)}}{\sqrt{n}}\Big)|\varphi\rangle = dx\,\varphi(x'(x),p)\frac{\partial x'}{\partial x}\exp\Big\{\sum_{n=1}^{\infty}\sum_{i=0}^{r}[\rho_i'(\rho)]^n p_i\frac{a^{\dagger(n)}}{\sqrt{n}}\Big\}|\varphi\rangle; \tag{23}$$

therefore,

$$S(z)|p\rangle = S(z)\int dx\,\varphi(x,p)\exp\Big(\sum_{n=1}^{\infty}\sum_{i=0}^{r}\rho_i{}^n p_i\frac{a^{\dagger(n)}}{\sqrt{n}}\Big)|\varphi\rangle$$

$$=\int dx'\,\varphi(x',p)\exp\Big(\sum_{n=1}^{\infty}\sum_{i=0}^{r}\rho_i{}'^n p_i\frac{a^{\dagger(n)}}{\sqrt{n}}\Big)|\varphi\rangle = |p\rangle, \tag{24}$$

as we wanted to prove.

Next we list some useful properties of the operators $S_s(z)$, $S_a(z)$, and $S_\pi(z)$ as defined in Eqs. (8)-(12). We define $z'$ by $z+z'-zz'=0$. Then

$$S_s(z)^{-1}=S_s(z'),\quad S_a(z)^{-1}=S_a(z'),\quad [S_a(z)S_\pi(z)]^{-1}=S_a(z')S_\pi(z'); \tag{25}$$

it follows that

$$S(z)^{-1}=S(z'). \tag{26}$$

The adjoint operators can be obtained immediately from Eqs. (9)-(12). In particular,

$$S_a{}^\dagger(z)a_\mu{}^{\dagger(i)}[S_a{}^\dagger(z)]^{-1}=\sum_j\Big(\frac{j}{i}\Big)^{1/2}a_\mu{}^{\dagger(j)}\binom{j-1}{i-1}z^{*j-i}(1-z^*)^i. \tag{27}$$

The linear dependences among vertex functions $f_\lambda(p)$ can be extracted from

$$\langle p|[1-S^\dagger(z)]|\lambda\rangle=0. \tag{28}$$

When $\langle p|a\rangle=0$, we shall say that the vector $|a\rangle$ is equivalent to zero, and write $|a\rangle\sim 0$. Equation (28) is valid for all $z$. From Eq. (13) we can cast this relation as follows:

$$(d/dz)S^\dagger(z)|_{z=0}|\lambda\rangle\sim 0. \tag{29}$$

These constitute all the linear relations among the vertex functions. It is amusing to note that by choosing

$$|\lambda\rangle=(n!)^{-1/2}a_{\mu_1}{}^{(1)+}\cdots a_{\mu_n}{}^{(1)+}|\varphi\rangle, \tag{30}$$

from Eq. (29) one reproduces those linear dependences discussed explicity in Ref. 4.

Finally, let us say a few words about the passage from the configuration of Fig. 1(a) to the conjugate configuration of Fig. 1(b). It has been shown by Caneschi, Schwimmer, and Veneziano[7] that

$$|\bar{p}\rangle=\Omega|p\rangle$$

with

$$\Omega=\exp(\textstyle\sum_n \Pi a^{(n)\dagger}/\sqrt{n})T \tag{31}$$

and $T$ defined by

$$Ta_\mu{}^{(i)\dagger}T^{-1}=\sum_j\Big(\frac{i}{j}\Big)^{1/2}a_\mu{}^{(j)\dagger}(-)^i\binom{j}{i}. \tag{32}$$

One has

$$T^2=1 \text{ and } \Omega^2=1. \tag{33}$$

Also,

$$T\exp(\textstyle\sum_n\Pi a^{(n)\dagger}/\sqrt{n})T^{-1}=\exp(-\sum_n\Pi a^{(n)\dagger}/\sqrt{n}). \tag{34}$$

By the insertion of the "twisting" operator $\Omega$ in Eq. (3), one has

$$A_{r+1,s+1}=\langle\bar{q}|\Omega^\dagger\Omega^\dagger\int dz\,z^{R-\alpha(s)-1}(1-z)^{\alpha(0)-1}\Omega\Omega|p\rangle=\langle q|\int dz\,z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}\Omega^\dagger z^R\Omega|\bar{p}\rangle. \tag{35}$$

This must be identical to the expression

$$A_{r+1,s+1}=\langle q|\int dz\,z^{-\alpha(s)-1}(1-z)^{\alpha(0)-1}z^R|\bar{p}\rangle \tag{3}$$

which we wrote starting directly from the configuration of Fig. 1(b). However, the direct proof for the equality of Eqs. (35) and (3) is by no means trivial, since the operators $z^R$ and $\Omega^\dagger z^R \Omega$ are different.

We shall proceed to show that these two operators are equal up to an $S(z)$ transformation (i.e., up to a generalized gauge transformation). In particular, we shall show that

$$S^\dagger(z)\Omega^\dagger z^R \Omega S(z) = z^R. \tag{36}$$

First we observe that

$$\begin{aligned}\Omega S(z) &= T\exp\Big(-\sum_n \frac{\Pi a^{(n)\dagger}}{\sqrt{n}}\Big) S_s(z) S_a(z) \exp\Big(\sum_n \frac{\Pi a^{(n)} z^n}{\sqrt{n}}\Big)\\ &= S_s(z) T S_a(z) \exp\Big(-\sum_n \frac{\Pi a^{(n)\dagger}}{\sqrt{n}}\Big)\exp\Big(\sum_n \frac{\Pi a^{(n)} z^n}{\sqrt{n}}\Big).\end{aligned} \tag{37}$$

By use of the identity

$$\sum_k (-)^k \binom{i}{k}\binom{i+j-k-1}{j-k} z^{j-k} = \sum_k (-)^k \binom{i}{k}\binom{j-1}{k-1} z^{j-k}(1-z)^k, \tag{38}$$

one can then show that

$$TS_a(z) = U_a(z), \tag{39}$$

where $U_a(z)$ is defined by $U_a(z)|\varphi\rangle = |\varphi\rangle$ and

$$U_a(z) a_\mu^{(i)\dagger} U_a(z)^{-1} = \sum_j a_\mu^{(j)\dagger}(ij)^{1/2} \sum_k \frac{(-)^k (i+j-k-1)!\, z^{i-k}}{k!\,(i-k)!\,(j-k)!}. \tag{40}$$

From the identity

$$\sum_{k=1}^\infty (ij)^{1/2} k z^k \Big(\sum_l \frac{(-)^l (i+k-l-1)!\, z^{i-l}}{l!\,(i-l)!\,(k-l)!}\Big)\Big(\sum_l \frac{(-)^l (j+k-l-1)!\, z^{j-l}}{l!\,(j-l)!\,(k-l)!}\Big) = \delta_{ij} z^i, \tag{41}$$

it follows that

$$U_a^\dagger(z) z^R U_a(z) = z^R. \tag{42}$$

At this point it is immediate to check that

$$(1-z)^{\Pi^2} \exp\Big[\sum_n \frac{\Pi a^{(n)\dagger} z^n}{\sqrt{n}}\Big] \exp\Big[-\sum_n \frac{\Pi a^{(n)}}{\sqrt{n}}\Big] z^R \exp\Big[-\sum_n \frac{\Pi a^{(n)\dagger}}{\sqrt{n}}\Big] \exp\Big[\sum_n \frac{\Pi a^{(n)} z^n}{\sqrt{n}}\Big] = z^R, \tag{43}$$

which completes the proof.

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