Supplemental Materials: Constraints on low-energy effective theories from weak cosmic censorship

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I. CORRECTIONS TO THE MAXWELL SOURCE AND STRESS TENSOR

We consider the most general fourth-derivative higher order corrections to Einstein-Maxwell theory, namely,

$$I = \int d^4x\, \sqrt{-g}\left(\frac{1}{2\kappa}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \Delta L\right) \tag{1}$$

where

$$\begin{aligned}\Delta L = \; & c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & + c_4RF_{\mu\nu}F^{\mu\nu} + c_5R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + c_6R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \\ & + c_7F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + c_8F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}\,.\end{aligned} \tag{2}$$

The field equations obtained by the variation of the action (1) with respect to A_μ and $g^{\mu\nu}$ are given respectively by

$$\nabla_\nu(F^{\mu\nu} - S^{\mu\nu}) = 0\,, \tag{3}$$

and

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} = \kappa(\tilde{T}_{\mu\nu} + \Delta T_{\mu\nu})\,. \tag{4}$$

In the above $\tilde{T}_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$ is the stress tensor of the Maxwell theory, and $\Delta T_{\mu\nu}$ and $S^{\mu\nu}$ are the corrections respectively to the stress tensor and Maxwell source field from the higher-dimension operators.

Here we list the details of the corrections to the Maxwell source field and stress tensor and , i.e., $S^{\mu\nu}$ in Eq. (9) and $\Delta T_{\mu\nu}$ in Eq. (10) of the main text:

$$\begin{aligned}S^{\mu\nu} = \; & 4c_4RF^{\mu\nu} + 2c_5(R^{\mu\rho}F_\rho{}^\nu - R^{\nu\rho}F_\rho{}^\mu) + 4c_6R^{\mu\nu\rho\sigma}F_{\rho\sigma} + \\ & + 8c_7F_{\rho\sigma}F^{\rho\sigma}F^{\mu\nu} + 8c_8F_{\rho\sigma}F^{\rho\nu}F^{\mu\sigma}\,,\end{aligned} \tag{5}$$

and

$$\begin{aligned}\Delta T_{\mu\nu} \;=\; & c_1\left(g_{\mu\nu}R^2 - 4RR_{\mu\nu} + 4\nabla_\nu\nabla_\mu R - 4g_{\mu\nu}\Box R\right) + \\ & + \; c_2\left(g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} + 4\nabla_\alpha\nabla_\nu R^\alpha_\mu - 2\Box R_{\mu\nu} - g_{\mu\nu}\Box R - 4R^\alpha_\mu R_{\alpha\nu}\right) + \\ & + \; c_3\Big(g_{\mu\nu}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 8\Box R_{\mu\nu} \\ & \quad + 4\nabla_\nu\nabla_\mu R + 8R^\alpha_\mu R_{\alpha\nu} - 8R^{\alpha\beta}R_{\mu\alpha\nu\beta}\Big) + \\ & + \; c_4\left(g_{\mu\nu}RF^2 - 4RF_\mu{}^\sigma F_{\nu\sigma} - 2F^2R_{\mu\nu} + 2\nabla_\mu\nabla_\nu F^2 - 2g_{\mu\nu}\Box F^2\right) + \\ & + \; c_5\Big(g_{\mu\nu}R^{\kappa\lambda}F_{\kappa\rho}F_\lambda{}^\rho - 4R_{\nu\sigma}F_{\mu\rho}F^{\sigma\rho} - 2R^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}) \\ & \quad - g_{\mu\nu}\nabla_\alpha\nabla_\beta(F^\alpha{}_\rho F^{\beta\rho} + 2\nabla_\alpha\nabla_\nu(F_{\mu\beta}F^{\alpha\beta}) - \Box(F_{\mu\rho}F_\nu{}^\rho)\Big) + \\ & + \; c_6\left(g_{\mu\nu}R^{\kappa\lambda\rho\sigma}F_{\kappa\lambda}F_{\rho\sigma} - 6F_{\alpha\nu}F^{\beta\gamma}R^\alpha{}_{\mu\beta\gamma} - 4\nabla_\beta\nabla_\alpha(F^\alpha{}_\mu F^\beta{}_\nu)\right) + \\ & + \; c_7\left(g_{\mu\nu}(F^2)^2 - 8F^2F_\mu{}^\sigma F_{\nu\sigma}\right) + \\ & + \; c_8\left(g_{\mu\nu}F^{\rho\kappa}F_{\rho\sigma}F^{\sigma\lambda}F_{\kappa\lambda} - 8F_\mu{}^\rho F_\nu{}^\sigma F_\rho{}^\kappa F_{\sigma\kappa}\right)\,.\end{aligned} \tag{6}$$

