Supporting Information for "Analytical Gradients for Molecular-Orbital-Based Machine Learning"

Sebastian J. R. Lee,[1] Tamara Husch,[1] Feizhi Ding,[1] and Thomas F. Miller III[1, a)]
Division of Chemistry and Chemical Engineering, California Institute of Technology, Pasadena, CA 91125, USA

(Dated: 17 February 2021)

I. DATA GENERATION

We directly adopted the coordinates of the QM7b-T (i), and the ISO17 data set (ii) from Ref. 1, and Ref. 2, respectively.

II. FEATURE DERIVATIVES

Tables S1–S3 specify how the partial derivative of the diagonal feature vector with respect to the Fock matrix and two center molecular orbital integrals are computed. Tables S4–S6 specify how the partial derivative of the off-diagonal feature vector with respect to the Fock matrix and two center molecular orbital integrals are computed.

TABLE S1. Partial derivative of the diagonal feature vector of FS 3 with respect to Fock matrix elements $\partial \mathbf{f}_i / \partial F_{pq}$.

Contributing feature values	Partial derivative
$\partial F_{ii}/\partial F_{pq}$	$\delta_{pi}\delta_{qi}$
$\partial \mathbf{F}_{ik}/\partial F_{pq}$	$\delta_{pi}\delta_{qk}\frac{F_{ik}}{\lvert F_{ik}\rvert}$
$\partial \mathbf{F}_{aa}/\partial F_{pq}$	$\delta_{pa}\delta_{qa}\langle ii\vert aa\rangle^3$
$\partial \mathbf{F}_{ab}/\partial F_{pq}$	$\langle aa\vert bb\rangle \Big(\delta_{pa}\delta_{qb}\frac{F_{ab}}{\lvert F_{ab}\rvert}\frac{\langle ii\vert aa\rangle}{2(F_{aa}-F_{ii})}\frac{\langle ii\vert bb\rangle}{2(F_{bb}-F_{ii})} - \lvert F_{ab}\rvert \frac{2(\delta_{pa}\delta_{qa}-\delta_{pi}\delta_{qi})\langle ii\vert aa\rangle}{(2F_{aa}-2F_{ii})^2}\frac{\langle ii\vert bb\rangle}{2(F_{bb}-F_{ii})}$ $-\lvert F_{ab}\rvert \frac{\langle ii\vert aa\rangle}{2(F_{aa}-F_{ii})}\frac{2(\delta_{pb}\delta_{qb}-\delta_{pi}\delta_{qi})\langle ii\vert bb\rangle}{(2F_{bb}-2F_{ii})^2}\Big)$
$\partial [\boldsymbol{K}^{aa}]_{bb}/\partial F_{pq}$	$-\langle aa\vert bb\rangle^4 \frac{2(\delta_{pa}\delta_{qa}-\delta_{pi}\delta_{qi})\langle ii\vert aa\rangle}{(2F_{aa}-2F_{ii})^2}\frac{\langle ii\vert bb\rangle}{2(F_{bb}-F_{ii})} - \langle aa\vert bb\rangle^4 \frac{\langle ii\vert aa\rangle}{2(F_{aa}-F_{ii})}\frac{2(\delta_{pb}\delta_{qb}-\delta_{pi}\delta_{qi})\langle ii\vert bb\rangle}{(2F_{bb}-2F_{ii})^2}$
$\partial [\boldsymbol{K}^{ab}]_{ab}/\partial F_{pq}$	$-\langle ab\vert ab\rangle^2 \frac{2(\delta_{pa}\delta_{qa}-\delta_{pi}\delta_{qi})\langle ia\vert ia\rangle}{(2F_{aa}-2F_{ii})^2}\frac{\langle ib\vert ib\rangle}{2(F_{bb}-F_{ii})} - \langle ab\vert ab\rangle^2 \frac{\langle ia\vert ia\rangle}{2(F_{aa}-F_{ii})}\frac{2(\delta_{pb}\delta_{qb}-\delta_{pi}\delta_{qi})\langle ib\vert ib\rangle}{(2F_{bb}-2F_{ii})^2}$

TABLE S2. Partial derivative of the diagonal feature vector of FS 3 with respect to Coulomb-type two-center molecular orbital integrals $\partial \mathbf{f}_i / \partial [\boldsymbol{K}^{pp}]_{qq}$.

