A counterexample to the existence of a general central limit theorem for pairwise independent identically distributed random variables

Benjamin Avanzi^{a,1}, Guillaume Boglioni Beaulieu^{b,1,*}, Pierre Lafaye de Micheaux^c, Frédéric Ouimet^{d,1}, Bernard Wong^{b,1}

^aCentre for Actuarial Studies, Department of Economics, University of Melbourne, VIC 3010, Australia.

^bSchool of Risk and Actuarial Studies, UNSW Sydney, NSW 2052, Australia.

^cSchool of Mathematics and Statistics, UNSW Sydney, NSW 2052, Australia.

^dPMA Department, California Institute of Technology, CA 91125, Pasadena, USA.

Abstract

The classical Central Limit Theorem (CLT) is one of the most fundamental results in statistics. It states that the standardized sample mean of a sequence of n *mutually* independent and identically distributed random variables with finite second moment converges in distribution to a standard Gaussian as n goes to infinity. In particular, *pairwise* independence of the sequence is generally not sufficient for the theorem to hold. We construct explicitly such a sequence of pairwise independent random variables having a common but arbitrary marginal distribution F (satisfying very mild conditions) and for which no CLT holds. We obtain, in closed form, the asymptotic distribution of the sample mean of our sequence, and find it is asymmetrical for any F . This is illustrated through several theoretical examples for various choices of F . Associated R codes are provided in a supplementary appendix online.

Keywords: central limit theorem, characteristic function, mutual independence, non-Gaussian asymptotic distribution, pairwise independence

2020 MSC: Primary : 62E20 Secondary : 60F05, 60E10

1. Introduction

The classical Central Limit Theorem (CLT) is a fundamental result in statistics, and its use by practitioners for building confidence intervals and testing hypotheses is widespread. Of course, to judiciously use any theorem, one must first understand all of its assumptions. In the case of the classical CLT, those include the assumption of *mutual* independence between the random variables considered, and we believe that this assumption is often neglected by practitioners. Hence, in this paper, our aim is to highlight just how crucial this assumption is. We do so by constructing explicitly a sequence of *pairwise* independent and identically distributed (p.i.i.d.) random variables (r.v.s) whose common margin F can be chosen arbitrarily (under very mild conditions) and for which the (standardized) sample mean is *not* asymptotically Gaussian. We give a closed-form expression for the limiting distribution of this sample mean. It is, to the best of our knowledge, the first example of this kind for which the asymptotic distribution of the sample mean is explicitly given, and known to be skewed and heavier tailed than a Gaussian distribution, for *any* choice of margin. Our sequence thus illustrates nicely why *mutual* independence is such a crucial assumption for the (classical) CLT to hold. It also allows us to quantify how far away from the Gaussian distribution one can get under the less restrictive assumption of *pairwise* independence.

‘A’ CLT for a sequence $\{X_j, j \geq 1\}$ of r.v.s. is a result establishing the convergence in distribution, under some conditions, of a normalized sum $(\sum_{j=1}^n X_j - a_n)/b_n$ to the standard Gaussian distribution. The established terminology is to refer (somewhat abusively) to ‘the’ CLT when the X_j ’s are *independent*, even though various sets of more or less restrictive conditions on $\{X_j, j \geq 1\}$ exist under which ‘a’ CLT holds. Moreover, most textbooks in mathematical statistics or introductory statistics only focus on the case of *independent* variables, even if we know (at least since Hoeffding [19]) that a CLT can possibly hold

*Corresponding author

Email addresses: b.avanzi@unimelb.edu.au (Benjamin Avanzi), g.boglioni@unsw.edu.au (Guillaume Boglioni Beaulieu), lafaye@unsw.edu.au (Pierre Lafaye de Micheaux), ouimetfr@caltech.edu (Frédéric Ouimet), bernard.wong@unsw.edu.au (Bernard Wong)

for dependent variables. As Stoyanov [31] puts it in his book: counterexamples can serve to “demonstrate the range of validity of the CLT and examine the importance of the conditions under which the CLT holds”. He and others (e.g., Bagui et al. [3]) give several counterexamples to CLTs for independent X_j ’s. But, surprisingly, counterexamples for *dependent* sequences are scarce in the literature. This lack of counterexamples might explain the commonly held belief—and often unverified assumption—among users of statistics that a large sample size is sufficient to ensure approximate normality of the sample mean, even though it is not. In many fields where applied statistics are used, articles typically discuss what the right sample size should be in order to confidently use ‘the’ CLT, but those articles do not also address the fundamental issue of *mutual* independence as a crucial assumption; examples can be found in biology [16], medicine [1, 18, 12], psychology [2], engineering [20], or economics [25]. In fact, after an extensive search, we have not found any article published in those fields that contains an *explicit* discussion of this crucial independence assumption.

Now, recall that the classical (or basic) CLT is stated for a sequence $\{X_j, j \geq 1\}$ of i.i.d. random variables with mean μ and standard deviation $0 < \sigma < \infty$ as follows:

$$S_n := \frac{\sum_{j=1}^n X_j - \mu n}{\sigma \sqrt{n}} \xrightarrow{d} Z, \quad \text{as } n \rightarrow \infty, \quad (1.1)$$

where the random variable Z has a standard Gaussian distribution, noted thereafter $N(0, 1)$, and ‘ $\xrightarrow{d}$ ’ denotes convergence in distribution. The first ‘i’ in the acronym i.i.d. stands for ‘independent’, which itself stands for ‘*mutually* independent’, while the last ‘i.d.’ stands for *identically distributed*.

It is known that *pairwise* independence among random variables is a necessary *but not sufficient* condition for them to be *mutually* independent. The earliest counterexample can be attributed to Bernšteĭn [4], followed by a few other authors, e.g., Geisser and Mantel [17], Pierce and Dykstra [26], Joffe [23], Bretagnolle and Kłopotowski [9], Derriennic and Kłopotowski [13]. However, from these illustrative examples alone it can be hard to understand how bad of a substitute to mutual independence pairwise independence is. One way to study this question is to consider those fundamental theorems of mathematical statistics that rely on the former assumption; do they ‘fail’ under the weaker assumption of *pairwise* independence? A definite answer to that question is beyond the scope of this work, as it depends on which theorem is considered. Nevertheless, note that the Law of Large Numbers, even if almost always stated for mutually independent r.v.s, does hold under pairwise independence, see Etemadi [15]. The same goes for the second Borel-Cantelli lemma, usually stated for mutually independent events but valid for pairwise independent events as well, see Erdős and Rényi [14]. The CLT (for i.d. r.v.s), however, does ‘fail’ under pairwise independence. Since it is arguably the most crucial result in all of statistics, and since pairwise independence and uncorrelatedness are concepts vastly used by practitioners, we will concentrate on this case from now on.

