Hydromagnetic Stability of a Streaming Cylindrical Incompressible Plasma*

N. J. ZABUSKY
Department of Physics, California Institute of Technology, Pasadena, California†
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A dispersion relation is derived and analyzed for the case where the equilibrium velocity of an incompressible, nonresistive, cylindrical plasma has a spiral motion along magnetic field lines. The symmetric hydromagnetic equations are used to derive the plasma hydromagnetic pressure. The dispersion relation is found by matching plasma and outer-region hydromagnetic pressures across a sharp-moving interface. The zeros of the dispersion relation are obtained by a sequence of mappings between three complex planes. The presence of flow introduces overstable modes. For $m = 0$ the time-divergences are removed by flow. For $m = 1$ the divergences are enhanced by flow such that the growth rates and oscillation frequencies increase linearly with the flow velocity. The smaller is the wavelength of the disturbance in the z direction, the larger are the overstable eigenvalues.

I. INTRODUCTION

THE criteria for the stability of an infinite cylindrical plasma have been determined by Kruskal and Tuck,[1] Rosenbluth,[2] and Roberts,[3] among others. All these analyses considered the plasma to be at rest in the equilibrium state. Kruskal and Tuck derived a dispersion relation, or characteristic eigenvalue equation, from a normal mode analysis and demonstrated that large longitudinal magnetic fields can remove sausage ($m = 0$) instabilities. Spiral ($m = 1$) instabilities of long wavelength (in the z direction) cannot be removed, since the strength of the longitudinal field required varies directly as the wavelength which is to be stabilized. Rosenbluth showed that a perfect conductor, external but close to the plasma, will remove this large wavelength instability.

The dispersion relation for an incompressible cylindrical plasma with an equilibrium velocity field is derived below. This relation is obtained by matching the complete stress tensor across a perturbed boundary which separates regions of different media and magnetic fields. The differential equation which characterizes the plasma is derived from the Elsasser[4] symmetric hydromagnetic equations.

The presence of flow causes oscillatory instabilities (complex eigenvalues). It will be shown that flow removes sausage ($m = 0$) instabilities while it enhances spiral ($m = 1$) instabilities. The latter effect is due to the *centrifugal forces* acting on the deformed cylinder.

Recently, Trehan[5] incorporated the effects of a finite plasma velocity into a stability-normal-mode analysis. However, the boundary conditions were treated in a manner which restricts the applicability of the results to certain special cases. In Sec. IX, it is shown that his boundary condition implicitly links the strength of the longitudinal magnetic fields with that of the velocity fields. This accounts for his conclusion that a spirally deformed plasma is stable when the flow parameter is large.

II. EQUATIONS FOR THE HYDROMAGNETIC APPROXIMATION

A. Differential Equations

The hydromagnetic equations for a compressible medium can be written as[6]

$$\partial_t \rho + \nabla\cdot(\rho \mathbf{V}) = 0, \tag{2.1}$$

$$\partial_t(\rho\mathbf{V}) + \nabla\cdot(\rho\mathbf{V}\mathbf{V}) = -\nabla p + \mathbf{j}\times\mathbf{B}, \tag{2.2}$$

$$D_t(p\rho^{-\gamma}) = 0, \tag{2.3}$$

where

$$D_t = \partial_t + \mathbf{V}\cdot\nabla, \tag{2.4}$$

and where we have made the standard hydromagnetic approximations, namely[7]: (1) the pressure tensor is diagonal with equal elements p; (2) the electric volume force $\sigma\mathbf{E}$, is negligible in comparison

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† Present address: Max-Planck-Institut für Physik und Astrophysik, Munich, Germany.