Contributing feature values	Partial derivative
$\partial \mathbf{F}_{aa}/\partial [\boldsymbol{K}^{pp}]_{qq}$	$3\delta_{pi}\delta_{qa}F_{aa}\langle ii\vert aa\rangle^2$
$\partial \mathbf{F}_{ab}/\partial [\boldsymbol{K}^{pp}]_{qq}$	$\lvert F_{ab}\rvert \frac{\delta_{pi}\delta_{qa}}{2(F_{aa}-F_{ii})}\langle aa\vert bb\rangle \frac{\langle ii\vert bb\rangle}{2(F_{bb}-F_{ii})} + \lvert F_{ab}\rvert \frac{\langle ii\vert aa\rangle}{2(F_{aa}-F_{ii})}\delta_{pa}\delta_{qb}\frac{\langle ii\vert bb\rangle}{2(F_{bb}-F_{ii})} + \lvert F_{ab}\rvert \frac{\langle ii\vert aa\rangle}{2(F_{aa}-F_{ii})}\langle aa\vert bb\rangle \frac{\delta_{pi}\delta_{qb}}{2(F_{bb}-F_{ii})}$
$\partial [\boldsymbol{K}^{ii}]_{ii}/\partial [\boldsymbol{K}^{pp}]_{qq}$	$3\delta_{pi}\delta_{qi}\langle ii\vert ii\rangle^2$
$\partial [\boldsymbol{K}^{ii}]_{kk}/\partial [\boldsymbol{K}^{pp}]_{qq}$	$3\delta_{pi}\delta_{qk}\langle ii\vert kk\rangle^2$
$\partial [\boldsymbol{K}^{ii}]_{aa}/\partial [\boldsymbol{K}^{pp}]_{qq}$	$3\delta_{pi}\delta_{qa}\langle ii\vert aa\rangle^2$
$\partial [\boldsymbol{K}^{aa}]_{aa}/\partial [\boldsymbol{K}^{pp}]_{qq}$	$3\delta_{pa}\delta_{qa}\langle aa\vert aa\rangle^2\langle ii\vert aa\rangle^3 + 3\delta_{pi}\delta_{qa}\langle aa\vert aa\rangle^3\langle ii\vert aa\rangle^2$
$\partial [\boldsymbol{K}^{aa}]_{bb}/\partial [\boldsymbol{K}^{pp}]_{qq}$	$4\delta_{pa}\delta_{qb}\langle aa\vert bb\rangle^3 \frac{\langle ii\vert aa\rangle}{2(F_{aa}-F_{ii})}\frac{\langle ii\vert bb\rangle}{2(F_{bb}-F_{ii})} + \langle aa\vert bb\rangle^4 \frac{\delta_{pi}\delta_{qa}}{2(F_{aa}-F_{ii})}\frac{\langle ii\vert bb\rangle}{2(F_{bb}-F_{ii})} + \langle aa\vert bb\rangle^4 \frac{\langle ii\vert aa\rangle}{2(F_{aa}-F_{ii})}\frac{\delta_{pi}\delta_{qb}}{2(F_{bb}-F_{ii})}$

a)Electronic mail: tfm@caltech.edu.

TABLE S3. Partial derivative of the diagonal feature vector of FS 3 with respect to exchange-type two-center molecular orbital integrals $\partial \mathbf{f}_i / \partial [\boldsymbol{K}^{pq}]_{pq}$.

