Regret-Optimal Filtering

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Abstract

We consider the problem of filtering in linear state-space models (e.g., the Kalman filter setting) through the lens of regret optimization. Specifically, we study the problem of causally estimating a desired signal generated by a linear state space model driven by process noise, and based on noisy observations of a related observation process. Different assumptions on the driving disturbance and the observation noise sequences give rise to different estimators: in the stochastic setting to the celebrated Kalman filter, and in the deterministic setting of bounded energy disturbances to H_∞ estimators. In this work, we formulate a novel criterion for estimator design which is based on the concept of regret. We define the regret as the difference in estimation error energy between a clairvoyant estimator that has access to all future observations (a so-called smoother) and a causal one that only has access to current and past observations. The regret-optimal estimator is chosen to minimize this worst-case difference across all bounded-energy noise sequences. The resulting estimator is adaptive in the sense that it aims to mimic the behavior of the clairvoyant estimator, irrespective of what the realization of the noise will be and thus nicely interpolates between the stochastic and deterministic approaches. We provide a solution for the regret estimation problem at two different levels. First, we provide a solution at the operator level by reducing it to the Nehari problem, i.e., the problem of approximating an anti-causal operator with a causal one. Second, when we have an underlying state-space model, we explicitly find the estimator that achieves the optimal regret. From a computational perspective, the regret-optimal estimator can be easily implemented by solving three Riccati equations and a single Lyapunov equation. For a state-space model of dimension n, the regret-optimal estimator has a state-space structure of dimension $3n$. We demonstrate the applicability and efficacy of the estimator in a variety of problems and observe that the estimator has average and worst-case performances that are simultaneously close to their optimal values. We therefore argue that regret-optimality is a viable approach to estimator design.

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1 Introduction

Filtering is the problem of estimating the current value of a desired signal, given current and past values of a related observation signal. It has numerous applications in signal processing, control, and learning and a rich history going back to Wiener, Kolomgorov, and Kalman. When the underlying desired and observation signals have state-space structures driven by white Gaussian noise, the celebrated Kalman filter gives the minimum mean-square error estimate of the current value of the desired signal, given the past and current of the observed signal Kalman (1960). When all that is known of the noise sources are their first and second-order statistics, the Kalman filter gives the linear least-mean-squares estimate. While these are very desired properties, the Kalman filter is predicated on knowing the underlying statistics and distributions of the signal. It can therefore have poor performance if the underlying signals have statistics and/or properties that deviate from those that are assumed. It is also not suitable for learning applications, since it has no possibility of "learning" the signal statistics.

Another approach to filtering that was developed in the 80's and 90's was H_∞ filtering, where the noise sources were considered adversarial and the *worst-case* estimation error energy was minimized (over all bounded energy noises). While H_∞ estimators are robust to lack of statistical knowledge of the underlying noise sources, and have some deep connections to classical learning algorithms (see, e.g. Hassibi et al. (1995)), they are often too conservative since they safeguard against the worst-case and do not exploit the noise structure.

1.1 Main contributions

The contributions can be summarized as follows:

- Motivated by the concept of regret in learning problems (e.g., Hazan (2019), Simchowitz (2020), Abbasi-Yadkori (2011),(2019)), we propose to adopt it for filtering problems so as to bridge between the philosophies of Kalman and H_∞ filtering. Specifically, we formulate a new design criterion for filtering which optimizes the difference in estimation error energies between a clairvoyant estimator that has access to the entire observations sequence (including future samples) and a causal one that does not have access to future observations. We show that the regret formulation is fundamentally different from the H_2 (e.g., the Kalman filter by Kalman (1960)) and H_∞ criteria (see the tutorial by Shaked et al. (1992)).

- We show that the regret-optimal estimation problem can be reduced to the classical Nehari problem in operator theory (Theorem 1). This is the problem of approximating an anti-causal operator with a causal one in the operator norm by Nehari (1957).

- When the underlying signals have a state space structure, we provide an explicit solution for the regret-optimal filter. The solution to the filtering problem is given as via simple steps; first, the optimal regret value is determined by solving two Riccati equations and a single Lyapunov, along with a bisection method with a simple condition that is given in Theorem 2. Then, the regret-optimal filter is given explicitly in a state-space form in Theorem 4.

- We present numerical examples that demonstrate the efficacy and applicability of the approach and observe that the regret-optimal filter has average and worst-case performances that are simultaneously close to their optimal values. We therefore argue that regret-optimality is a viable approach to estimator design.

2 The Setting and Problem Formulation

2.1 Notation

Linear operators are denoted by calligraphic letters, e.g., $\mathcal{X}$. Finite-dimensional vectors and matrices are denoted with small and capital letters, respectively, e.g., x and X. Subscripts are used to denote time indices e.g., x_i, and boldface letters denote the set of finite-dimensional vectors at all times, e.g., $\mathbf{x} = \{x_i\}_i$.