Révész and Wschebor [29] were the first to provide a pairwise independent sequence for which S_n does not converge in distribution to a $N(0, 1)$. For their sequence, which is binary (i.e., two-state), S_n converges to a standardized χ_1^2 distribution. Romano and Siegel [30, Example 5.45] provide a two-state, and Bradley [6] a three-state, pairwise independent sequence for which S_n converges in probability to 0. Janson [22] provides a broader class of pairwise independent counterexamples, most defined with X_j ’s having a continuous margin and for which S_n converges in probability to 0. The author also constructs a pairwise independent sequence of $N(0, 1)$ r.v.s for which S_n converges to the random variable $S = R \cdot Z$, with R a r.v. whose distribution can be arbitrarily chosen among those with support $[0, 1]$, and Z a $N(0, 1)$ r.v. independent of R . The r.v. S can be seen as ‘better behaved’ than a $N(0, 1)$, in the sense that it is symmetric with a variance smaller than 1 (regardless of the choice of R). Cuesta and Matrán [11, Section 2.3] construct a sequence $\{X_j, j \geq 1\}$ of r.v.s taking values uniformly on the integers $\{0, 1, \dots, p-1\}$, with p a prime number, for which S_n is ‘worse behaved’ than a $N(0, 1)$. Indeed, their S_n converges in distribution to a mixture (with weights $(p-1)/p$ and $1/p$ respectively) of the constant 0 and of a centered Gaussian r.v. with variance p . This distribution is symmetric but it has heavier tails than that of a $N(0, 1)$.

Other authors go beyond *pairwise* independence and study CLTs under ‘ K -tuplewise independence’, for $K \geq 3$. A random sequence $\{X_j, j \geq 1\}$ is said to be K -tuplewise independent if for every choice of K distinct integers $j_1, \dots, j_K$, the random variables $X_{j_1}, \dots, X_{j_K}$ are mutually independent. Kantorovitz [24] provides an example of a triplewise independent two-state sequence for which S_n converges to a ‘misbehaved’ distribution—that of $Z_1 \cdot Z_2$, where Z_1 and Z_2 are independent $N(0, 1)$. Pruss [28] presents a sequence of K -tuplewise independent random variables $\{X_j, j \geq 1\}$ taking values in $\{-1, 1\}$ for which the asymptotic distribution of S_n is never Gaussian, for whichever choice of K . Bradley and Pruss [8]

extend this construction to a *strictly stationary* sequence of K -tuplewise independent r.v.s whose margin is uniform on the interval $[-\sqrt{3}, \sqrt{3}]$. Weakley [34] further extends this construction by allowing the X_j 's to have any symmetrical distribution (with finite variance). Takeuchi [32] showed that even if K grows linearly with the sample size n , a CLT need not be valid.

In the body of research discussed above, a non-degenerate and explicit limiting distribution for S_n is obtained only for very specific choices of margin for the X_j 's. In this paper, we allow this margin to be almost any non-degenerate distribution, yet we still obtain explicitly the limiting distribution of S_n . This distribution depends on the choice of the margin, but it is always skewed and heavier tailed than a Gaussian. By the generality of our construction (the class of marginals allowed is very broad), and the fact we explicitly find the asymptotic distribution of the standardized mean, this work raises new awareness on the dangers of using a CLT on a sample that is only *pairwise* independent.

The rest of the paper is organized as follows. In Section 2, we construct our pairwise independent sequence $\{X_j, j \geq 1\}$. In Section 3, we derive explicitly the asymptotic distribution of the standardized mean of that sequence, while in Section 4, we study key properties of such a distribution. In Section 5, we conclude.

2. Construction of the Gaussian-Chi-squared pairwise independent sequence

In this section, we build a sequence $\{X_j, j \geq 1\}$ of p.i.i.d. (but not mutually independent) r.v.s for which a CLT does *not* hold. We show in Section 4 that the asymptotic distribution of the (standardized) sample mean of this sequence can be conveniently written as that of the sum of a Gaussian r.v. and of an independent scaled Chi-squared r.v. Importantly, the r.v.s forming this sequence have a common (but arbitrary) marginal distribution F satisfying the following condition:

Condition 1. For any r.v. $W \sim F$, the variance $\text{Var}(W)$ is finite and there exists a Borel set A for which $\mathbb{P}(W \in A) = \ell^{-1}$, for some integer $\ell \geq 2$, and $\mathbb{E}[W|A] \neq \mathbb{E}[W|A^c]$.

As long as the variance is finite, the restriction on F includes *all* distributions with an absolutely continuous part on some interval. It also includes *almost all* discrete distributions with at least one weight of the form ℓ^{-1} ; see Remark 2. Also, note that, for a given F , many choices for A (with possibly different values of ℓ) could be available, depending on F .

We begin our construction of the sequence $\{X_j, j \geq 1\}$ by letting F be a distribution satisfying Condition 1. For a r.v. $W \sim F$, let A be any Borel set such that

$$\mathbb{P}(W \in A) = \ell^{-1}, \quad \text{for some integer } \ell \geq 2. \quad (2.1)$$

Then, for an integer $m \geq 2$, let $M_1, \dots, M_m$ be a sequence of i.i.d. r.v.s with discrete distribution on the set $\{1, 2, \dots, \ell\}$ and defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $i = 1, 2, \dots, \ell$, let

$$p_i := \mathbb{P}(M_j = i) = \ell^{-1}, \quad \text{for } j = 1, 2, \dots, m, \quad (2.2)$$

i.e. the M 's are discrete uniforms on the set $\{1, \dots, \ell\}$. For all pairs (M_i, M_j) , $1 \leq i < j \leq m$, define a r.v. $D_{i,j}$ as

$$D_{i,j} = \begin{cases} 1, & \text{if } M_i = M_j, \\ 0, & \text{otherwise.} \end{cases} \quad (2.3)$$

The $D_{i,j}$ are p.i.i.d (but not mutually independent) with $\mathbb{P}(D_{i,j} = 1) = \ell^{-1}$; see Remark 1. For convenience, we refer to these $n = \binom{m}{2}$ random variables $D_{1,2}, D_{1,3}, \dots, D_{1,m}, D_{2,3}, D_{2,4}, \dots, D_{m-1,m}$ simply as

$$D_1, \dots, D_n, \quad (2.4)$$

where for $1 \leq i < j \leq m$, $D_{k(i,j)} := D_{i,j}$ with $k(i,j) = [i(2m-1) - i^2]/2 + j - m$. Note that when $\ell = 2$ and $p_1 = p_2 = 1/2$, the M_j 's are (shifted) Bernoulli(1/2) r.v.s, and the sequence (2.4) is equivalent to a pairwise independent sequence first mentioned in Geisser and Mantel [17] and for which we already know that a CLT does not hold, see Révész and Wschebor [29].