Contributing feature values	Partial derivative
$\partial [\boldsymbol{K}^{ik}]_{ik} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$\delta_{pi}\delta_{qk}$
$\partial [\boldsymbol{K}^{ia}]_{ia} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$\delta_{pi}\delta_{qa}$
$\partial [\boldsymbol{K}^{ab}]_{ab} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$2\delta_{pa}\delta_{qb} \langle ab \vert ab \rangle \frac{\langle ia \vert ia \rangle}{2(F_{aa}-F_{ii})} \frac{\langle ib \vert ib \rangle}{2(F_{bb}-F_{ii})} + \langle ab \vert ab \rangle^2 \frac{\delta_{pi}\delta_{qa}}{2(F_{aa}-F_{ii})} \frac{\langle ib \vert ib \rangle}{2(F_{bb}-F_{ii})} + \langle ab \vert ab \rangle^2 \frac{\langle ia \vert ia \rangle}{2(F_{aa}-F_{ii})} \frac{\delta_{pi}\delta_{qb}}{2(F_{bb}-F_{ii})}$

TABLE S4. Partial derivative of the off-diagonal feature vector of FS 3 with respect to Fock matrix elements $\partial \mathbf{f}_i / \partial F_{pq}$.

Contributing feature values	Partial derivative
$\partial F_{\tilde{i}\tilde{i}} / \partial F_{pq}$	$G_{ij} \left(\frac{1}{2}\delta_{pi}\delta_{qi} + \frac{1}{2}\delta_{pj}\delta_{qj} + \delta_{pi}\delta_{qj} \frac{F_{ij}}{\lvert F_{ij} \rvert} \right)$
$\partial F_{\tilde{i}\tilde{j}} / \partial F_{pq}$	$G_{ij} \left(\frac{1}{2}\delta_{pi}\delta_{qi} - \frac{1}{2}\delta_{pj}\delta_{qj} \right) \frac{F_{ii}-F_{jj}}{\lvert F_{ii}-F_{jj} \rvert}$
$\partial F_{\tilde{j}\tilde{j}} / \partial F_{pq}$	$G_{ij} \left(\frac{1}{2}\delta_{pi}\delta_{qi} + \frac{1}{2}\delta_{pj}\delta_{qj} - \delta_{pi}\delta_{qj} \frac{F_{ij}}{\lvert F_{ij} \rvert} \right)$
$\partial \mathbf{F}_{\tilde{i}k} / \partial F_{pq}$	$G_{ij} \left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qk} \frac{F_{ik}}{\lvert F_{ik} \rvert} + \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qk} \frac{F_{jk}}{\lvert F_{jk} \rvert} \right)$
$\partial \mathbf{F}_{\tilde{j}k} / \partial F_{pq}$	$G_{ij} \left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qk} \frac{F_{ik}}{\lvert F_{ik} \rvert} - \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qk} \frac{F_{jk}}{\lvert F_{jk} \rvert} \right) \frac{F_{ik}-F_{jk}}{\lvert F_{ik}-F_{jk} \rvert}$
$\partial \mathbf{F}_{aa} / \partial F_{pq}$	$G_{ij} \left(\delta_{pa}\delta_{qa} \lvert \langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3 \rvert / 2 \right)$
$\partial \mathbf{F}_{ab} / \partial F_{pq}$	$G_{ij} \langle aa \vert bb \rangle \frac{\frac{1}{2}\lvert \langle ii \vert aa \rangle - \langle jj \vert aa \rangle \rvert}{2F_{aa}-F_{ii}-F_{jj}} \frac{\frac{1}{2}\lvert \langle ii \vert bb \rangle - \langle jj \vert bb \rangle \rvert}{2F_{bb}-F_{ii}-F_{jj}} \left(\delta_{pa}\delta_{qb} \frac{F_{ab}}{\lvert F_{ab} \rvert} - \lvert F_{ab} \rvert \frac{2\delta_{pa}\delta_{qa}-\delta_{pi}\delta_{qi}-\delta_{pj}\delta_{qj}}{2F_{aa}-F_{ii}-F_{jj}} - \lvert F_{ab} \rvert \frac{2\delta_{pb}\delta_{qb}-\delta_{pi}\delta_{qi}-\delta_{pj}\delta_{qj}}{2F_{bb}-F_{ii}-F_{jj}} \right)$
$\partial [\boldsymbol{K}^{aa}]_{bb} / \partial F_{pq}$	$-G_{ij} \langle aa \vert bb \rangle^4 \frac{\frac{1}{2}\lvert \langle ii \vert aa \rangle - \langle jj \vert aa \rangle \rvert}{2F_{aa}-F_{ii}-F_{jj}} \frac{\frac{1}{2}\lvert \langle ii \vert bb \rangle - \langle jj \vert bb \rangle \rvert}{2F_{bb}-F_{ii}-F_{jj}} \left(\frac{2\delta_{pa}\delta_{qa}-\delta_{pi}\delta_{qi}-\delta_{pj}\delta_{qj}}{2F_{aa}-F_{ii}-F_{jj}} + \frac{2\delta_{pb}\delta_{qb}-\delta_{pi}\delta_{qi}-\delta_{pj}\delta_{qj}}{2F_{bb}-F_{ii}-F_{jj}} \right)$
$\partial [\boldsymbol{K}^{ab}]_{ab} / \partial F_{pq}$	$-G_{ij} \langle ab \vert ab \rangle^2 \frac{\frac{1}{2}\lvert \langle ia \vert ia \rangle - \langle ja \vert ja \rangle \rvert}{2F_{aa}-F_{ii}-F_{jj}} \frac{\frac{1}{2}\lvert \langle ib \vert ib \rangle - \langle jb \vert jb \rangle \rvert}{2F_{bb}-F_{ii}-F_{jj}} \left(\frac{2\delta_{pa}\delta_{qa}-\delta_{pi}\delta_{qi}-\delta_{pj}\delta_{qj}}{2F_{aa}-F_{ii}-F_{jj}} + \frac{2\delta_{pb}\delta_{qb}-\delta_{pi}\delta_{qi}-\delta_{pj}\delta_{qj}}{2F_{bb}-F_{ii}-F_{jj}} \right)$