1 M. Kruskal and J. L. Tuck, Proc. Roy. Soc. (London) **A245**, 222 (1958).

2 M. Rosenbluth Proc. Third Intern. Cong. Phenomena Ionized Gases, Venice (1957), p. 888.

3 P. H. Roberts, Astrophys. J. **124**, 430 (1956).

4 W. N. Elsasser, Phys. Rev. **79**, 183 (1950).

5 S. K. Trehan, Astrophys. J. **128**, 475 (1959).

6 See reference cited in footnote 1, Eqs. 1, 2, and 8.

7 The conditions under which these approximations are valid are given in Sec. 2 of I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. (London) **A244**, 17 (1958).

with the magnetic volume force; (3) displacement currents are negligible; (4) the thermal conductivity and the electrical resistivity are assumed to vanish; and (5) an adiabatic condition approximates the energy Eq. (2.3). For a perfectly conducting medium the Ohm's law equation reduces to

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = 0. \tag{2.5}$$

Maxwell's equations (neglecting displacement currents) complete the set:

$$\nabla\cdot\mathbf{B} = 0, \tag{2.6}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j}, \tag{2.7}$$

$$\nabla \times \mathbf{E} = -\partial_t\mathbf{B}. \tag{2.8}$$

When dealing with problems in which the veclocity fields are of comparable magnitude to the magnetic fields, a transformed set of equations introduced by Elsasser[8] is more convenient to deal with. For an *incompressible* fluid ($\rho = \rho_p =$ const) one adds and subtracts the momentum conservation Eq. (2.2) with the curl of Ohm's Law (2.5) and obtains the symmetric[9] hydromagnetic equations

$$\partial_t\mathbf{Q}_- + \nabla\cdot(\mathbf{Q}_+\mathbf{Q}_-) = -\nabla\pi, \tag{2.9}$$

$$\partial_t\mathbf{Q}_+ + \nabla\cdot(\mathbf{Q}_-\mathbf{Q}_+) = -\nabla\pi. \tag{2.10}$$

The corresponding mass-flux continuity equation is

$$\nabla\cdot\mathbf{Q}_\pm = 0, \tag{2.11}$$

where

$$\mathbf{Q}_\pm = \mathbf{V} \pm \mathbf{V}_A. \tag{2.12}$$

The normalized hydromagnetic pressure π is defined as

$$\pi = (1/\rho_p)(p + B^2/2\mu_0), \tag{2.13}$$

and the magnetic field has been normalized as

$$\mathbf{V}_A = \mathbf{B}/(\mu_0\rho_p)^{-\frac{1}{2}}. \tag{2.14}$$

These equations are symmetric with respect to the interchange $(-) \leftrightarrow (+)$.

[8] These equations were also presented independently by S. Lundquist, Arkiv Fysik **5**, 297 (1952).

[9] For a compressible medium one introduces

$$\mathbf{R}_\pm = (\rho/\rho_p)\mathbf{V} \pm \mathbf{V}_A,$$

and obtains the quasi-symmetric pair of equations,

$$\partial_t\mathbf{R}_- + \tfrac{1}{2}\nabla\cdot\{(\mathbf{R}_+ + \mathbf{Q}_+)\mathbf{Q}_- + (\mathbf{R}_+ - \mathbf{Q}_+)\mathbf{Q}_-\} = -\nabla\pi,$$

$$\partial_t\mathbf{R}_+ + \tfrac{1}{2}\nabla\cdot\{(\mathbf{R}_- + \mathbf{Q}_-)\mathbf{Q}_+ + (\mathbf{R}_- - \mathbf{Q}_-)\mathbf{Q}_+\} = -\nabla\pi,$$

where $\mathbf{Q}_\pm$ and π are given in (2.12) and (2.13). These equations are symmetric with respect to the interchange $(-)\leftrightarrow(+)$ but not $\mathbf{R}\leftrightarrow\mathbf{Q}$.