From the sequence $D_1, \dots, D_n$, we now construct a new pairwise independent sequence $X_1, \dots, X_n$ such that $X_k \sim F$ for all $k = 1, \dots, n$. Define U and V to be the truncated versions of W , respectively off and on the set A :

$$U \stackrel{d}{=} W \mid \{W \in A^c\}, \quad V \stackrel{d}{=} W \mid \{W \in A\}, \quad (2.5)$$

and denote

$$\mu_U := \mathbb{E}[U], \quad \mu_V := \mathbb{E}[V]. \quad (2.6)$$

Then, consider n independent copies of U , and independently n independent copies of V :

$$U_1, \dots, U_n, \stackrel{\text{i.i.d.}}{\sim} F_U, \quad V_1, \dots, V_n \stackrel{\text{i.i.d.}}{\sim} F_V, \quad (2.7)$$

both defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Finally, for $\omega \in \Omega$ and for $k = 1, \dots, n$, construct

$$X_k(\omega) = \begin{cases} U_k(\omega), & \text{if } D_k(\omega) = 0, \\ V_k(\omega), & \text{if } D_k(\omega) = 1. \end{cases} \quad (2.8)$$

By conditioning on D_k , one can check that

$$F_{X_k}(x) = (1 - \ell^{-1})F_{U_k}(x) + \ell^{-1}F_{V_k}(x) = F(x). \quad (2.9)$$

In the next section, we will derive the asymptotic distribution of the sample mean of those X 's, and see that it is *not* Gaussian.

The fact we do not have a CLT for the sequence in (2.4) can be explained heuristically as follows. Within the sequence $D_1, \dots, D_n$, there can be a 'very high' proportion of 1's. This occurs if the sequence $M_1, \dots, M_m$ contains a large proportion of equal variables. However, by definition of $D_{i,j}$, in order to have a very large proportion of 0's among the D 's, one would require a large proportion of pairs (M_i, M_j) , $1 \leq i < j \leq m$, to be such that $M_i \neq M_j$. This is impossible, since *all* the possible pairs are used to form the sequence of D 's. This very asymmetrical situation makes the asymptotic distribution of the standardized sample mean of the D 's highly skewed to the right.

Remark 1. In Condition 1, the restriction $\mathbb{P}(W \in A) = \ell^{-1}$ for some integer ℓ may seem arbitrary. Likewise, in (2.2) the choice $p_i = \ell^{-1}$ for $i = 1, \dots, \ell$ may also seem arbitrary. We establish here that none of these choices are arbitrary. Indeed, assume first that the only restriction on $p_1, p_2, \dots, p_\ell \in (0, 1)$ is that

$$\begin{aligned} (1) & : p_1 + p_2 + \dots + p_\ell = 1, \\ (2) & : p_1^2 + p_2^2 + \dots + p_\ell^2 = w, \\ (3) & : p_1^3 + p_2^3 + \dots + p_\ell^3 = w^2, \end{aligned} \quad (2.10)$$

for some $w \in (0, 1)$. Condition (1) is necessary for the distribution (2.2) to be well-defined, and conditions (2) and (3) are rewritings of

$$\mathbb{P}(D_{i,j} = 1) = w, \quad 1 \leq i < j \leq m, \quad (2.11)$$

and

$$\mathbb{P}(D_{i,j} = 1, D_{j,k} = 1) = \mathbb{P}(D_{i,j} = 1)\mathbb{P}(D_{j,k} = 1), \quad 1 \leq i < j < k \leq m, \quad (2.12)$$

which are sufficient to guarantee that the D 's are identically distributed and pairwise independent. Now, the solution $p_i = \ell^{-1}$ to (2.10) is unique. Indeed, by squaring condition (2) in (2.10) then applying the Cauchy-Schwarz inequality, one gets

$$w^2 = \left(\sum_{i=1}^{\ell} p_i^{3/2} p_i^{1/2} \right)^2 \leq \sum_{i=1}^{\ell} p_i^3 \sum_{i=1}^{\ell} p_i = \sum_{i=1}^{\ell} p_i^3 \quad (2.13)$$

where the last equality comes from condition (1) in (2.10). Then, condition (3) requires that we have the equality in (2.13), and this happens if and only if $p_i^{3/2} = \lambda p_i^{1/2}$ for all $i \in \{1, \dots, \ell\}$ and for some $\lambda \in \mathbb{R}$. In turn, this implies $p_i = \lambda = \ell^{-1}$ because of (1) and since $p_i > 0$, which then implies $w = \ell^{-1}$ by (2). This reasoning shows that we cannot extend our method to an arbitrary $\mathbb{P}(W \in A) \in (0, 1)$ in (2.1).

Remark 2. There is no easy characterization of all discrete distributions with finite variance such that $\mathbb{P}(W \in A) = \ell^{-1}$ for some Borel set A and some integer $\ell \geq 2$, but for which the last part of Condition 1 is not satisfied. However, the proportion of such distributions can be expected to be very small. As a simple but convincing example, consider the set of discrete distributions on three values $-\infty < x < y < z < \infty$ with weights $p_x, p_y, p_z \in (0, 1)$. The variance is finite and say one of the three p 's has the form ℓ^{-1} for some integer ℓ . The only way that $\mathbb{E}[W|A] = \mathbb{E}[W|A^c]$ is satisfied is by having A contain y and only y so that we must have $p_y = \ell^{-1}$, $p_x = p$ and $p_z = (1 - p - \ell^{-1})$ for some parameter $p \in (0, 1)$, and

$x p_x + z p_z = y(1 - \ell^{-1})$. In other words, once ℓ is fixed, there is only freedom in the choice of x, z and p . If we remove the restriction $\mathbb{E}[W|A] = \mathbb{E}[W|A^c]$ (i.e. $x p_x + z p_z = y(1 - \ell^{-1})$), it gives us at least one more dimension of freedom in the selection of x, y, z, p_x, p_y, p_z . Hence, in this case, the proportion is actually '0'. An analogous argument can be made for other discrete distributions of this kind. The restriction $\mathbb{E}[W|A] = \mathbb{E}[W|A^c]$ will always remove a dimension of freedom in the choice of the range of values or the weights.

3. Main result

We now state our main result.