2.2 The setting and problem formulation

We consider a general estimation problem

$$\begin{aligned} \mathbf{y} &= \mathcal{H}\,\mathbf{w} + \mathbf{v} \\ \mathbf{s} &= \mathcal{L}\,\mathbf{w} \end{aligned} \tag{1}$$

where $\mathcal{H}$ and $\mathcal{L}$ are strictly causal operators, the sequence $\mathbf{w}$ denotes an exogenous disturbance w that generates a hidden state as the output of an operator $\mathcal{H}$, $\mathbf{y}$ denotes the observations process and $\mathbf{s}$ denotes the signal that should be estimated. Note that we did not specify yet the estimator operation or the time horizon. The setting is quite general and includes for instance the state-space model that will be presented in the next section.

A linear estimator is a linear mapping from the observations to the signal space $\mathbf{s}$ and is denoted as $\hat{\mathbf{s}} = \mathcal{K}\,\mathbf{y}$. Then, for any $\mathcal{K}$, the estimation error of the signal is

$$\begin{aligned} \mathbf{e} &= \mathbf{s} - \hat{\mathbf{s}} \\ &= \begin{pmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \mathbf{v} \end{pmatrix} \\ &\triangleq T_{\mathcal{K}} \begin{pmatrix} \mathbf{w} \\ \mathbf{v} \end{pmatrix} \end{aligned} \tag{2}$$

Note that the estimation error is a function of the driving disturbance $\mathbf{w}$ and the observation noise $\mathbf{v}$. The squared error can then be expressed as

$$\begin{aligned} \mathbf{e}(\mathbf{w}, \mathbf{v}, \mathcal{K}) &\triangleq \mathbf{e}^*\mathbf{e} \\ &= \begin{pmatrix} \mathbf{w}^* & \mathbf{v}^* \end{pmatrix} T_{\mathcal{K}}^* T_{\mathcal{K}} \begin{pmatrix} \mathbf{w} \\ \mathbf{v} \end{pmatrix}. \end{aligned} \tag{3}$$

Different assumptions on the driving disturbance and the observation noise sequences give rise to different estimators: in the stochastic setting to the celebrated Kalman filter, and in the deterministic setting of bounded energy disturbances to H_∞ estimators. A common characteristic of these two paradigms is that if we do not restrict the constructed estimators to be causal, then there exists a single linear estimator that attains the minimal Frobenius and operator norms simultaneously. This known fact is summarized in the following lemma.

Lemma 1 (The non-causal estimator)**.** *For the H_2 and the H_∞ problems, the optimal non-causal estimator is $\mathcal{K}_0 = \mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-1}$.*

Note that the non-causal estimator cannot be implemented in practice even for simple operator $\mathcal{L}, \mathcal{H}$ since it requires access to future instances of the observations. However, the fact that there is a single estimator that simultaneously optimizes these two norms naturally leads to our new approach of regret optimization. Specifically, we will aim at constructing a causal (or strictly causal) estimator that performs as close as possible to the non-causal estimator in Lemma 1.

The optimal regret can now be defined as

$$\begin{aligned} \text{regret}^* &= \min_{\text{causal } \mathcal{K}} \max_{\mathbf{w},\mathbf{v}\in\ell_2,\mathbf{w},\mathbf{v}\neq 0} \frac{|\mathbf{e}(\mathbf{w},\mathbf{v},\mathcal{K}) - \mathbf{e}(\mathbf{w},\mathbf{v},\mathcal{K}_0)|}{\|\mathbf{w}\|^2 + \|\mathbf{v}\|^2} \\ &= \min_{\text{causal } \mathcal{K}} \|T_{\mathcal{K}}^* T_{\mathcal{K}} - T_{\mathcal{K}_0}^* T_{\mathcal{K}_0}\|. \end{aligned} \tag{4}$$

In words, the defined regret metric measures the worst-case deviation of the estimation error from the estimation error of the non-causal estimator across all bounded-energy disturbances sequences. It is illuminating to compare now the regret criterion with the traditional H_∞ estimation:

$$\underbrace{\inf_{\text{causal } K} \|T_K^* T_K\|}_{H_\infty \text{ estimation}} \quad , \quad \underbrace{\inf_{\text{causal } K} \|T_K^* T_K - T_{K_0}^* T_{K_0}\|}_{\text{regret-optimal estimation}}.$$

The difference is now transparent; in H_∞ estimation, one attempts to minimize the worst-case gain from the disturbances energy to the estimation error, whereas in regret-optimal estimation one attempts to minimize the worst-case gain from the disturbance energy to the regret. It is this latter fact that makes the regret-optimal estimator more adaptive since it has as its baseline the best that any noncausal estimator can do, whereas the H_∞ estimator has no baseline to measure itself against. This fact will be illustrated in Section 4, where we will show that the regret definition results in an estimator that interpolates between the H_2 and the H_∞ design criteria.

Simplifying the optimal regret to have a simple formula is a difficult task and. Therefore, in this paper, we define a sub-optimal problem of determining whether the optimal regret is below, above or equal to a given threshold γ. This is made precise in the following problem definition.

Problem 1 (The γ-optimal regret estimation problem)**.** *For a fixed γ, if exists, find a causal estimator $\mathcal{K}$ such that*

$$\|T_{\mathcal{K}}^* T_{\mathcal{K}} - T_{\mathcal{K}_0}^* T_{\mathcal{K}_0}\|_\infty \leq \gamma^2. \tag{5}$$

A *γ-optimal estimator* is referred to as any solution to Problem 1.