B. Boundary Equations

1. *Hydromagnetic Boundary Conditions*

When electromagnetic radiation effects are ignored, only three of the boundary conditions[10] are required to set up the dispersion relation:

continuity of normal flux: $[\mathbf{n}\cdot\mathbf{B}] = 0$ (2.15)

continuity of normal velocity: $[\mathbf{n}\cdot\mathbf{V}] = 0$ (2.16)

continuity of stress: $[\mathbf{n}\pi - \mathbf{B}(\mathbf{B}\cdot\mathbf{n})] = 0,$ (2.17)

where the brackets indicate the difference between plasma quantities and outerregion quantities. **n** is a unit vector normal to the surface of discontinuity, $f(x, y, z, t) = 0$, and is $(+)$ when pointing into the plasma; that is,

$$\mathbf{n} = \nabla f/|\nabla f|. \tag{2.18}$$

If we describe the interface by saying that a particle once on the surface remains there,[11] then the surface is described by the equation

$$D_t f = 0. \tag{2.19}$$

If we take the gradient of (2.19) and use (2.18), it can be shown that **n** satisfies

$$\partial_t\mathbf{n} + \mathbf{n}\{\mathbf{n}\cdot[(\mathbf{V}\cdot\nabla)\mathbf{n}]\} + v_n(\mathbf{n}\cdot\nabla)\mathbf{n} = \mathbf{n}\times(\mathbf{n}\times\nabla v_n), \tag{2.20}$$

where

$$v_n = \mathbf{V}\cdot\mathbf{n}, \tag{2.21}$$

This equation is equivalent to those given previously,[12] although the form in which it is written is different. The second term on the left of (2.20) is of second-order in a first-order perturbation analysis and thus is neglected.

III. EQUILIBRIUM CONFIGURATION

Figure 1 depicts the equilibrium state. Here we have a plasma of constant density ρ_p surrounded by a compressible, nonconducting gas of constant density ρ_0. The magnetic field within and external to the plasma has longitudinal and azimuthal com-

[10] T. G. Northrop, Phys. Rev. **103**, 1150 (1956).

[11] This describes a surface of major discontinuity where the density and tangential velocity are discontinuous across the boundary. In other physical circumstances one may deal with surfaces which drift with respect to the particles of the medium—the so-called surfaces of minor discontinuity. In discontinuities of order "n", the density and the acceleration and their first $(n - 1)$ time and space derivatives are continuous across the surface. See J. Hadamard, *Leçons sur la Propagation des Ondes et les Equations de l'Hydrodynamique* (Cie, Paris, 1903). In particular: Chap. 2; Chap. 5, Sec. 256–258; and Chap. 7.

[12] See Eq. (9b) of reference cited in footnote 1.

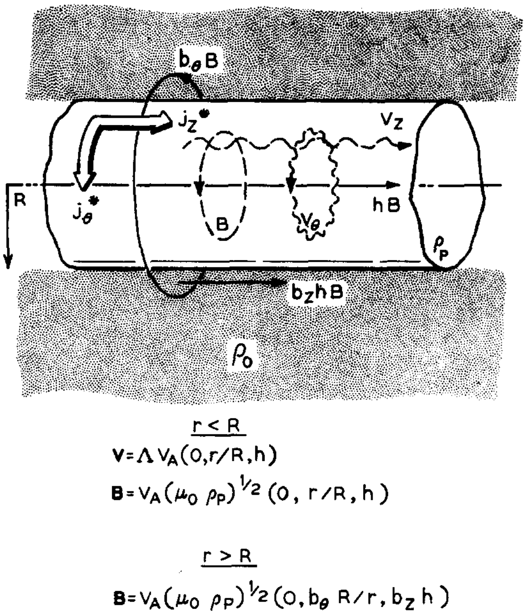


FIG. 1. The equilibrium configuration.

ponents. The current sheets, $j_\theta{}^*$ and $j_z{}^*$, at the plasma interface, cause jump discontinuities in the longitudinal and azimuthal components, respectively. The velocity field is in the same direction as the magnetic field.