Theorem 1. Let $X_1, \dots, X_n$ be random variables defined as in (2.8) and denote their mean and variance by μ and σ^2 , respectively. Then,

- (a) $X_1, \dots, X_n$ are pairwise independent;
- (b) As $m \rightarrow \infty$ (and hence as $n \rightarrow \infty$), the standardized sample mean $S_n := (\sum_{k=1}^n X_k - \mu n)/\sigma\sqrt{n}$ converges in distribution to a random variable

$$S := \sqrt{1 - r^2} Z + r \chi, \quad (3.1)$$

where $Z \sim N(0, 1)$, χ is independently distributed as a standardized $\chi_{\ell-1}^2$ and $r := \sqrt{\ell^{-1}(1 - \ell^{-1})}(\mu_V - \mu_U)/\sigma$ with μ_U, μ_V defined in (2.6).

Remark 3. Interestingly, since a standardized chi-squared distribution converges to a standard Gaussian as its degree of freedom tends to infinity, we see that $S \xrightarrow{d} N(0, 1)$ as $\ell \rightarrow \infty$.

Remark 4. When removing the restriction $\mathbb{E}[W|A] \neq \mathbb{E}[W|A^c]$ in Condition 1, the case $r = 0$ (i.e. $\mu_U = \mu_V$) is possible, so our construction also provides a new instance of a pairwise independent (but not mutually independent) sequence for which a CLT does hold.

Proof of Theorem 1. Proving (a) is straightforward. Simple calculations show that $D_1, \dots, D_n$ are pairwise independent; recall (2.12). Now, for any $k, k' \in \{1, 2, \dots, n\}$ with $k \neq k'$, the r.v.s

$$D_k, U_k, V_k, D_{k'}, U_{k'}, V_{k'}$$

are mutually independent and one can write $X_k = g(D_k, U_k, V_k)$ and $X_{k'} = g(D_{k'}, U_{k'}, V_{k'})$, for g a Borel-measurable function. Since X_k and $X_{k'}$ are integrable, the result follows from Pollard [27, Section 4.1, Corollary 2].

The proof of (b) is more involved. We prove (3.1) by obtaining the limit of the characteristic function of S_n , and then by invoking Lévy's continuity theorem. Namely, we show that, for all $t \in \mathbb{R}$,

$$\varphi_{S_n}(t) \xrightarrow{m \rightarrow \infty} \varphi_{\sqrt{1-r^2}Z}(t) \cdot \varphi_{r\chi}(t) = \exp\left(-\frac{1}{2}(1-r^2)t^2 - itr\sqrt{(\ell-1)/2}\right) \left(1 - itr\sqrt{2/(\ell-1)}\right)^{-(\ell-1)/2}. \quad (3.2)$$

First, let us define by $N_i = N_i(m)$ the number of M_j 's equal to i ($i = 1, 2, \dots, \ell$) within the sample $\{M_j; j = 1, \dots, m\}$. Then, $\mathbf{N} := (N_1, \dots, N_\ell) \sim \text{Multinomial}(m, (\ell^{-1}, \dots, \ell^{-1}))$. Importantly, if $\mathbf{N}$ is known, then the number $p(\mathbf{N})$ of 1's in the sequence $\{D_j; j = 1, \dots, n\}$ can be deduced as

$$\begin{aligned} p(\mathbf{N}) &= \sum_{i=1}^{\ell-1} \binom{N_i}{2} \mathbb{1}_{\{N_i \geq 2\}} + \left(m - \sum_{i=1}^{\ell-1} N_i\right) \mathbb{1}_{\{m - \sum_{i=1}^{\ell-1} N_i \geq 2\}} \\ &= \sum_{i=1}^{\ell-1} \frac{N_i(N_i - 1)}{2} + \frac{(m - \sum_{i=1}^{\ell-1} N_i)(m - \sum_{i=1}^{\ell-1} N_i - 1)}{2} \\ &= \frac{1}{2} \sum_{i=1}^{\ell-1} N_i^2 + \frac{1}{2} \sum_{i=1}^{\ell-1} \sum_{i'=1}^{\ell-1} N_i N_{i'} - m \sum_{i=1}^{\ell-1} N_i + \frac{m(m-1)}{2} \\ &= \frac{1}{2} \sum_{i=1}^{\ell-1} (N_i - m\ell^{-1})^2 + \frac{1}{2} \sum_{i=1}^{\ell-1} \sum_{i'=1}^{\ell-1} (N_i - m\ell^{-1})(N_{i'} - m\ell^{-1}) - \frac{\ell(\ell-1)}{2} m^2 \ell^{-2} + \frac{m(m-1)}{2} \end{aligned}$$

$$= \frac{m\ell^{-1}}{2} \sum_{i=1}^{\ell-1} \sum_{i'=1}^{\ell-1} \left(\frac{1}{\ell^{-1}} \mathbb{1}_{\{i=i'\}} + \frac{1}{\ell^{-1}} \right) \frac{(N_i - m\ell^{-1})}{\sqrt{m}} \frac{(N_{i'} - m\ell^{-1})}{\sqrt{m}} - \frac{m}{2} (1 - \ell^{-1}) + n\ell^{-1}, \quad (3.3)$$

where $\mathbb{1}_B$ denotes the indicator function on the set B . The covariances of a Multinomial($m, (p_1, p_2, \dots, p_\ell)$) distribution are well known to be $m\Sigma$ where $\Sigma_{i,i'} = p_i \mathbb{1}_{\{i=i'\}} - p_i p_{i'}$, for $1 \leq i, i' \leq \ell - 1$, and it is also known that $(\Sigma^{-1})_{i,i'} = p_i^{-1} \mathbb{1}_{\{i=i'\}} + p_\ell^{-1}$, $1 \leq i, i' \leq \ell - 1$; see [33, eq. 21]. Therefore, with $p_i = \ell^{-1}$ for all i , we see from (3.3) that

$$\begin{aligned} \frac{p(\mathbf{N}) - n\ell^{-1}}{\sqrt{n\ell^{-1}(1 - \ell^{-1})}} &= \sqrt{\frac{m}{m-1}} \left[\frac{\sum_{i=1}^{\ell-1} \sum_{i'=1}^{\ell-1} (\Sigma^{-1})_{i,i'} \frac{(N_i - m\ell^{-1})}{\sqrt{m}} \frac{(N_{i'} - m\ell^{-1})}{\sqrt{m}}}{\sqrt{2(\ell-1)}} - \sqrt{\frac{\ell-1}{2}} \right] \\ &\xrightarrow{d} \frac{\xi - (\ell-1)}{\sqrt{2(\ell-1)}}, \quad \text{where } \xi \sim \chi_{\ell-1}^2. \end{aligned} \quad (3.4)$$