Finally, we define a fundamental problem which will serve as the main tool in the derivations.

Problem 2 (The Nehari problem)**.** *Given an anti-causal and bounded operator $\mathcal{U}$, find a causal operator $\mathcal{K}$ such that $\|\mathcal{K} - \mathcal{U}\|$ is minimized.*

This problem is well known as the Nehari problem. In the general operator notation, it is difficult to derive an explicit formulae for the approximation $\mathcal{K}$ and the minimal value of a valid γ_N. However, when there is a state-space structure to the operator $\mathcal{U}$, then the problem has a closed-form solution that will be presented in Section 6.

2.3 The state-space model

The setting defined above in its operator notation is general and cannot have an explicit structured solution. In many cases, including our problem, imposing a state space structure for the problem provides means to obtain explicit estimators. In the state-space setting, the equations in (1) are simplify to

$$\begin{aligned} x_{i+1} &= Fx_i + Gw_i \\ y_i &= Hx_i + v_i \\ s_i &= Lx_i, \end{aligned} \tag{6}$$

where x_i is the hidden state, y_i is the observation and s_i corresponds to the signal that needs to be estimated. We also make the standard assumption that the pair (F, H) is detectable. To recover the state-space setting from its operator notation counterpart in (1), choose $\mathcal{H}$ and $\mathcal{L}$ as Toeplitz operators with Markov parameters HF^iG and LF^iG, respectively.

A causal estimator is defined as a sequence of mappings $\pi_i(\cdot)$ with the estimation being $\hat{s}_i = \pi_i(\{y_j\}_{j\leq i})$. The estimation error at time i is

$$e_i = s_i - \hat{s}_i. \tag{7}$$

In a similar fashion, we can define a strictly causal estimator as a sequence of strictly causal mappings, i.e., $\hat{s}_i = \pi_i(\{y_j\}_{j<i})$. Due to lack of space in this paper, we will not present the solution for the strictly causal setting which follows from the same steps that will be taken for the causal scenario.

Note that we did not specify the time horizon of the problem so that the formulation here and in the previous section hold for finite, one-sided infinite and doubly infinite time horizons. However, to simplify the derivations of the state-space, we will focus here on the case of doubly-infinite time horizon where the total estimation error energy can be expressed as $\sum_{i=-\infty}^{\infty} e_i^* e_i$.

In this case, it is also convenient to define the causal transfer matrices

$$\begin{aligned} H(z) &= H(zI - F)^{-1}G, \\ L(z) &= L(zI - F)^{-1}G. \end{aligned} \tag{8}$$

that describe the filters whose input is the disturbance w and their outputs are the observation y and the target signal s, respectively. We now proceed to show the main results of this paper.

3 Main results

In this section, we present our main results. We first provide the reduction of the general regret estimation problem to a Nehari problem. In Section 3.1, we provide an explicit solution for the state-space setting in the causal scenario.

Theorem 1 (Reduction to the Nehari Problem)**.** *A γ-optimal estimator exists if and only if there exists a solution to the Neahri problem*

$$\min_{causal\ \mathcal{K}} \|\{\nabla_\gamma \mathcal{K}_0 \Delta\}_- - \mathcal{K}\| \leq 1, \tag{9}$$

where $\{\cdot\}_-$ denotes the strictly anticausal part of its argument, and Δ, ∇_γ are causal operators that are obtained from the canonical factorizations

$$\begin{aligned} \Delta\Delta^* &= I + \mathcal{H}\mathcal{H}^* \\ \nabla_\gamma^* \nabla_\gamma &= \gamma^{-2}(I + \gamma^{-2}\mathcal{L}(I + \mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^*). \end{aligned} \tag{10}$$

Let $(\gamma^*, \mathcal{K}_N)$ be a solution that achieves the upper bound in the Nehari problem (9)*, then a regret-optimal estimator is given by*

$$\mathcal{K} = \nabla_{\gamma^*}^{-1}(\mathcal{K}_N + \{\nabla_{\gamma^*}\mathcal{K}_0\Delta\}_+)\Delta^{-1} \tag{11}$$

where $\{\cdot\}_+$ denoted the causal part of an operator.

For general operators $\mathcal{L}, \mathcal{H}$, the reduction in Theorem 1 does not give practical means to derive an implementable estimator. However, it provides the outline of the necessary technical steps in order to have explicit characterizations in the state-space setting. Specifically, in the state-space setting we will need to perform two canonical spectral factorizations (Eq. 10) and a decomposition of the operator $\nabla_{\gamma^*}\mathcal{K}_0\Delta$ into causal and anticausal operators. The proof of Theorem 1 appears in Section 5.

3.1 Solution for the state-space setting

We now proceed to particularize our results to the state-space representation of the estimation problem. Towards our main objective to derive the regret-optimal estimator, we will solve the sub-optimal problem, i.e., for a given γ. Thus, our results are presented in two steps. First, we provide a simple condition to verify whether the value of γ is valid or not. Then, assuming that the threshold γ have been optimized, we present the regret-optimal estimator.