Note that $F^2 = F_{\rho\sigma}F^{\rho\sigma}$ and $\Box = \nabla_a\nabla^a$.

II. CORRECTIONS TO THE REISSNER-NORDSTRÖM BLACK HOLE

The functions $\lambda(r)$ and $\nu(r)$ are related to the components of Ricci curvature tensor $R_{\mu\nu}$ via

$$\frac{1}{2}\left(R^t_t - R^r_r\right) - R^\theta_\theta = \frac{1}{r^2}\frac{d}{dr}\left[r(e^{-\lambda(r)} - 1)\right] , \tag{7}$$
$$R^t_t - R^r_r = -\frac{e^{-\lambda(r)}}{r}\left[\nu'(r) + \lambda'(r)\right] .$$

To solve for λ and ν explicitly, we need an additional boundary condition. Assuming that at $r \to \infty$ the metric approaches the Schwarzschld solution, the results are then given by

$$e^{-\lambda(r)} = 1 - \frac{\kappa M}{4\pi r} - \frac{1}{r}\int_r^\infty dr\, r^2\left[\frac{1}{2}\left(R^t_t - R^r_r\right) - R^\theta_\theta\right] , \tag{8}$$
$$\nu(r) = -\lambda(r) + \int_r^\infty dr\, r\left(R^t_t - R^r_r\right)e^{\lambda(r)} .$$

We further take the trace-reverse of Eq. (10) from the main text and obtain that

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) , \tag{9}$$

where T is the trace of the total energy-momentum tensor $T_{\mu\nu}$, and is given by $T = T^t_t + T^r_r + 2T^\theta_\theta$. Plugging the trace-reversed Einstein field equation into the integral expression (8), we get

$$e^{-\lambda(r)} = 1 - \frac{\kappa M}{4\pi r} - \frac{\kappa}{r}\int_r^\infty dr\, r^2 T^t_t , \tag{10}$$
$$\nu(r) = -\lambda(r) + \kappa\int_r^\infty dr\, r\left(T^t_t - T^r_r\right)e^{\lambda(r)} .$$

Once we know the diagonal components of the energy-momentum tensor, it will be straightforward to compute the corrections to the spherically symmetric static spacetime as induced by $T_{\mu\nu}$.

We now take our background spacetime to be Reissner-Nordström black hole in four-dimension. That is,

$$e^{\nu^{(0)}} = e^{-\lambda^{(0)}} = 1 - \frac{\kappa M}{4\pi r} + \frac{\kappa Q^2}{32\pi^2 r^2} , \tag{11}$$
$$F^{(0)}_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{Q}{4\pi r^2} dt \wedge dr .$$