TABLE S5. Partial derivative of the off-diagonal feature vector of FS 3 with respect to Coulomb-type two-center molecular orbital integrals $\partial \mathbf{f}_i / \partial [\boldsymbol{K}^{pp}]_{qq}$.

Contributing feature values	Partial derivative
$\partial \mathbf{F}_{aa} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij}F_{aa} \left(\frac{1}{2}\delta_{pi}\delta_{qa} \langle ii \vert aa \rangle^2 - \frac{1}{2}\delta_{pj}\delta_{qa} \langle jj \vert aa \rangle^2 \right) \frac{\langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3}{\lvert \langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3 \rvert}$
$\partial \mathbf{F}_{ab} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$G_{ij} \frac{\frac{1}{2}\lvert \langle ii \vert aa \rangle - \langle jj \vert aa \rangle \rvert}{2F_{aa}-F_{ii}-F_{jj}} \frac{\frac{1}{2}\lvert \langle ii \vert bb \rangle - \langle jj \vert bb \rangle \rvert}{2F_{bb}-F_{ii}-F_{jj}} \lvert F_{ab} \rvert \left(\frac{(\delta_{pi}\delta_{qa}-\delta_{pj}\delta_{qa})(\langle ii \vert aa \rangle - \langle jj \vert aa \rangle)}{\lvert \langle ii \vert aa \rangle - \langle jj \vert aa \rangle \rvert^2} \langle aa \vert bb \rangle + \delta_{pa}\delta_{qb} + \langle aa \vert bb \rangle \frac{(\delta_{pi}\delta_{qb}-\delta_{pj}\delta_{qb})(\langle ii \vert bb \rangle - \langle jj \vert bb \rangle)}{\lvert \langle ii \vert bb \rangle - \langle jj \vert bb \rangle \rvert^2} \right)$
$\partial [\boldsymbol{K}^{\tilde{i}\tilde{i}}]_{\tilde{i}\tilde{i}} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\frac{1}{2}\delta_{pi}\delta_{qi} \langle ii \vert ii \rangle^2 + \frac{1}{2}\delta_{pj}\delta_{qj} \langle jj \vert jj \rangle^2 + \delta_{pi}\delta_{qj} \langle ii \vert jj \rangle^2 \right)$
$\partial [\boldsymbol{K}^{\tilde{i}\tilde{i}}]_{\tilde{j}\tilde{j}} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\frac{1}{2}\delta_{pi}\delta_{qi} \langle ii \vert ii \rangle^2 - \frac{1}{2}\delta_{pj}\delta_{qj} \langle jj \vert jj \rangle^2 \right) \frac{\langle ii \vert ii \rangle^3 - \langle jj \vert jj \rangle^3}{\lvert \langle ii \vert ii \rangle^3 - \langle jj \vert jj \rangle^3 \rvert}$
$\partial [\boldsymbol{K}^{\tilde{j}\tilde{j}}]_{\tilde{j}\tilde{j}} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\frac{1}{2}\delta_{pi}\delta_{qi} \langle ii \vert ii \rangle^2 + \frac{1}{2}\delta_{pj}\delta_{qj} \langle jj \vert jj \rangle^2 - \delta_{pi}\delta_{qj} \langle ii \vert jj \rangle^2 \right)$