Throughout the derivations, there are three Riccati and a single Lyapunov equations. The first Riccati equation is the standard one from the Kalman filter, i.e.,

$$P = GG^* + FPF^* - FPH^*(I + HPH^*)^{-1}HPF^*. \tag{12}$$

The stabilizing solution is denoted as P, its feedback gain as $K_P = FPH^*(I + HPH^*)^{-1}$ and its closed-loop system as $F_P = F - GK_P$.

The remaining two Riccati equation depend on the parameter γ and therefore should be part of the optimization on γ. Define the γ-dependent Riccati equations as

$$\begin{aligned} W &= H^*H + \gamma^{-2}L^*L + F^*WF - K_W^* R_W K_W \\ Q &= -GR_W^{-1}G^* + F_W Q F_W^* - K_Q R_Q^{-1} K_Q^*, \end{aligned} \tag{13}$$

with

$$\begin{aligned} K_W &= R_W^{-1}G^*WF; \quad R_W = I + G^*WG \\ K_Q &= F_W Q L^* R_Q^{-1}; \quad R_Q = \gamma^2 I + LQL^*. \end{aligned} \tag{14}$$

Additionally, define the corresponding closed-loop systems $F_Q = F_W - K_Q L$ and $F_W = F - GK_W$, and the factorizations $R_W = R_W^{*/2} R_W^{1/2}$ and $R_Q = R_Q^{1/2} R_Q^{*/2}$. Note that the Riccati equation for Q depends on the solution to Riccati equation for W.

Finally, define U as the solution to the Lyapunov equation

$$U = K_Q L P F_P^* + F_Q U F_P^*. \tag{15}$$

We are now ready to present the condition for the existence of a regret-optimal estimator.

Theorem 2 (Condition for Estimator Existence)**.** *A γ-optimal estimator exists if and only if*

$$\bar{\sigma}(Z_\gamma \Pi) \leq 1, \tag{16}$$

where Z_γ and Π are the solutions to the Lyapunov equations

$$\begin{aligned} \Pi &= F_P^* \Pi F_P + H^*(I + HPH^*)^{-1}H \\ Z_\gamma &= F_P Z_\gamma F_P^* + F_P(P - U)^* L^* R_Q^{-1} L (P - U) F_P^*. \end{aligned} \tag{17}$$

A regret-optimal estimator that attains the optimal regret can be found by optimizing over γ in (16) so that the maximal singular value is arbitrarily close to 1. From now on, we assume that the value of γ is fixed after the optimization which fix in turn the γ-dependent quantities (W, Q, U, Z_γ).

A key element in our solution to the regret-optimal estimator is a solution to the Nehari problem in Theorem 1. Recall that its solution provides the best approximation (in the operator norm) for the anticausal part of the transfer function $\nabla_\gamma^{-1}(z)L(z)H^*(z^{-*})\Delta^{-*}(z^{-*})$. We denote this anticausal part as $T(z)$ which appears explicitly below in (37). By having the operator $T(z)$, we can provide a solution for the Nehari problem.

Lemma 2. *For any γ, the optimal solution to the Nehari problem with $T(z)$ in (37) is*

$$K_N(z) = \tilde{\Pi}(I + F_N(zI - F_N)^{-1})G_N, \tag{18}$$

where

$$\begin{aligned} G_N &= (I - F_P Z_\gamma F_P^* \Pi)^{-1} F_P Z_\gamma H^*(I + HPH^*)^{-*/2} \\ F_N &= F_P - G_N(I + HPH^*)^{-1/2} H \\ \tilde{\Pi} &= R_Q^{-1/2} L(P - U) F_P^* \Pi \end{aligned} \tag{19}$$

where (Z_γ, Π) are defined in (17).

Although the solution to the Nehari problem is given for any value of γ, it should be clear that it should be chosen accordingly with Theorem 2 in order to result in a γ-optimal estimator.

The following theorem reveals the structure of the regret-optimal estimator in the frequency domain.

Theorem 3 (The Regret-Optimal Estimator in Frequency Domain)**.** *Given the optimal threshold γ^*, a regret-optimal estimator for the causal scenario is given by*

$$K(z) = \nabla_{\gamma^*}^{-1}(z)[K_N(z) + S(z)]\Delta^{-1}(z) + K_{H_2}(z),$$

with

$$\begin{aligned} \nabla_\gamma^{-1}(z) &= (I + L(zI - F_W)^{-1}K_Q)R_Q^{1/2} \\ S(z) &= -R_Q^{-1/2} L[(zI - F_Q)^{-1}F_Q + I]UH^*(I + HPH^*)^{-*/2} \\ \Delta^{-1}(z) &= (I + HPH^*)^{-1/2}(I + H(zI - F)^{-1}K_P)^{-1}, \end{aligned} \tag{20}$$

where all constants are defined in (12)-(15), $K_N(z)$ is given in (18) and $K_{H_2}(z)$ is the causal H_2 (Kalman) filter:

$$\begin{aligned} K_{H_2}(z) &= LPH^*(I + HPH^*)^{-1} \\ &\quad + L(I - PH^*(I + HPH^*)^{-1}H)(zI - F_P)^{-1}K_P. \end{aligned}$$

It is interesting to note that the causal Kalman filter naturally appears as part of our solution to the regret-optimal estimation. This implies that the regret-optimal estimator is a sum of two terms; the first is a Kalman filter which is designed to minimize the Frobenius norm of the operator T_K, while the other term is resulted from the Nehari and guarantees that the regret criterion is minimized.