Here $\nu^{(0)}(r)$ and $\lambda^{(0)}(r)$ refer to the metric components in the unperturbed black hole spacetime, and $F^{(0)}_{\mu\nu}$ is the background electromagnetic energy-momentum tensor. Considering the action in Eq. (2) of the main text, we treat the corrections from higher-dimension operators as perturbations. For convenience, we also introduce a power counting parameter ε, and consider a one-parameter family of actions I_ε, which is given by

$$I_\varepsilon = \int d^4x\, \sqrt{-g}(L_0 + \varepsilon\Delta L) . \tag{12}$$

The original action will be recovered after setting $\varepsilon = 1$. We then expand everthing into powers series in ε. For instance,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \mathcal{O}(\varepsilon^2) , \qquad F_{\mu\nu} = F^{(0)}_{\mu\nu} + \varepsilon f^{(1)}_{\mu\nu} + \mathcal{O}(\varepsilon^2) . \tag{13}$$

At order ε^1, the stress energy tensor is given by

$$T^{(1)}_{\mu\nu} = \widetilde{T}_{\mu\nu}[g^{(0)}, f^{(1)}, F^{(0)}] + \widetilde{T}_{\mu\nu}[h^{(1)}, F^{(0)}, F^{(0)}] + \Delta T_{\mu\nu}[g^{(0)}, F^{(0)}] . \tag{14}$$

Noting that in order to compute the corrections to the metric, we need to calculate ${T_\mu}^\nu$ instead of $T_{\mu\nu}$. At order ε^1, ${T_\mu}^{\nu(1)}$ is given by

$${T_\mu}^{\nu(1)} = {\widetilde{T}_\mu}{}^\nu[g^{(0)}, F^{(1)}] + \Delta {T_\mu}^\nu[g^{(0)}, F^{(0)}] . \tag{15}$$

We solve for the corrections to Maxwell equations, and obtain that the nonzero components of $f^{(1)}_{\mu\nu}$ are

$$f^{(1)}_{tr} = -f^{(1)}_{rt} = \frac{1}{32\pi^3 r^6}\left(c_5\kappa Q^3 - 16\pi c_6\kappa MQr + 6c_6\kappa Q^3 + 8c_7 Q^3 + 4c_8 Q^3\right) . \tag{16}$$

This corresponds to the gauge field A_a given by

$$A_t = -\frac{q}{r} + \frac{2q^3}{5r^5}\left(c_5\kappa + 6c_6\kappa - \frac{5c_6\kappa mr}{q^2} + 8c_7 + 4c_8\right), \qquad A_r = A_\theta = A_\phi = 0 . \tag{17}$$

With the corrections to $F_{\mu\nu}$, we can solve for the corrected energy-momentum tensor ${T_\mu}^{\nu(1)}$. We then find the corrected metric tensor component to be

$$\begin{aligned}
e^{-\lambda} =& 1 - \frac{\kappa m}{r} + \frac{\kappa q^2}{2r^2} + c_2\left(\frac{3\kappa^3 mq^2}{r^5} - \frac{6\kappa^3 q^4}{5r^6} - \frac{4\kappa^2 q^2}{r^4}\right) \\
&+ c_3\left(\frac{12\kappa^3 mq^2}{r^5} - \frac{24\kappa^3 q^4}{5r^6} - \frac{16\kappa^2 q^2}{r^4}\right) + c_4\left(\frac{14\kappa^2 mq^2}{r^5} - \frac{6\kappa^2 q^4}{r^6} - \frac{16\kappa q^2}{r^4}\right) \\
&+ c_5\left(\frac{5\kappa^2 mq^2}{r^5} - \frac{11\kappa^2 q^4}{5r^6} - \frac{6\kappa q^2}{r^4}\right) + c_6\left(\frac{7\kappa^2 mq^2}{r^5} - \frac{16\kappa^2 q^4}{5r^6} - \frac{8\kappa q^2}{r^4}\right) \\
&+ c_7\left(-\frac{4\kappa q^4}{5r^6}\right) + c_8\left(-\frac{2\kappa q^4}{5r^6}\right) , \\
e^{+\nu} =& 1 - \frac{\kappa m}{r} + \frac{\kappa q^2}{2r^2} + c_2\left(\frac{\kappa^3 mq^2}{r^5} - \frac{\kappa^3 q^4}{5r^6} - \frac{2\kappa^2 q^2}{r^4}\right) \qquad (18) \\
&+ c_3\left(\frac{4\kappa^3 mq^2}{r^5} - \frac{4\kappa^3 q^4}{5r^6} - \frac{8\kappa^2 q^2}{r^4}\right) + c_4\left(-\frac{6\kappa^2 mq^2}{r^5} + \frac{4\kappa^2 q^4}{r^6} + \frac{4\kappa q^2}{r^4}\right) \\
&+ c_5\left(\frac{4\kappa^2 q^4}{5r^6} - \frac{\kappa^2 mq^2}{r^5}\right) + c_6\left(\frac{\kappa^2 mq^2}{r^5} - \frac{\kappa^2 q^4}{5r^6} - \frac{2\kappa q^2}{r^4}\right) \\
&+ c_7\left(-\frac{4\kappa q^4}{5r^6}\right) + c_8\left(-\frac{2\kappa q^4}{5r^6}\right) . \qquad (19)
\end{aligned}$$