$\partial [\boldsymbol{K}^{\tilde{i}\tilde{i}}]_{kk} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qk} \langle ii \vert kk \rangle^2 + \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qk} \langle jj \vert kk \rangle^2 \right)$
$\partial [\boldsymbol{K}^{\tilde{j}\tilde{j}}]_{kk} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qk} \langle ii \vert kk \rangle^2 - \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qk} \langle jj \vert kk \rangle^2 \right) \frac{\langle ii \vert kk \rangle^3 - \langle jj \vert kk \rangle^3}{\lvert \langle ii \vert kk \rangle^3 - \langle jj \vert kk \rangle^3 \rvert}$
$\partial [\boldsymbol{K}^{\tilde{i}\tilde{i}}]_{aa} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qa} \langle ii \vert aa \rangle^2 + \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qa} \langle jj \vert aa \rangle^2 \right)$
$\partial [\boldsymbol{K}^{\tilde{j}\tilde{j}}]_{aa} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qa} \langle ii \vert aa \rangle^2 - \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qa} \langle jj \vert aa \rangle^2 \right) \frac{\langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3}{\lvert \langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3 \rvert}$
$\partial [\boldsymbol{K}^{aa}]_{aa} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$3G_{ij} \left(\delta_{pa}\delta_{qa} \langle aa \vert aa \rangle^2 \frac{1}{2} \lvert \langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3 \rvert + \langle aa \vert aa \rangle^3 \frac{1}{2} (\delta_{pi}\delta_{qa} \langle ii \vert aa \rangle^2 - \delta_{pj}\delta_{qa} \langle jj \vert aa \rangle^2) \frac{\langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3}{\lvert \langle ii \vert aa \rangle^3 - \langle jj \vert aa \rangle^3 \rvert} \right)$
$\partial [\boldsymbol{K}^{aa}]_{bb} / \partial [\boldsymbol{K}^{pp}]_{qq}$	$G_{ij} \frac{\frac{1}{2}\lvert \langle ii \vert aa \rangle - \langle jj \vert aa \rangle \rvert}{2F_{aa}-F_{ii}-F_{jj}} \frac{\frac{1}{2}\lvert \langle ii \vert bb \rangle - \langle jj \vert bb \rangle \rvert}{2F_{bb}-F_{ii}-F_{jj}} \left(4 \langle aa \vert bb \rangle^3 \delta_{pa}\delta_{qb} + \langle aa \vert bb \rangle^4 \frac{(\delta_{pi}\delta_{qa}-\delta_{pj}\delta_{qa})(\langle ii \vert aa \rangle - \langle jj \vert aa \rangle)}{\lvert \langle ii \vert aa \rangle - \langle jj \vert aa \rangle \rvert^2} + \langle aa \vert bb \rangle^4 \frac{(\delta_{pi}\delta_{qb}-\delta_{pj}\delta_{qb})(\langle ii \vert bb \rangle - \langle jj \vert bb \rangle)}{\lvert \langle ii \vert bb \rangle - \langle jj \vert bb \rangle \rvert^2} \right)$