At this point, the frequency-domain results can be converted into a simple state-space.

Theorem 4 (The Causal Regret-Optimal Estimator)**.** *Given the optimal threshold γ^*, a regret-optimal estimator for the causal scenario is given by*

$$\begin{aligned} \xi_{i+1} &= \tilde{F}\xi_i + \tilde{G}y_i \\ \hat{s}_i &= \tilde{H}\xi_i + \tilde{J}y_i. \end{aligned} \tag{21}$$

where the matrices are given by

$$\tilde{F} = \begin{pmatrix} F_P & 0 & 0 \\ \tilde{F}_{2,1} & F_N & 0 \\ \tilde{F}_{3,1} & \tilde{F}_{3,2} & F_W \end{pmatrix}; \tag{22}$$

$$\begin{aligned} \tilde{H} &= \begin{pmatrix} \tilde{H}_1 & R_Q^{1/2}\tilde{\Pi}F_N & L \end{pmatrix} \\ \tilde{G} &= \begin{pmatrix} K_P \\ G_N(I + HPH^*)^{-1/2} \\ \tilde{G}_3 \end{pmatrix} \\ \tilde{J} &= L(P - U)H^*(I + HPH^*)^{-1} \\ &\quad + R_Q^{1/2}\tilde{\Pi}G_N(I + HPH^*)^{-1/2}, \end{aligned} \tag{23}$$

and the explicit constants are

$$\begin{aligned} \tilde{F}_{2,1} &= -G_N(I + HPH^*)^{-1/2}H \\ \tilde{F}_{3,1} &= F_W UH^*(I + HPH^*)^{-1}H \\ &\quad - K_Q R_Q^{1/2}\tilde{\Pi}G_N(I + HPH^*)^{-1/2}H \\ \tilde{F}_{3,2} &= K_Q R_Q^{1/2}\tilde{\Pi}F_N \\ \tilde{H}_1 &= L - L(P - U)H^*(I + HPH^*)^{-1}H \\ &\quad - LK_Q R_Q^{1/2}\tilde{\Pi}G_N(I + HPH^*)^{-1/2}H \\ \tilde{G}_3 &= K_Q R_Q^{1/2}\tilde{\Pi}G_N(I + HPH^*)^{-1/2} \\ &\quad - F_W UH^*(I + HPH^*)^{-1}. \end{aligned} \tag{24}$$

with the Riccati variables defined in (12)-(15) and the variables $(F_N, G_N, \tilde{\Pi})$ defined in (19).

By Theorem 4, given the optimal threshold γ^*, the regret-optimal estimator can be easily implemented. Note that the $\gamma-$dependent variables should be computed only throughout the process of determining γ^* but not throughout the estimation process itself. Thus, from computational perspective, the filter requires the same resources as the standard Kalman filter. Its internal state inherits the finite dimension of the original state space but has an increased dimension with a factor of three.

4 Numerical examples

We have performed two numerical experiments to evaluate the performance of the regret-optimal estimator compared to the traditional H_2 and H_∞ estimators. As mentioned earlier, the performance of any (linear) estimator is governed by the transfer operator T_K that maps the disturbance sequences $\mathbf{w}$ and $\mathbf{v}$ to the errors sequences $\mathbf{e}$. It will be useful to represent this operator via its transfer function in the z-domain, i.e.,

$$T_K(z) = \left[\begin{array}{cc} L(z) - K(z)H(z) & -K(z) \end{array}\right].$$

The squared Frobenius norm of T_K, which is what the H_2 estimator minimizes, is given by

$$\|T_K\|_F^2 = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{trace}\left(T_K^*(e^{j\omega})T_K(e^{j\omega})\right) d\omega,$$

and the squared operator norm of T_K, which is what H_∞ estimators minimize, by

$$\|T_K\|^2 = \max_{0\le\omega\le 2\pi} \bar{\sigma}\left(T_K^*(e^{j\omega})T_K(e^{j\omega})\right),$$

where $\bar{\sigma}(\cdot)$ denotes the maximal singular value of a matrix.

4.1 Scalar systems

We start with a simple scalar system to illustrate the results. For scalar systems, $T_K(z)$ is a 1-by-2 vector so we have that

$$\|T_K\|_F^2 = \frac{1}{2\pi}\int_0^{2\pi} \left\|T_K(e^{j\omega})\right\|^2 d\omega.$$

Consider now a simple stable scalar state-space with $F = 0.9$, $H = L = G = 1$. For such a system, we have constructed the optimal H_2, H_∞, and non-causal estimators, as well as the regret-optimal estimator. Plotting the value of $\|T_K(e^{j\omega})\|^2$, as a function of frequency, is quite illuminating as it allows one to assess and compare the performance of the respective estimators across the full range of input disturbances. This is done in Figure 1.