In the above we have defined the reduced quantities $m = M/4\pi$ and $q = Q/4\pi$. Note that the R^2-term in the action has no contributions to the equation of motion at leading order in ε. The contributions from $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\theta}R^{\mu\nu\rho\theta}$ can be canceled out by choosing $c_2 = -4c_3$. This directly confirms that the Gauss-Bonnet term is a topological invariant and does not influence the equation of motion. Due to the fact that only the tr- and $rt-$component of $F_{\mu\nu}$ are nonzero, the term $F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$ always have twice the contributions from $F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}$ towards the equation of motion.

III. EXPLICIT FORMS OF $\mathbf{Q}_\xi$ AND $\mathbf{C}_a$ FOR THE HIGHER THEORY

The Lagrangian 4-form $\boldsymbol{L}$ for the higher theory can be written as $\boldsymbol{L} = \boldsymbol{L}_0 + \sum_i c_i \boldsymbol{L}_i$. In this appendix, by following the canonical method developed by Iyer and Wald, we derive and present the Noether charge and constraint associated with each term in $\boldsymbol{L}$.

Variation of the Lagrangian 4-form $\boldsymbol{L}_0$ yields

$$\delta\boldsymbol{L}_0 = \delta g_{ab}\left(-\frac{1}{2\kappa}G_{ab} + \frac{1}{2}T^{\text{EM}}_{ab}\right)\boldsymbol{\epsilon} + \delta A_a\left(\nabla_b F^{ba}\right)\boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_0 , \tag{20}$$

where $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ is the Einstein tensor, and T^{EM}_{ab} is the electro-magnetic stress-energy tensor, which is defined by

$$T^{\text{EM}}_{ab} = F_{ac}{F_b}^c - \frac{1}{4}g_{ab}F_{de}F^{de} . \tag{21}$$

The symplectic potential can be written as

$$\boldsymbol{\Theta}_0 = \boldsymbol{\Theta}^{\text{GR}} + \boldsymbol{\Theta}^{\text{EM}} , \tag{22}$$

where

$$\Theta^{\rm GR}_{abc}\left(\phi,\delta\phi\right) = \frac{1}{2\kappa}\epsilon_{dabc}g^{de}g^{fg}\left(\nabla_g\delta g_{ef} - \nabla_e\delta g_{fg}\right)\,, \tag{23}$$

$$\Theta^{\rm EM}_{abc}\left(\phi,\delta\phi\right) = -\epsilon_{dabc}F^{de}\delta A_e\,. \tag{24}$$

Let ξ^a be any smooth vector field on the spacetime. We find that the Noether charges associated with the vector field are respectively,

$$\left(Q^{\rm GR}_\xi\right)_{ab} = -\frac{1}{2\kappa}\epsilon_{abcd}\nabla^c\xi^d\,, \tag{25}$$

$$\left(Q^{\rm EM}_\xi\right)_{ab} = -\frac{1}{2}\epsilon_{abcd}F^{cd}A_e\xi^e\,. \tag{26}$$