TABLE S6. Partial derivative of the off-diagonal feature vector of FS 3 with respect to exchange-type two-center molecular orbital integrals $\partial \mathbf{f}_i / \partial [\boldsymbol{K}^{pq}]_{pq}$.

Contributing feature values	Partial derivative
$\partial [\boldsymbol{K}^{\tilde{i}\tilde{j}}]_{\tilde{i}\tilde{j}} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$G_{ij}\left(\frac{1}{2}\delta_{pi}\delta_{qi} - \frac{1}{2}\delta_{pj}\delta_{qj}\right)\frac{\langle ii\|ii\rangle - \langle jj\|jj\rangle}{\|\langle ii\|ii\rangle - \langle jj\|jj\rangle\|}$
$\partial [\boldsymbol{K}^{\tilde{i}k}]_{\tilde{i}k} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$G_{ij}\left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qk} + \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qk}\right)$
$\partial [\boldsymbol{K}^{\tilde{j}k}]_{\tilde{j}k} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$G_{ij}\left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qk} - \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qk}\right)\frac{\langle ik\|ik\rangle - \langle jk\|jk\rangle}{\|\langle ik\|ik\rangle - \langle jk\|jk\rangle\|}$
$\partial [\boldsymbol{K}^{\tilde{i}a}]_{\tilde{i}a} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$G_{ij}\left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qa} + \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qa}\right)$
$\partial [\boldsymbol{K}^{\tilde{j}a}]_{\tilde{j}a} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$G_{ij}\left(\frac{1}{\sqrt{2}}\delta_{pi}\delta_{qa} - \frac{1}{\sqrt{2}}\delta_{pj}\delta_{qa}\right)\frac{\langle ia\|ia\rangle - \langle ja\|ja\rangle}{\|\langle ia\|ia\rangle - \langle ja\|ja\rangle\|}$
$\partial [\boldsymbol{K}^{ab}]_{ab} / \partial [\boldsymbol{K}^{pq}]_{pq}$	$G_{ij}\frac{\frac{1}{2}\|\langle ia\|ia\rangle - \langle ja\|ja\rangle\|}{2F_{aa}-F_{ii}-F_{jj}}\frac{\frac{1}{2}\|\langle ib\|ib\rangle - \langle jb\|jb\rangle\|}{2F_{bb}-F_{ii}-F_{jj}}\Big(2\langle ab\|ab\rangle\,\delta_{pa}\delta_{qb} + \langle ab\|ab\rangle^2\frac{(\delta_{pi}\delta_{qa}-\delta_{pj}\delta_{qa})(\langle ia\|ia\rangle - \langle ja\|ja\rangle)}{\|\langle ia\|ia\rangle - \langle ja\|ja\rangle\|^2}$ $+\langle ab\|ab\rangle^2\frac{(\delta_{pi}\delta_{qb}-\delta_{pj}\delta_{qb})(\langle ib\|ib\rangle - \langle jb\|jb\rangle)}{\|\langle ib\|ib\rangle - \langle jb\|jb\rangle\|^2}\Big)$

TABLE S7. Partial derivative of the off-diagonal feature vector of FS 3 with respect to the centroid distance between orbital p and orbital q $\partial \mathbf{f}_{ij} / \partial R^n_{pq}$.