As can be seen, the non-causal estimator outperforms the other three estimators across all frequencies. The H_2 estimator minimizes the Frobenius norm, i.e., the average performance over iid w, which is the area under the curve. However, in doing so, it sacrifices the worst-case performance and so has a relatively large peak at low frequencies. The H_∞ estimator minimizes the operator norm, i.e., the worst-case performance,

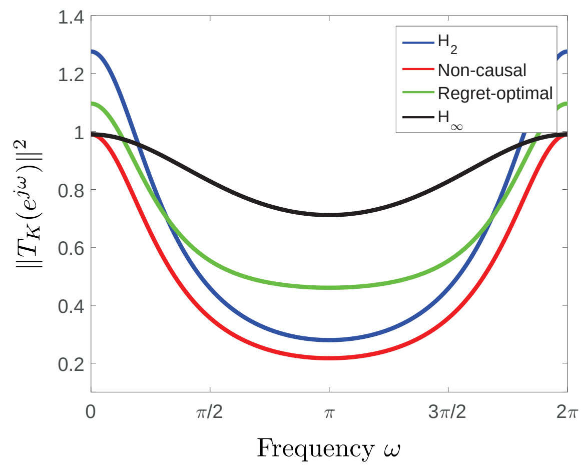
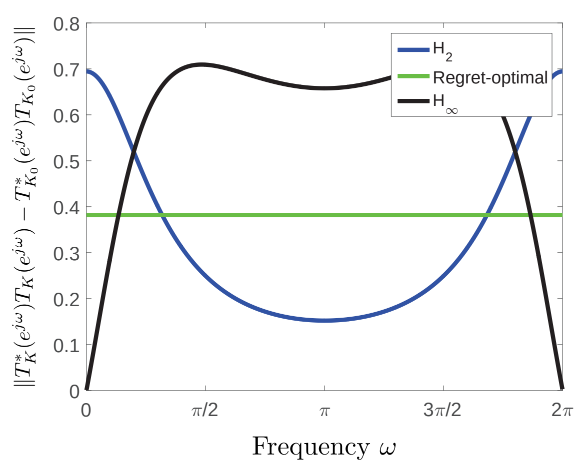


Figure 1: The squared operator norm as a function of the frequency parameter for the scalar system in Section 4.1. The norm is compared between the H_2, H_∞, non-causal and our new regret-optimal estimator. As can be seen, the non-causal estimator achieves the best performance at all frequencies. As expected, among all causal estimators, the H_∞ estimator achieves the lowest peak, and the H_2 estimator attains the smallest area under its curve (i.e., integral). Out new estimator attains the best of the two worlds as it achieves a lower peak than the H_2 estimator, and a comparable area with the H_2 estimator. Precise comparison of the resulted norms appears in Table 1.

Table 1: Performance for the Scalar Example

	$\|T_K\|_F^2$	$\|T_K\|^2$	Regret
Noncausal estimator	0.46	0.99	0
Regret-optimal	0.65	1.1	0.38
H_2 estimator	0.6	1.27	0.7
H_∞ estimator	0.94	0.99	0.71



which is the peak of the curve. (Here we can see that the H_∞-optimal estimator has the same peak as the non-causal estimator meaning that it attains the same worst-case performance.) However, in doing so, it sacrifices the average performance and has a relatively large area under the curve. Recall that the regret-optimal estimator aims to mimic the non-causal behavior. In doing so, it achieves the best of both worlds: it has an area under the curve that is close to that of the H_2-optimal estimator (0.6 vs 0.65), and it has a peak that significantly improves upon the the peak of the H_2-optimal estimator. The precise norms are presented in Table 1.

It is also illuminating to examine our new regret criterion in Fig. 4.1. We plot the regret of the causal estimators with respect to the non-causal estimator. It can be seen that at low frequencies, the H_∞ estimator has the lowest regret, while at mid-frequencies it is the H_2 estimator. However, their peak is almost twice that of the regret-optimal estimator that maintains almost a constant regret across all frequencies.

4.2 Tracking example

Here, we will study a one-dimensional tracking problem whose state-space model is

$$\begin{pmatrix} x_{i+1} \\ \nu_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & \Delta T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ \nu_i \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta T \end{pmatrix} a_i$$

$$y_i = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ \nu_i \end{pmatrix} + v_i \tag{25}$$

$$s_i = x_{i+1}, \tag{26}$$

where x_i corresponds to the position, ν_i corresponds to velocity and a_i to the exogenous acceleration. The desired signal is the position of the object *at the next time step* $s_i = x_{i+1}$, and the observations signal is the noisy position $y_i = x_i + v_i$, where v_i is measurement noise. The frequency reponse of the various estimators is presented in Fig. 2 and Table 2 summarizes their performance.