The equations of motion and constraints are given by

$$\boldsymbol{E}_0\delta\phi = -\boldsymbol{\epsilon}\left(\frac{1}{2}T^{ab}\delta g_{ab} + j^a\delta A_a\right)\,, \tag{27}$$

$$C_{bcda} = \epsilon_{ebcd}\left(T^e{}_a + j^eA_a\right)\,, \tag{28}$$

where we have defined $T_{ab} = \frac{1}{\kappa}\left(G_{ab} - \kappa T^{\rm EM}_{ab}\right)$ as the non-electromagnetic stress energy tensor, and $j^a = \nabla_b F^{ab}$ is the charge-current of the Maxwell sources.

We similarly obtain the Noether charges and constraints for all higher-derivative terms. The results are presented below.

a. $\boldsymbol{L}_1$ Variation of $\boldsymbol{L}_1$ yields

$$\delta\boldsymbol{L}_1 = \delta g_{ab}(E_1)^{ab}\boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_1\,, \tag{29}$$

where we have defined

$$(E_1)^{ab} = \frac{1}{2}g^{ab}R^2 - 2RR^{ab} + 2\nabla^b\nabla^aR - 2g^{ab}\nabla_c\nabla^cR\,. \tag{30}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^1_\xi)_{ab} = \epsilon_{abcd}\left(-4\xi^c\,\nabla^dR + 2R\,\nabla^d\xi^c\right)\,. \tag{31}$$

The constraints are given by

$$C_{bcda} = -2\epsilon_{ebcd}\left(E_1\right)^e{}_a\,. \tag{32}$$

b. $\boldsymbol{L}_2$ Variation of $\boldsymbol{L}_2$ yields

$$\delta\boldsymbol{L}_2 = \delta g_{ab}(E_2)^{ab}\boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_2\,, \tag{33}$$

where we have defined

$$(E_2)^{ab} = \frac{1}{2}g^{ab}R_{cd}R^{cd} + \nabla_c\nabla^bR^{ac} + \nabla_c\nabla^aR^{bc} - g^{ab}\nabla_d\nabla_cR^{cd} - \nabla^c\nabla_cR^{ab} - 2R^{ac}R^b{}_c\,. \tag{34}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^2_\xi)_{ab} = \epsilon_{abcd}\left(4\xi^{[f}\,\nabla^{c]}R_f{}^d + R_f{}^d\nabla^f\xi^c + R_f{}^c\nabla^d\xi^f\right)\,. \tag{35}$$

The constraints are given by

$$C_{bcda} = -2\epsilon_{ebcd}(E_2)^e{}_a\,. \tag{36}$$

c. $\boldsymbol{L}_3$ Variation of $\boldsymbol{L}_3$ yields

$$\delta \boldsymbol{L}_3 = \delta g_{ab} c_3 (E_3)^{ab} \boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_3 \,, \tag{37}$$

where we have defined

$$(E_3)^{ab} = \frac{1}{2} g^{ab} R^2 + 2 g^{ab} R_{cd} R^{cd} + 2 R^{ab} R - 8 R_{cd} R^{acbd} + 2 \nabla^b \nabla^a R - 4 \Box R^{ab} \,. \tag{38}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^3_\xi)_{ab} = \epsilon_{abcd} \left(-4 \xi^e \nabla_f R_e{}^{fcd} + 2 R_{ef}{}^{cd} \nabla^f \xi^e \right) \,. \tag{39}$$