Now, let

$$\tilde{U}_k := \frac{\sigma_U}{\sigma} \cdot \frac{U_k - \mu_U}{\sigma_U} \quad \text{and} \quad \tilde{V}_k := \frac{\sigma_V}{\sigma} \cdot \frac{V_k - \mu_V}{\sigma_V}, \quad (3.5)$$

then we can write

$$S_n = \frac{1}{\sqrt{n}} \left(r \frac{(p(\mathbf{N}) - n\ell^{-1})}{\sqrt{\ell^{-1}(1 - \ell^{-1})}} + \sum_{\substack{k=1 \\ D_k=0}}^n \tilde{U}_k + \sum_{\substack{k=1 \\ D_k=1}}^n \tilde{V}_k \right), \quad (3.6)$$

since, from (2.9), we know that

$$\mu = (1 - \ell^{-1})\mu_U + \ell^{-1}\mu_V. \quad (3.7)$$

With the notation $t_n := t/\sqrt{n}$, the mutual independence between the U_k 's, the V_k 's and $\mathbf{M} := \{M_j\}_{j=1}^m$ yields, for all $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\exp(itS_n)|\mathbf{M}] &= \exp\left(itr \frac{(p(\mathbf{N}) - n\ell^{-1})}{\sqrt{n\ell^{-1}(1 - \ell^{-1})}}\right) \prod_{\substack{k=1 \\ D_k=0}}^n \mathbb{E}[\exp(it_n \tilde{U}_k)|\mathbf{M}] \prod_{\substack{k=1 \\ D_k=1}}^n \mathbb{E}[\exp(it_n \tilde{V}_k)|\mathbf{M}] \\ &= \exp\left(itr \frac{(p(\mathbf{N}) - n\ell^{-1})}{\sqrt{n\ell^{-1}(1 - \ell^{-1})}}\right) [\varphi_{\tilde{U}}(t_n)]^{n(1-\ell^{-1})} [\varphi_{\tilde{V}}(t_n)]^{n\ell^{-1}} \left[\frac{\varphi_{\tilde{V}}(t_n)}{\varphi_{\tilde{U}}(t_n)} \right]^{p(\mathbf{N}) - n\ell^{-1}} \\ &= \exp\left(itr \frac{(p(\mathbf{N}) - n\ell^{-1})}{\sqrt{n\ell^{-1}(1 - \ell^{-1})}}\right) \cdot [\varphi_{\tilde{U}}(t_n)]^{n(1-\ell^{-1})} [\varphi_{\tilde{V}}(t_n)]^{n\ell^{-1}} \\ &\quad \cdot \left[\frac{[\varphi_{\tilde{V}}(t_n)]^n \cdot \exp\left(\frac{1}{2} \cdot \frac{\sigma_V^2}{\sigma^2} t^2\right)}{[\varphi_{\tilde{U}}(t_n)]^n \cdot \exp\left(\frac{1}{2} \cdot \frac{\sigma_U^2}{\sigma^2} t^2\right)} \right]^{\frac{p(\mathbf{N}) - n\ell^{-1}}{n}} \cdot \left[\frac{\exp\left(-\frac{1}{2} \cdot \frac{\sigma_V^2}{\sigma^2} t^2\right)}{\exp\left(-\frac{1}{2} \cdot \frac{\sigma_U^2}{\sigma^2} t^2\right)} \right]^{\frac{p(\mathbf{N}) - n\ell^{-1}}{n}}. \end{aligned} \quad (3.8)$$

(The reader should note that, for n large enough, the manipulations of exponents in the second and third equality above are valid because the highest powers of the complex numbers involved have their principal argument converging to 0. This stems from the fact that $0 \leq p(\mathbf{N}) \leq n$, and the quantities $[\varphi_{\tilde{U}}(t_n)]^n$ and $[\varphi_{\tilde{V}}(t_n)]^n$ both converge to real exponentials as $n \rightarrow \infty$, by the classical CLT.) We now evaluate the four terms on the right-hand side of (3.8). For the first term in (3.8), the continuous mapping theorem and (3.4) yield

$$\exp\left(itr \frac{(p(\mathbf{N}) - n\ell^{-1})}{\sqrt{n\ell^{-1}(1 - \ell^{-1})}}\right) \xrightarrow{d} \exp\left(itr \frac{\xi - (\ell-1)}{\sqrt{2(\ell-1)}}\right), \quad \text{as } m \rightarrow \infty. \quad (3.9)$$

For the second term in (3.8), the classical CLT yields

$$\begin{aligned} [\varphi_{\tilde{U}}(t_n)]^{n(1-\ell^{-1})} [\varphi_{\tilde{V}}(t_n)]^{n\ell^{-1}} &\xrightarrow{m \rightarrow \infty} \exp\left(-\frac{1}{2} \cdot (1 - \ell^{-1}) \frac{\sigma_U^2}{\sigma^2} t^2\right) \exp\left(-\frac{1}{2} \cdot \ell^{-1} \frac{\sigma_V^2}{\sigma^2} t^2\right) \\ &= \exp\left(-\frac{1}{2} (1 - r^2) t^2\right), \end{aligned} \quad (3.10)$$

where in the last equality we used that, from (2.9), we have

$$\sigma^2 = \mathbb{E}[X^2] - \mu^2 = (1 - \ell^{-1})\sigma_U^2 + \ell^{-1}\sigma_V^2 + \ell^{-1}(1 - \ell^{-1})(\mu_U - \mu_V)^2. \quad (3.11)$$

For the third term in (3.8), the quantity inside the bracket converges to 1 by the classical CLT. Hence, the elementary bound

$$|\exp(z) - 1| \leq |z| + \sum_{j=2}^{\infty} \frac{|z|^j}{j!} \leq |z| + \frac{|z|^2}{2(1 - |z|)} \leq \frac{1 + \ell^{-1}}{2\ell^{-1}}|z|, \quad \text{for all } |z| \leq 1 - \ell^{-1}, \quad (3.12)$$

and the fact that $\left| \frac{p(\mathbf{N}) - n\ell^{-1}}{n} \right| \leq 1 - \ell^{-1}$ yield, as $m \rightarrow \infty$,

$$\left| \frac{\left[\frac{[\varphi_{\tilde{V}}(t_n)]^n \cdot \exp\left(\frac{1}{2} \cdot \frac{\sigma_V^2}{\sigma^2} t^2\right)}{[\varphi_{\tilde{U}}(t_n)]^n \cdot \exp\left(\frac{1}{2} \cdot \frac{\sigma_U^2}{\sigma^2} t^2\right)} \right]^{\frac{p(\mathbf{N}) - n\ell^{-1}}{n}}}{1} - 1 \right| \leq \frac{1 - \ell^{-2}}{2\ell^{-1}} \left| \text{Log} \left[\frac{[\varphi_{\tilde{V}}(t_n)]^n \cdot \exp\left(\frac{1}{2} \cdot \frac{\sigma_V^2}{\sigma^2} t^2\right)}{[\varphi_{\tilde{U}}(t_n)]^n \cdot \exp\left(\frac{1}{2} \cdot \frac{\sigma_U^2}{\sigma^2} t^2\right)} \right] \right| \rightarrow 0. \quad (3.13)$$

