SPECTRAL INTENSITIES OF RADIATION FROM NON-HARMONIC AND APERIODIC SYSTEMS

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In order to find expressions for the spectral intensities and the spatial distribution of radiation from the most general kind of quantum dynamical system, it is necessary to find the matrix components of the components of acceleration of the system. These can easily be found in the following way. We have, if A is the acceleration in the X_k direction and H is the total energy,

$$A_k = \dot{v}_k = -\frac{1}{ih}(v_k H - H v_k) \qquad (1)$$

$$v_k = \dot{X}_k = -\frac{1}{ih}(X_k H - H X_k) \qquad (2)$$

which give

$$h^2 A_k = 2HX_kH - X_kH^2 - H^2X_k. \qquad (3)$$

If $H = T + V$ where T is the kinetic energy and V the potential energy, then

$$h^2A_k = 2TX_kT - X_kT^2 - T^2X_k - X_kV^2 - V^2X_k + 2VX_kV$$
$$+ 2TX_kV + 2VX_kT - X_kTV - TVX_k - X_kVT - VTX_k. \qquad (4)$$

If V does not contain the X_k-component of velocity explicitly, V and X_k commute and (4) then reduces to

$$h^2A_k = 2TX_kT - X_kT^2 - T^2X_k + TX_kV + VX_kT - X_kTV - VTX_k. \qquad (5)$$

If $T = \frac{1}{2m}(P_1^2 + P_2^2 + P_3^2) = -\frac{h^2}{2m}\nabla^2$ then as $\nabla^2X = X\nabla^2 + 2\frac{\partial}{\partial x}$, $\nabla^4X = X\nabla^4 + 4\frac{\partial}{\partial x}\nabla^2$, $\nabla^2X\nabla^2 = X\nabla^4 + 2\frac{\partial}{\partial x}\nabla^2$, the sum of the first three terms on the right-hand side of (5) vanish. When (5) is completely reduced it gives

$$A_k = -\frac{1}{m}\frac{\partial V}{\partial X_k} \qquad (6)$$

a result which might have been expected in the first place. Then the matrix components of the acceleration in the X_k direction are given by

$$A_k(r,s) = -\frac{1}{m}\int \frac{\partial V}{\partial x_k}\psi_r^*\psi_s dq \qquad (7)$$

where the integration is performed over coördinate space. This general formula can apply to any system where the potential energy does not involve X and the kinetic energy is $\frac{1}{2m}$ $(P_1^2 + P_2^2 + P_3^2)$. Let $A = (A_1^2 + A_2^2 + A_3^2)^{1/2}$ be the resultant acceleration. If the potential energy does not involve any of the velocities, then the matrix components of A are given by

$$A(r,s) = -\frac{1}{m}\int\left[\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2 + \left(\frac{\partial V}{\partial x_3}\right)^2\right]^{1/2} \psi_r^*\psi_s dq. \quad (8)$$

Classically an electron with resultant acceleration A radiates energy at the rate $\frac{2e^2A^2}{3C^2}$ so that it should be expected that the total energy radiated by r,s quantum jumps should be

$$P(r,s) = \frac{2e^2}{3m^2c^2}\left|\int\left[\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right) + \left(\frac{\partial V}{\partial x_3}\right)^2\right]^{1/2} \psi_r^*\psi_s dq\right|^2. \quad (9)$$

Also, classically, the intensity of radiation in a direction making an angle θ with the resultant acceleration is, in the case of slow velocities,

$$I = \frac{A^2e^2 \sin^2\theta}{4\pi r^2c^3}. \quad (10)$$

Let l_1, l_2, l_3 be the direction cosines of the direction in which the intensity is desired. Then (10) can be rewritten classically as

$$I_{l_1,l_2,l_3} = \frac{e^2}{4\pi r^2c^3}\left[A^2 - (l_1A_1 + l_2A_2 + l_3A_3)^2\right]. \quad (11)$$

Then as $l_1A_1(r,s) + l_2A_2(r,s) + l_3A_3(r,s) = -\frac{1}{m}\int\left[l_1\frac{\partial V}{\partial x_1} + l_2\frac{\partial V}{\partial x_2} + l_3\frac{\partial V}{\partial x_3}\right]\psi_r^*\psi_s dq$, we have for the intensity in the l_1, l_2, l_3 direction of radiation from r,s jumps

$$I_{l_1,l_2,l_3}(r,s) = \frac{e^2}{4\pi m^2c^3r^2}\left\{\left|\int\left[\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2 + \left(\frac{\partial V}{\partial x_3}\right)^2\right]^{1/2} \psi_r^*\psi_s\right|^2 - \left|\int\left[l_1\frac{\partial V}{\partial x_1} + l_2\frac{\partial V}{\partial x_2} + l_3\frac{\partial V}{\partial x_3}\right]\psi_r^*\psi_s dq\right|^2\right\}. \quad (12)$$

In a uniform field of force F along the X-axis where $V = eFX$, then from (7) $A_1(r,s) = 0$ when $r \neq s$ so that A_1 is a diagonal matrix as we should expect, as it is a constant of the motion. Then from (9) it is seen that a uniformly accelerated electron abstracts a constant stream of energy from the field so that the spectrum consists of a single line of zero frequency.

For a hydrogen-like atom of nuclear charge Z the potential energy is $V = -eZ/r$ and $\frac{\partial V}{\partial z} = eZ \cos \theta/r^2$. When the wave equation is separated in spherical polar coördinates so that the unnormalized wave functions are given by

$$\psi_{n,l,m}(r,\theta,\phi) = x(r)e^{im\phi}P_l^m(\cos \theta)$$

where

$$x(r) = \left(\frac{2r}{a_0 n}\right)^l \cdot e^{-r/a_0 n} L_{l+n}^{2l+1}\left(\frac{2r}{a_0 n}\right)$$

and where $a_0 = h^2/4\pi^2 meZ$, we get from (7) and results assembled by Kupper*

$$A_3(n,l,m;n',l',m') = -\frac{eZ}{m}\left(\frac{l^2 - m^2}{4l^2 - 1}\right)^{1/2} \cdot \frac{1}{N_r}\int_0^\infty x_n(r)x_{n'}(r)dr \quad (13)$$

where N_r is a normalizing factor. The factor of this expression depending on n,n' differs from that which would be obtained by the Heisenberg amplitude rule. The latter rule is only strictly accurate when applied to an harmonic oscillator.

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THE AURORA RED LINE

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Recently[1] the author reported some experiments concerning the excitation of the aurora green line when oxygen was mixed with active nitrogen. In that report it was mentioned that a red line having a wave-length of 6654.8 A.U. was also excited and its agreement with an unclassified oxygen line prompted the suggestion that the line 6654.8 may be an oxygen line. One would therefore be justified in concluding that this line should be observed in the light of the night sky and in the Aurora Borealis.

Following a suggestion by Dr. G. Cario, the author reëxamined the plates on which the red line was photographed and the conclusion has been reached that this "line" is in reality a band belonging to the first positive group of nitrogen. In the following discussion the reasons for this conclusion will be presented.

While studying the conditions under which the green line was excited it was noted that the red line diminished in intensity very little, as the