The constraints are given by

$$C_{bcda} = -2 \epsilon_{ebcd} (E_3)^e{}_a \,. \tag{40}$$

d. $\boldsymbol{L}_4$ Variation of $\boldsymbol{L}_4$ yields

$$\delta \boldsymbol{L}_4 = \delta g_{ab} (E^g_4)^{ab} \boldsymbol{\epsilon} + \delta A_a (E^A_4)^a \boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_4 \,, \tag{41}$$

where we have defined the equation of motions for g_{ab} and A_a respectively as

$$(E^g_4)^{ab} = \left[-R^{ab} + \frac{1}{2} g^{ab} R - g^{ab} \nabla^2 + \nabla^{(a} \nabla^{b)} \right] F^2 - 2 R F^{ac} F_b{}^c \,, \tag{42}$$

$$(E^A_4)^a = 4 \nabla_b \left(R F^{ab} \right) \,. \tag{43}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^4_\xi)_{ab} = \epsilon_{abcd} \left(F^2 \nabla^d \xi^c - 2 \xi^c \nabla^d F^2 + 2 R F^{cd} A_e \xi^e \right) \,. \tag{44}$$

The constraints are given by

$$C_{bcda} = -2 \epsilon_{ebcd} (E^g_4)^e{}_a - \epsilon_{ebcd} (E^A_4)^e A_a \,. \tag{45}$$

e. $\boldsymbol{L}_5$ Variation of $\boldsymbol{L}_5$ yields

$$\delta \boldsymbol{L}_5 = \delta g_{ab} (E^g_5)^{ab} \boldsymbol{\epsilon} + \delta A_a (E^A_5)^a \boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_5 \,, \tag{46}$$

where we have defined the equation of motions for g_{ab} and A_a respectively as

$$\begin{aligned} (E^g_5)^{ab} = {} & 2 F^{(bc} F_c{}^d R^{a)}{}_d - F^{ac} F^{bd} R_{cd} + \frac{1}{2} F_c{}^e F^{cd} g^{ab} R_{de} \\ & - \nabla^{(a} F^{b)c} \nabla_d F_c{}^d - F^{cd} \nabla_d \nabla^{(a} F^{b)}{}_c - F^{(bc} \nabla_d \nabla^{a)} F_c{}^d - F^{(bc} \Box F^{a)}{}_c \\ & - \nabla^{(b} F_{cd} \nabla^d F^{a)c} - F^{cd} g^{ab} \nabla_{(d} \nabla_{e)} F_c{}^e - \nabla_d F^b{}_c \nabla^d F^{ac} \\ & + \frac{1}{2} g^{ab} \nabla_c F^{cd} \nabla_e F_d{}^e - \frac{1}{2} g^{ab} \nabla_d F_{ce} \nabla^e F^{cd} \,, \end{aligned} \tag{47}$$

$$(E^A_5)^a = 2 \nabla_c \left(R^{bc} F^a{}_b + F^{bc} R^a{}_b \right) \,. \tag{48}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^5_\xi)_{ab} = \epsilon_{abcd} \left[-2 \xi^e A_e F^{fc} R_f{}^d - 2 \xi^c F^{f(e} \nabla_e F_f{}^{d)} + \xi^e \nabla^d \left(F^{fc} F_{ef} \right) + F_f{}^d F_e{}^f \nabla^{[c} \xi^{e]} \right] \,. \tag{49}$$

The constraints are given by

$$C_{bcda} = -2 \epsilon_{ebcd} (E^g_5)^e{}_a - \epsilon_{ebcd} (E^A_5)^e A_a \,. \tag{50}$$

f. $\boldsymbol{L}_6$ Variation of $\boldsymbol{L}_6$ yields

$$\delta \boldsymbol{L}_6 = \delta g_{ab}(E^g_6)^{ab}\boldsymbol{\epsilon} + \delta A_a (E^A_6)^a \boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_6\,, \tag{51}$$

where we have defined the equation of motions for g_{ab} and A_a respectively as

$$\begin{aligned}(E^g_6)^{ab} &= \frac{1}{2}F^{cd}F^{ef}g^{ab}R_{cdef} - 3F^{(ac}F^{de}R^{b)}{}_{cde} \\ &\quad - 2F^{(ac}\nabla_c\nabla_d F^{b)d} - 2F^{(ac}\nabla_d\nabla_c F^{b)d} - 4\nabla_c F^{(ac}\nabla_d F^{b)d}\,,\end{aligned} \tag{52}$$