For the fourth term in (3.8), the continuous mapping theorem and $\frac{p(\mathbf{N}) - n\ell^{-1}}{n} \xrightarrow{\mathbb{P}} 0$ (recall (3.4)) yield

$$\left[\frac{\exp\left(-\frac{1}{2} \cdot \frac{\sigma_V^2}{\sigma^2} t^2\right)}{\exp\left(-\frac{1}{2} \cdot \frac{\sigma_U^2}{\sigma^2} t^2\right)} \right]^{\frac{p(\mathbf{N}) - n\ell^{-1}}{n}} \xrightarrow{\mathbb{P}} 1, \quad \text{as } m \rightarrow \infty. \quad (3.14)$$

By combining (3.9), (3.10), (3.13) and (3.14), Slutsky's theorem implies, for all $t \in \mathbb{R}$,

$$\mathbb{E}[\exp(itS_n)|\mathbf{M}] \xrightarrow{d} \exp\left(itr \frac{\xi - (\ell - 1)}{\sqrt{2(\ell - 1)}}\right) \exp\left(-\frac{1}{2}(1 - r^2)t^2\right), \quad \text{as } m \rightarrow \infty. \quad (3.15)$$

Since the sequence $\{\mathbb{E}[\exp(itS_n)|\mathbf{M}]\}_{m \in \mathbb{N}}$ is uniformly integrable (it is bounded by 1), Theorem 25.12 in [5] shows that we also have the mean convergence

$$\mathbb{E}[\mathbb{E}[\exp(itS_n)|\mathbf{M}]] \rightarrow \mathbb{E}\left[\exp\left(itr \frac{\xi - (\ell - 1)}{\sqrt{2(\ell - 1)}}\right)\right] \exp\left(-\frac{1}{2}(1 - r^2)t^2\right), \quad \text{as } m \rightarrow \infty, \quad (3.16)$$

which proves (3.2). The conclusion follows. $\square$

4. Properties of S

Recall that F denotes the marginal distribution of the r.v.s $X_1, \dots, X_n$ in (2.8). Theorem 1 states that S_n , the standardized sample mean of these r.v.s, converges in distribution to a r.v. S whose characteristic function is given by (3.2). When the choice $\ell \geq 2$ is fixed, the distribution of S has only one parameter, r , defined as

$$r = \frac{\sqrt{\ell^{-1}(1 - \ell^{-1})}(\mu_V - \mu_U)}{\sigma} = \frac{\mu_V - \mu}{\sigma\sqrt{\ell - 1}}, \quad (4.1)$$

where the second equality stems from (3.7). Hence, r depends on the margin F (through the quantities A , μ_U , μ_V and σ). The behavior of S with respect to F (via r) is now studied. From (3.11), we see that

$$r^2 = 1 - \frac{(1 - \ell^{-1})\sigma_U^2 + \ell^{-1}\sigma_V^2}{\sigma^2}, \quad \text{and thus } 0 \leq r^2 \leq 1. \quad (4.2)$$

Example 5 (and several other examples in the Supplementary Materials) show how the critical points $r^2 = 0, 1$ can be achieved or approached when F is discrete or absolutely continuous. See also the Supplementary Materials for R computing codes to generate observations from these examples. (Note that these examples could serve as scenarios of dependence to compare, via Monte-Carlo simulations, various tests of independence; see e.g., [21].)

Example 5 (r arbitrarily close to 0 when F is absolutely continuous). Let $\ell = 2$, $A = [1, \infty)$ and let $W \sim f$ where f is the density of a Log-Normal(0, β) distribution. Note that $\text{median}(W) = 1$, $\mathbb{E}[W] = \exp(\beta/2)$, and $\text{Var}[W] = [\exp(\beta) - 1]\exp(\beta)$. Furthermore,

$$\mu_V = \int_1^\infty 2xf(x)dx = \sqrt{\frac{2}{\pi\beta}} \int_1^\infty \exp\left(-\frac{(\log x)^2}{2\beta}\right) dx = \exp(\beta/2) \left[1 + \text{Erf}(\sqrt{\beta/2})\right], \quad (4.3)$$

where the integral on the second line was solved with *Mathematica*. Hence, from (4.1),

$$r = \frac{\mu_V - \mathbb{E}[W]}{\sqrt{\text{Var}[W]}} = \frac{\exp(\beta/2)\text{Erf}(\sqrt{\beta/2})}{\sqrt{[\exp(\beta) - 1]\exp(\beta)}} = \frac{\text{Erf}(\sqrt{\beta/2})}{\sqrt{\exp(\beta) - 1}}, \quad (4.4)$$

and it is straightforward to see that $r \rightarrow 0$ as $\beta \rightarrow \infty$.

Next, recall that the characteristic function on the right-hand side of (3.2) is that of

$$S = \sqrt{1 - r^2}Z + r\chi, \quad (4.5)$$

where the r.v.s $Z \sim N(0, 1)$ and $\chi \sim [\chi_{\ell-1}^2 - (\ell - 1)]/\sqrt{2(\ell - 1)}$ are independent. This makes it clear that, when $\ell \geq 2$ is fixed, r completely determines the shape of S ; the closer r gets to 0, the closer the distribution of S is to a standard Gaussian, while the closer r gets to ± 1 , the closer the distribution of S is to a standardized $\pm\chi_{\ell-1}^2$. This shift from a Gaussian distribution towards a $\chi_{\ell-1}^2$ distribution is represented graphically in Figure 4.1 (where $\ell = 2$ and r varies). On the other hand, regardless of r , if ℓ increases then S gets closer to a $N(0, 1)$, as illustrated in Figure 4.2 (where $r = 0.9$ and ℓ varies). These figures illustrate clearly that pairwise independence might be a very poor substitute to mutual independence as an assumption in CLTs.

Figure 4.1: Density (left) and CDF (right) of S for fixed $\ell = 2$ and varying r ($r = 0.6, 0.8, 0.95$), compared to those of a $N(0, 1)$. This illustrates that CLTs can ‘fail’ substantially under pairwise independence.