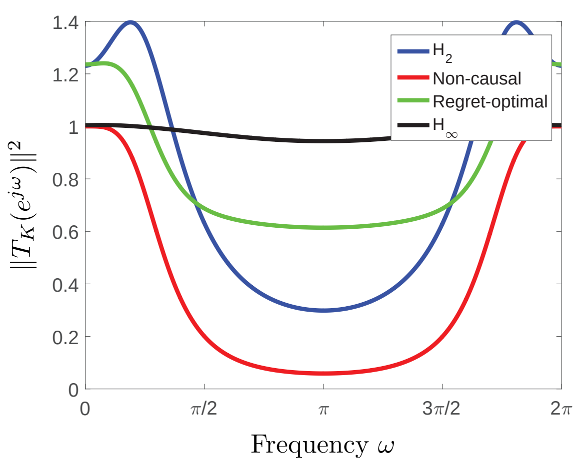


Figure 2: The frequency response of the various estimators for the tracking example in Section 4.2. Comparison of the corresponding norms appears in Table 2.

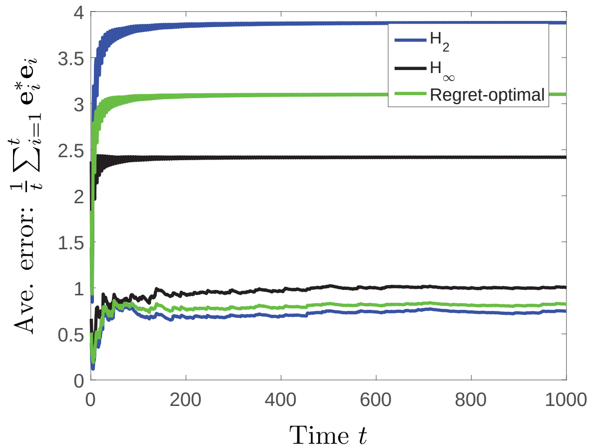


Figure 3: Time-averaged estimation error energy as a function of time for the tracking example with two different disturbances. In the bottom three curves, the state-space model is driven with Gaussian disturbances. In the top three curves, it is driven with an adversarial disturbance.

Table 2: Performance for the Tracking Experiment

	$\|T_K\|_F^2$	$\|T_K\|^2$	Regret
Noncausal estimator	0.39	1	0
Regret-optimal	0.82	1.24	0.65
H_2 estimator	0.77	1.4	1.02
H_∞ estimator	0.97	1	0.95

The time domain performance of the various filters is given in Fig. 3. We plot the time-averaged estimation error energy as a function of time for the H_2, H_∞, and regret-optimal filters for two different types of noise. One is the white Gaussian noise for which the H_2 filter is the optimal, and one is an adversarial noise for which the H_∞ filter is the best. As can be seen, the regret-optimal filter has a performance that interpolates nicely between these filters and achieves good performance across a range of disturbances.

5 Proof of Theorem 1

Recall that we aim to solve the sub-optimal problem

$$T_{\mathcal{K}}^* T_{\mathcal{K}} - T_{\mathcal{K}_0}^* T_{\mathcal{K}_0} \preceq \gamma^2 I. \tag{27}$$

By the Schur complement and the *Matrix inversion lemma* (in its operator form), we can write

$$T_{\mathcal{K}}(\gamma^{-2} I - \gamma^{-2} T_{\mathcal{K}_0}^* (I + \gamma^{-2} T_{\mathcal{K}_0} T_{\mathcal{K}_0}^*)^{-1} \gamma^{-2} T_{\mathcal{K}_0}) T_{\mathcal{K}}^* \preceq I.$$

It can now be shown that for any $\mathcal{K}$,

$$T_{\mathcal{K}}T_{\mathcal{K}_0}^* = \begin{pmatrix}\mathcal{L}-\mathcal{K}\mathcal{H} & -\mathcal{K}\end{pmatrix}\begin{pmatrix} I \\ -\mathcal{H}\end{pmatrix}(I+\mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^* = T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*. \quad (28)$$

Combining the simplified condition with (28) gives

$$\begin{aligned} T_{\mathcal{K}}T_{\mathcal{K}}^* &\preceq \gamma^2 I + \gamma^{-2}T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*(I+\gamma^{-2}T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*)^{-1}T_{\mathcal{K}_0}T_{\mathcal{K}_0}^* \\ &= \gamma^2 I + T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*(\gamma^2 I+T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*)^{-1}T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*. \end{aligned} \quad (29)$$

By the completion of square, we also have

$$T_{\mathcal{K}}T_{\mathcal{K}}^* = (\mathcal{K}-\mathcal{K}_0)(I+\mathcal{H}\mathcal{H}^*)(\mathcal{K}-\mathcal{K}_0)^* + T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*$$

and rearranging the RHS of the condition gives

$$\begin{aligned} &\gamma^2 I + T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*(\gamma^2 I+T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*)^{-1}T_{\mathcal{K}_0}T_{\mathcal{K}_0}^* - T_{\mathcal{K}_0}T_{\mathcal{K}_0}^* \\ &= \gamma^2(I+\gamma^{-2}T_{\mathcal{K}_0}T_{\mathcal{K}_0}^*)^{-1}. \end{aligned} \quad (30)$$

To conclude, the condition can be written as

$$\begin{aligned} &(\mathcal{K}-\mathcal{K}_0)(I+\mathcal{H}\mathcal{H}^*)(\mathcal{K}-\mathcal{K}_0)^* \\ &\quad\preceq \gamma^2(I+\gamma^{-2}\mathcal{L}(I+\mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^*)^{-1} \end{aligned} \quad (31)$$