$$(E^A_6)^a = 4\nabla_d\left(F^{bc}R^{ad}{}_{bc}\right)\,. \tag{53}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^6_\xi)_{ab} = \epsilon_{abcd}\left[2\xi^e A_e F^{fg}R_{fg}{}^{cd} - 2\xi^e\nabla_f\left(F^{cd}F_e{}^f\right) + F^{cd}F_{ef}\nabla^f\xi^e\right]\,. \tag{54}$$

The constraints are given by

$$C_{bcda} = -2\epsilon_{ebcd}(E^g_6)^e{}_a - \epsilon_{ebcd}(E^A_6)^e A_a\,. \tag{55}$$

g. $\boldsymbol{L}_7$ Variation of $\boldsymbol{L}_7$ yields

$$\delta \boldsymbol{L}_7 = \delta g_{ab}(E^g_7)^{ab}\boldsymbol{\epsilon} + \delta A_a (E^A_7)^a \boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_7\,, \tag{56}$$

where we have defined the equation of motions for g_{ab} and A_a respectively as

$$(E^g_7)^{ab} = \frac{1}{2}g^{ab}F^2F^2 - 4F^{ac}F^b{}_cF^2\,, \tag{57}$$

$$(E^A_7)^a = 8\nabla_b\left(F^{ab}F^2\right)\,. \tag{58}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^7_\xi)_{ab} = \epsilon_{abcd}\left(4\xi^e A_e F^{cd}F^2\right)\,. \tag{59}$$

The constraints are given by

$$C_{bcda} = -2\epsilon_{ebcd}(E^g_7)^e{}_a - \epsilon_{ebcd}(E^A_7)^e A_a\,. \tag{60}$$

h. $\boldsymbol{L}_8$ Variation of $\boldsymbol{L}_8$ yields

$$\delta \boldsymbol{L}_8 = \delta g_{ab}(E^g_8)^{ab}\boldsymbol{\epsilon} + \delta A_a (E^A_8)^a \boldsymbol{\epsilon} + \mathbf{d}\boldsymbol{\Theta}_8\,, \tag{61}$$

where we have defined the equation of motions for g_{ab} and A_a respectively as

$$(E^g_8)^{ab} = \frac{1}{2}g^{ab}F_c{}^dF_d{}^eF_e{}^fF_f{}^c - 4F^{ac}F^{bd}F_c{}^eF_{de}\,, \tag{62}$$

$$(E^A_8)^a = -8\nabla_d\left(F^a{}_bF^b{}_cF^{cd}\right)\,. \tag{63}$$

The Noether charge associated with the vector field ξ^a is

$$(Q^8_\xi)_{ab} = \epsilon_{abcd}\left(4\xi^e A_e F_f{}^dF_g{}^cF^{gf}\right)\,. \tag{64}$$

The constraints are given by

$$C_{bcda} = -2\epsilon_{ebcd}(E^g_8)^e{}_a - \epsilon_{ebcd}(E^A_8)^e A_a\,. \tag{65}$$

Finally, the above results can be summarized in the following compact form:

$$(\mathbf{Q}_\xi)_{c_3c_4} = \epsilon_{abc_3c_4}\left(M^{abc}\,\xi_c - E^{abcd}\,\nabla_{[c}\,\xi_{d]}\right)\,, \tag{66}$$

where

$$M^{abc} \equiv -2\nabla_d E^{abcd} + E^{ab}_F A^c\,, \tag{67}$$

and

$$(\mathbf{C}^d)_{abc} = \epsilon_{eabc}(2E^{pqre}R_{pqr}{}^d + 4\nabla_f\nabla_h E^{efdh} + 2E^{eh}_F F^d{}_h - 2A^d\nabla_h E^{eh}_F - g^{ed}\boldsymbol{L}) \tag{68}$$

with

$$E^{abcd} \equiv \frac{\delta \boldsymbol{L}}{\delta R_{abcd}}, \qquad E^{ab}_F \equiv \frac{\delta \boldsymbol{L}}{\delta F_{ab}}\,. \tag{69}$$