In terms of moments, simple calculations with *Mathematica* yield that

$$\mathbb{E}[S] = 0, \quad \mathbb{E}[S^2] = 1, \quad \mathbb{E}[S^3] = \sqrt{\frac{8}{\ell - 1}} r^3 \quad \text{and} \quad \mathbb{E}[S^4] = 3 + \frac{12}{\ell - 1} r^4, \quad (4.6)$$

so that upper bounds on the skewness and kurtosis of S are $\sqrt{8/(\ell - 1)}$ and $3 + 12/(\ell - 1)$, respectively. The limiting r.v. S can therefore be much more skewed and heavy-tailed than the standard Gaussian distribution, which is also confirmed by Figure 4.1.

Lastly, let us comment on the r parameter, and explain why a r close to 1 yields a more ‘drastic’ departure from normality. First, recall that a CLT does not apply to the sequence of pairwise independent r.v.s $D_1, \dots, D_n$ given in (2.4) because the proportion of 1’s in that sequence can be very large, whereas the proportion of 0’s can never be large. Consequently, the distribution of the asymptotic sample mean of this sequence is asymmetrical (skewed to the right). When we ‘assign’ an arbitrary margin to the D ’s in order to create our sequence $\{X_j, j \geq 1\}$, we can attenuate (to a certain degree) this asymmetry. Consider for example the case $\ell = 2$ and $A = [\tilde{w}, \infty)$, where $\tilde{w}$ denotes the median of an absolutely

Figure 4.2: Density (left) and CDF (right) of S for fixed $r = 0.9$ and varying ℓ ($\ell = 3, 6, 15$), compared to those of a $N(0, 1)$. This illustrates that S converges to a $N(0, 1)$ as ℓ grows.

continuous distribution F . In this case, the X 's, as opposed to the D 's, take a continuous range of values, and hence X 's 'above the median' can be quite close to their mean (whereas the D 's are all either 'much bigger' or 'much smaller' than their mean). The parameter $r = (\mu_V - \mu)/\sigma$ measures to what extent this 'attenuation of asymmetry' happens. Indeed, if r is close to 0, the X 's observations above the median are not too far away from the mean (on average). This implies that, even if the proportion of observations above the median is huge, it will not overly boost the overall mean of the sample, and the distribution of this mean will not be overly asymmetrical.

To give a concrete example (again with $\ell = 2$, $A = [\tilde{w}, \infty)$), let $X \sim \text{Log-Normal}(\alpha, \beta)$. In that case, simple calculations (see Example 5 for details) yield $r = \text{Erf}(\sqrt{\beta/2})/\sqrt{\exp(\beta) - 1}$, a decreasing function of β . On the other hand, it is well known that the kurtosis of X is an increasing function of β . So, increasing β makes X heavier tailed, while giving a lower value of r . For ease of interpretation, and since r is invariant to shifting and scaling, consider the r.v. $Y = (X - \mu)/\sigma$, which has the same value of r and the same kurtosis as X . For Y , it is clear from (4.1) that r is just the *mean of Y given that it exceeds its median* (i.e., $\mathbb{E}[Y|Y > \text{median}(Y)]$). As β increases, the right tail gets heavier. To compensate the more extreme values on the right, while keeping the mean of Y equal to 0, the median is forced to move further away to the left of the mean. Hence, the mean of observations above the median, i.e. r , also gets smaller.

5. Conclusion

We constructed a pairwise independent sequence of identically distributed r.v.s $\{X_j, j \geq 1\}$ having *any* distribution that satisfies Condition 1, and for which no CLT holds. We obtained the asymptotic distribution of the standardized sample mean S_n of our sequence, and found it to be always 'worse behaved' than a Gaussian. Furthermore, the extent of this departure from normality depends on the initial common margin of the X_j 's. This is in contradiction with any CLT under which, regardless of the margin, S_n always converges to a Gaussian.

A corollary of our main result is that there exists a sequence of pairwise independent Gaussian r.v.s (and hence uncorrelated) for which the limiting distribution of S_n is substantially 'worse behaved' than a Gaussian, being asymmetric and heavier tailed. To our knowledge, no other such example exists in the literature. Given the widespread use of CLTs, even in standard parametric statistical techniques such as tests and confidence intervals for means and variances [see, e.g., 10], this constitutes a serious warning to practitioners of statistics who may think that, to invoke 'the' CLT, all one needs is for the original random variables $\{X_j, j \geq 1\}$ to be approximately Gaussian or to have a large enough sample size. Mutual independence is a crucial assumption that should not be forgotten, nor misunderstood.

As a final comment, note that some authors have studied CLTs under K -tuplewise independence (for $K > 2$); see, e.g., Pruss [28], Bradley and Pruss [8], Bradley [7], Weakley [34], Takeuchi [32]. It would be interesting to generalize our construction in that direction. One might wonder if, as K increases, the distribution of S_n would necessarily get closer to that of a Gaussian. An articulated answer to this question is not trivial and is left for future research.

Acknowledgements

B. A. and B. W. are supported by Australian Research Council's Linkage (LP130100723) and Discovery (DP200101859) Projects funding schemes. G. B. B. acknowledges financial support from UNSW Sydney under a University International Postgraduate Award, from UNSW Business School under a supplementary scholarship, and from the FRQNT (B2). F. O. is supported by a postdoctoral fellowship from the NSERC (PDF) and the FRQNT (B3X supplement).

Conflict of interest

The authors have no conflict of interest to disclose.