By defining the canonical factorizations

$$\begin{aligned} \Delta\Delta^* &= I+\mathcal{H}\mathcal{H}^* \\ \nabla_\gamma^*\nabla_\gamma &= \gamma^{-2}(I+\gamma^{-2}\mathcal{L}(I+\mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^*). \end{aligned} \quad (32)$$

and applying the Schur complement again gives that

$$(\mathcal{K}\Delta-\mathcal{K}_0\Delta)^*\nabla_\gamma^*\nabla_\gamma(\mathcal{K}\Delta-\mathcal{K}_0\Delta) \preceq I. \quad (33)$$

Note that $\nabla_\gamma\mathcal{K}\Delta$ is a causal operator. Now, let $\nabla_\gamma\mathcal{K}_0\Delta = \mathcal{S}+\mathcal{T}$ where $\mathcal{S}$ is a causal operator and $\mathcal{T}$ is a strictly anticausal operator (both operators depend on γ implicitly). Then, if $\mathcal{K}_N$ is a solution to the Nehari problem $\|\mathcal{K}_N - \mathcal{T}\| \le 1$, then a γ-optimal estimator is given by $\nabla^{-1}(\mathcal{K}_N+\mathcal{S})\Delta^{-1}$.

6 Proof Outline of the State-Space

In this section, we present the main lemmas that constitute the explicit solution for the state-space setting. As written above, there are three technical lemmas to obtain a Nehari problem. The solution to the Nehari problem is known and appears in the supplementary material. Proofs of the technical lemmas appear in the supplementary material as well.

The first factorization appears as follows.

Lemma 3. *The transfer function $I+H(z)H^*(z^{-*})$ can be factored as*

$$\Delta(z)\Delta^*(z^{-*}) = I + H(z)H^*(z^{-*})$$

with

$$\Delta(z) = (I+H(zI-F)^{-1}K_P)(I+HPH^*)^{1/2} \quad (34)$$

where $(I+HPH^*)^{1/2}(I+HPH^*)^{*/2} = I+HPH^*$, $K_P = FPH^*(I+HPH^*)^{-1}$ and P is the stabilizing solution to the Riccati equation

$$P = GG^* + FPF^* - FPH^*(I+HPH^*)^{-1}HPF^*.$$

Moreover, the transfer function $\Delta^{-1}(z)$ is bounded on $|z|\ge 1$.

In the second factorization, the expression we aim to factor is positive but the order of its causal and anticausal components are in the reversed order. This is resolved with an additional Riccati equation.

Lemma 4. *For any $\gamma > 0$, the factorization $\nabla_\gamma^*(z^{-*})\nabla_\gamma(z) = \gamma^{-2}(I+\gamma^{-2}L(z)(I+H^*(z^{-*})H(z))^{-1}L^*(z^{-*}))$ holds with*

$$\nabla_\gamma(z) = R_Q^{-1/2}(I - L(zI-F_Q)^{-1}K_Q), \quad (35)$$

where $R_Q = R_Q^{1/2}R_Q^{*/2}$, Q is a solution to the Riccati equation

$$Q = -GR_W^{-1}G^* + F_W Q F_W^* - K_Q R_Q K_Q^*,$$

and $K_Q = F_W QL^*R_Q^{-1}$ and $R_Q = \gamma^2 I + LQL^*$ and the closed-loop system $F_Q = F_W - K_Q L$. The constants (F_W, K_W) are obtained from the solution W to the Riccati equation

$$W = H^*H + L_\gamma^* L_\gamma + F^*WF - K_W^* R_W K_W, \quad (36)$$

and $K_W = R_W^{-1}G^*WF$ and $R_W = I+G^*WG$ with $R_W = R_W^{*/2}R_W^{1/2}$ and $F_W = F - GK_W$.

The following lemma is the required decomposition.

Lemma 5. *The product of the transfer matrices $\nabla_\gamma(z)L(z)H^*(z^{-*})\Delta^{-*}(z^{-*})$ can be written as the sum of an anticausal transfer function*

$$\begin{aligned} T(z) &= R_Q^{-1/2}L(P-U)F_P^* \\ &\quad\cdot(z^{-1}I-F_P^*)^{-1}H^*(I+HPH^*)^{-*/2}. \end{aligned} \quad (37)$$

and a causal transfer function

$$\begin{aligned} S(z) &= \nabla_\gamma(z)L[(zI-F)^{-1}F+I]PH^*(I+HPH^*)^{-*/2} \\ &\quad - R_Q^{-1/2}L[(zI-F_Q)^{-1}F_Q+I]UH^*(I+HPH^*)^{-*/2}, \end{aligned}$$

where U solves $U = K_Q LPF_P^* + F_Q UF_P^*$.

It can be shown that the first line of $S(z)$ is $\nabla_\gamma(z)K_{H_2}(z)\Delta(z)$ where $K_{H_2}(z)$ is the optimal H_2 estimator. By having the decomposition and the anticausal $T(z)$ in part, we can apply the results of the Nahari problem to obtain Lemma 2.

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