Contributing feature values	Partial derivative
$\partial \mathbf{f}_{ij} / \partial R^n_{pq}$	$-\frac{R_{ij}^4}{R_0^6}G_{ij}\mathbf{f}_{ij}(R^n_{ii} - R^n_{jj})(\delta_{pi}\delta_{qi} - \delta_{pj}\delta_{qj})$

III. ANALYSIS OF OPTIMIZED ISO17 STRUCTURES

As stated in the main text we optimized the constitutional isomers in ISO17 with MP2, with MOB-ML, with HF, and with B3LYP-D3 and compared the resulting RMSD of each structure (Figure 1). Here we further analyze the quality of the structures by comparing bond length mean absolute errors (MAE) and angle MAE of each structure shown in Figure 1.

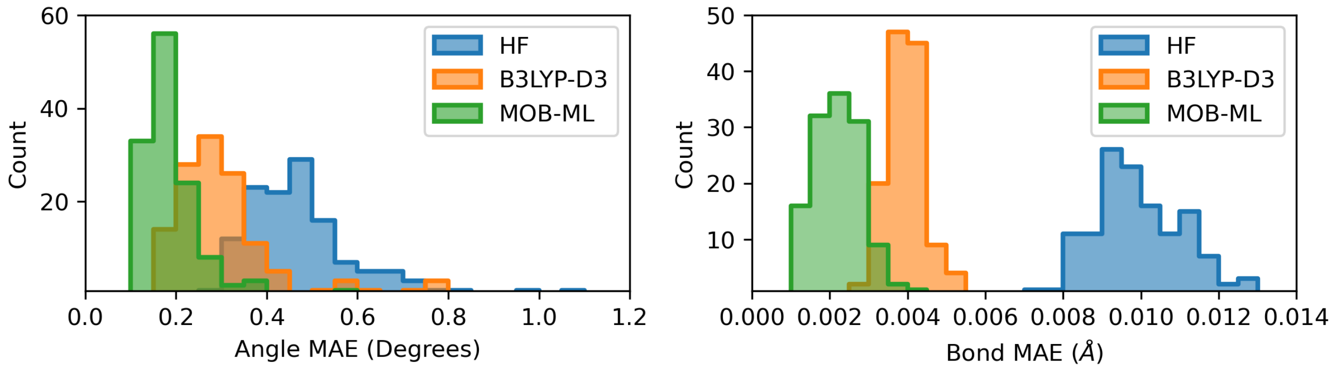

FIG. S1. Histogrammed angle mean absolute errors (left panel) and bond mean absolute errors (right panel) of HF structures (blue), B3LYP-D3 structures (orange), and MOB-ML structures (green) with respect to MP2 structures for the unique isomers in the ISO17 data set. The MOB-ML model was trained on 220 randomly selected QM7b-T structures as listed in Table 1 in the main text.

Figure S1 shows that the MOB-ML optimized structures are very similar to the reference MP2 optimized structures with a bond length MAE of 0.002 Å and an angle MAE of 0.19 degrees. The MOB-ML optimized structures are significantly closer to the reference MP2 structures than the HF-optimized structures which have an average bond length MAE of 0.010 Å and angle MAE of 0.48 degrees. Moreover, the MOB-ML structures are more similar to the reference MP2 structures than those obtained from B3LYP-D3. The B3LYP-D3 structures have an average bond length MAE of 0.004 Å and angle MAE of 0.31 degrees with respect to the MP2 reference structures.

[1] L. Cheng, M. Welborn, A. S. Christensen, and T. F. Miller III, "A universal density matrix functional from molecular orbital-based machine learning: Transferability across organic molecules," J. Chem. Phys. **150**, 131103 (2019).

[2] K. T. Schütt, P.-J. Kindermans, H. E. Sauceda, S. Chmiela, A. Tkatchenko, and K.-R. Müller, "SchNet: A continuous-filter convolutional neural network for modeling quantum interactions," arXiv:1706.08566 [physics, stat] (2017), arXiv: 1706.08566.