References

- [1] Altman, D.G., Bland, J.M., 1995. Statistics notes: The normal distribution. *BMJ* 310, 298. doi:[10.1136/bmj.310.6975.298](https://doi.org/10.1136/bmj.310.6975.298), [arXiv:https://www.bmj.com/content/310/6975/298.full.pdf](https://arxiv.org/abs/https://www.bmj.com/content/310/6975/298.full.pdf). doi:[10.1136/bmj.310.6975.298](https://doi.org/10.1136/bmj.310.6975.298).
- [2] Anderson, C.J., 2010. Central limit theorem, in: *The Corsini Encyclopedia of Psychology*. American Cancer Society, pp. 1–2. doi:[10.1002/9780470479216.corpsy0160](https://doi.org/10.1002/9780470479216.corpsy0160).
- [3] Bagui, S.C., Bhaumik, D.K., Mehra, K.L., 2013. A few counter examples useful in teaching central limit theorems. *The American Statistician* 67, 49–56. doi:[10.1080/00031305.2012.755361](https://doi.org/10.1080/00031305.2012.755361).
- [4] Bernšteĭn, S.N., 1927. *Theory of probability* (in Russian). Moscow. MR0169758.
- [5] Billingsley, P., 1995. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. third ed., John Wiley & Sons, Inc., New York. MR1324786.
- [6] Bradley, R.C., 1989. A stationary, pairwise independent, absolutely regular sequence for which the central limit theorem fails. *Probab. Theory Related Fields* 81, 1–10. MR981565.
- [7] Bradley, R.C., 2010. A strictly stationary, “causal,” 5-tuplewise independent counterexample to the central limit theorem. *ALEA Lat. Am. J. Probab. Math. Stat.* 7, 377–450. MR2741193.
- [8] Bradley, R.C., Pruss, A.R., 2009. A strictly stationary, N -tuplewise independent counterexample to the central limit theorem. *Stochastic Process. Appl.* 119, 3300–3318. MR2568275.
- [9] Bretagnolle, J., Kłopotowski, A., 1995. Sur l'existence des suites de variables aléatoires s à s indépendantes échangeables ou stationnaires. *Ann. Inst. H. Poincaré Probab. Statist.* 31, 325–350. MR1324811.
- [10] Coeurjolly, J.F., Drouilhet, R., Lafaye de Micheaux, P., Robineau, J.F., 2009. asympTest: A simple R package for classical parametric statistical tests and confidence intervals in large samples. *The R Journal* 1, 26–30. doi:[10.32614/RJ-2009-015](https://doi.org/10.32614/RJ-2009-015).
- [11] Cuesta, J.A., Matrán, C., 1991. On the asymptotic behavior of sums of pairwise independent random variables. *Statist. Probab. Lett.* 11, 201–210. MR1097975.
- [12] Cundill, B., Alexander, N.D., 2015. Sample size calculations for skewed distributions. *BMC Med. Res. Methodol.* 15, 28. doi:[10.1186/s12874-015-0023-0](https://doi.org/10.1186/s12874-015-0023-0).
- [13] Derriennic, Y., Kłopotowski, A., 2000. On Bernstein's example of three pairwise independent random variables. *Sankhyā Ser. A* 62, 318–330. MR1803459.
- [14] Erdős, P., Rényi, A., 1959. On Cantor's series with convergent $\sum 1/q_n$. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 2, 93–109. MR126414.
- [15] Etemadi, N., 1981. An elementary proof of the strong law of large numbers. *Z. Wahrsch. Verw. Gebiete* 55, 119–122. MR606010.
- [16] Fay, D.S., Gerow, K., 2013. A biologist's guide to statistical thinking and analysis, in: *WormBook: the online review of C. elegans biology*. doi:[10.1895/wormbook.1.159.1](https://doi.org/10.1895/wormbook.1.159.1).
- [17] Geisser, S., Mantel, N., 1962. Pairwise independence of jointly dependent variables. *Ann. Math. Statist.* 33, 290–291. MR137188.
- [18] Ghasemi, A., Zahediasl, S., 2012. Normality tests for statistical analysis: A guide for non-statisticians. *Int. J. Endocrinol. Metab.* 10, 486–489. doi:[10.5812/ijem.3505](https://doi.org/10.5812/ijem.3505).
- [19] Hoeffding, Wassily; Robbins, H., 1948. The central limit theorem for dependent random variables. *Duke Mathematical Journal* 15, 773–780. doi:[10.1215/s0012-7094-48-01568-3](https://doi.org/10.1215/s0012-7094-48-01568-3). doi:[10.1215/s0012-7094-48-01568-3](https://doi.org/10.1215/s0012-7094-48-01568-3).
- [20] Huberts, L.C.E., Schoonhoven, M., Goedhart, R., Diko, M.D., Does, R.J.M.M., 2018. The performance of control charts for large non-normally distributed datasets. *Qual. Reliab. Eng. Int.* 34, 979–996. doi:<https://doi.org/10.1002/qre.2287>. doi:[10.1002/qre.2287](https://doi.org/10.1002/qre.2287).
- [21] Hušková, M., Meintanis, S.G., 2008. Testing procedures based on the empirical characteristic functions. I. Goodness-of-fit, testing for symmetry and independence. *Tatra Mt. Math. Publ.* 39, 225–233. MR2452040.
- [22] Janson, S., 1988. Some pairwise independent sequences for which the central limit theorem fails. *Stochastics* 23, 439–448. MR943814.
- [23] Joffe, A., 1974. On a set of almost deterministic k -independent random variables. *Ann. Probability* 2, 161–162. MR356150.
- [24] Kantorovitz, M.R., 2007. An example of a stationary, triplewise independent triangular array for which the CLT fails. *Statist. Probab. Lett.* 77, 539–542. MR2344639.
- [25] Kresojević, B., Gajić, M., 2019. Application of the t-test in health insurance cost analysis: Large data sets. *Economics* 7, 157–167. doi:[10.2478/eoik-2019-0024](https://doi.org/10.2478/eoik-2019-0024).
- [26] Pierce, D.A., Dykstra, R.L., 1969. Independence and the normal distribution. *The American Statistician* 23, 39–39. doi:[10.1080/00031305.1969.10481871](https://doi.org/10.1080/00031305.1969.10481871).
- [27] Pollard, D., 2002. *A user's guide to measure theoretic probability*. volume 8 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge. MR1873379.

- [28] Pruss, A.R., 1998. A bounded N -tuplewise independent and identically distributed counterexample to the CLT. *Probab. Theory Related Fields* 111, 323–332. [MR1640791](#).
- [29] Révész, P., Wschebor, M., 1965. On the statistical properties of the Walsh functions. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 9, 543–554. [MR0199637](#).
- [30] Romano, J.P., Siegel, A.F., 1986. Counterexamples in probability and statistics. The Wadsworth & Brooks/Cole Statistics/Probability Series, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA. [MR831223](#).
- [31] Stoyanov, J.M., 2013. Counterexamples in probability. Dover Publications, Inc., Mineola, NY. [MR03837562](#).
- [32] Takeuchi, K., 2019. A family of counterexamples to the central limit theorem based on binary linear codes. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* E102.A, 738–740. [doi:10.1587/transfun.E102.A.738](#).
- [33] Tanabe, K., Sagae, M., 1992. An exact Cholesky decomposition and the generalized inverse of the variance-covariance matrix of the multinomial distribution, with applications. *J. Roy. Statist. Soc. Ser. B* 54, 211–219. [MR1157720](#).
- [34] Weakley, L.M., 2013. Some strictly stationary, N -tuplewise independent counterexamples to the central limit theorem. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Indiana University, [MR3167384](#).