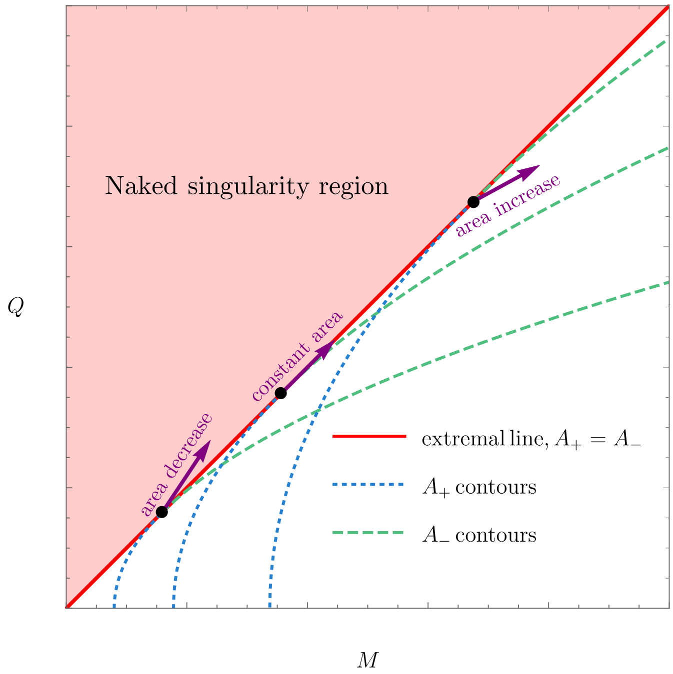

FIG. 1: Extremality contour and constant area contours. Extremal black holes live on the red solid line which divides the whole parameter space into the naked singularity region and the non-extremal black hole region. The constant area contours are always tangent to the extremal line. A small perturbation around an extremal point then shifts the spacetime to one of the following: (i) a naked singularity when the horizon area is decreased; (ii) another extremal solution when the area is unchanged; and (iii) a nonextremal black hole when the area is increased.

IV. PROOF THAT CONSTANT AREA DIRECTION IS ALONG THE EXTREMALITY CURVE

Suppose the radius, hence area A of the horizon is determined implicitly by the following equation

$$F(M, Q, A) = 0\,. \tag{70}$$

Extremality condition requires, in addition, that

$$\partial_A F(M, Q, A) = 0\,. \tag{71}$$

This is because the two roots of $1/g_{rr}$ coincide at this location.

Extremal black holes is a one-parameter family, with $Q_{\rm ext}(M)$, $A_{\rm ext}(M)$ determined jointly by Eqs. (70) and (71). In practice, when $Q < Q_{\rm ext}(M)$, we will have contours of constant A (as shown in Fig. 1), determined by

$$\partial_M F dM + \partial_Q F dQ = 0\,, \tag{72}$$

or

$$(dQ/dM)_A = -\partial_M F/\partial_Q F\,. \tag{73}$$

On the other hand, we can find out the direction of the extremality curve in the (M, Q, A) space. The tangent vector satisfies

$$\partial_M F \Delta M + \partial_M F \Delta Q + \partial_A F \Delta A = 0\,. \tag{74}$$

However, because we have $\partial_A F$ on that curve, we have $\partial_A F = 0$ and also

$$(dQ/dM)_{\rm ext} = -\partial_M F/\partial_Q F\,. \tag{75}$$

This means, on the extremality contour, the direction at which area remains constant is the same as the contour itself. This does not mean that the contour all has the same area — instead, constant area contours reach the extremality contour in a tangential way, as shown in the figure.