The study of the equilibrium configurations $(\partial_t = 0)$ is simplified by assuming only radially dependent quantities $(\partial_\theta = \partial_z = 0)$. This causes $j_r = (1/\mu_0)\,(\nabla \times \mathbf{B})_r = 0$.[13] Thus the r component of (2.5) gives us the relation

$$V_\theta B_z = V_z B_\theta, \quad \text{or} \quad V_\theta V_{Az} = V_z V_{A\theta}. \tag{3.1}$$

This is the mathematical statement of the fact that fluid streamlines adhere to flux lines in the nonresistive case. For this problem we take the flow parameter Λ to be a constant:

$$\Lambda = V_\theta/V_{A\theta} = V_z/V_{Az}.$$

We also assume that $V_{A\theta}$ varies linearly with r and that $h = V_{Az}/V_{A\theta}$ $(r = R)$ is a constant. These results are summarized by the following vector relations:

$r < R$

$$\begin{aligned} \mathbf{V} &= \Lambda V_A(0, r/R, h) \\ \mathbf{B} &= V_A(\mu_0\rho_p)^{\frac{1}{2}}(0, r/R, h), \end{aligned} \tag{3.2}$$

$r > R$

$$\begin{aligned} \mathbf{V} &= 0 \\ \mathbf{B} &= V_A(\mu_0\rho_p)^{\frac{1}{2}}(0, b_\theta R/r, b_z h). \end{aligned} \tag{3.3}$$

[13] When the subscripts r, θ, and z follow letters, they indicate the vector components of these quantities, excepting, of course, the partial derivative ∂. The subscripts p and o will designate plasma and outer-region quantities, respectively.

The constant V_A is the Alfvén velocity associated with the normalizing plasma density, ρ_p, and the internal azimuthal magnetic field at $r = R$.

In order to find the kinetic pressure distribution the equilibrium assumptions, (3.2) and (3.3) are substituted into the r component of (2.2) and we obtain, after normalizing,

$$\frac{p}{p_p} = 1 - \frac{\gamma M_A{}^2}{2} x^2[2 - \Lambda^2], \tag{3.4}$$

where p_p is the normalizing pressure, and $x = r/R$; $M_A = V_A/c_s$; $c_s{}^2 = \gamma p_p/\rho_p$. Thus the normalized hydromagnetic pressure becomes

$$\pi_p = \frac{p_m}{\rho_p} - \frac{V_A{}^2 x^2}{2}(1 - \Lambda^2), \tag{3.5}$$

where

$$p_m = p_p + B_z{}^2/2\mu_0.$$

If we define

$$\beta_i = 2p_i/\rho_p V_A{}^2, \tag{3.6}$$

then at $x = 1$, (3.4) imposes the requirement

$$\beta_p \geq 2 - \Lambda^2. \tag{3.7}$$

If (3.2) and (3.3) are substituted into the *pressure continuity condition* (2.17), we obtain (after normalizing) the condition

$$\beta_p = \beta_0 + b_\theta{}^2 + h^2(b_z{}^2 - 1) + (1 - \Lambda^2). \tag{3.8}$$

The incompressible approximation is valid only for fluid velocities which are much smaller than sonic velocities; That is

$$M^2 = V^2/c_s{}^2 < 1. \tag{3.9}$$

If we take the plasma sonic velocity as

$$c_s{}^2 = \gamma p_p/\rho_p, \tag{3.10}$$

then we can write

$$(V_A/c_s)^2 = B^2/\gamma\mu_0 p_p = 2/\gamma\beta_p, \tag{3.11}$$

where β_p is defined in (3.6). If (3.11) is divided by (3.9), we obtain the result

$$\Lambda^2 = \tfrac{1}{2}\gamma\beta_p M^2. \tag{3.12}$$

For $\gamma = \frac{5}{3}$ and $\beta_p = 2$, $\Lambda^2 = \frac{5}{3}M^2$. Thus, one can apply the following results only when $\Lambda < 1$.

IV. NORMAL-MODE ANALYSIS—OUTER MEDIUM SOLUTIONS

In performing a normal-mode analysis one must first linearize. Each "total" quantity (subscript T)

is replaced by the sum of its equilibrium value and a small (perturbation) quantity designated by a tilde,

$$q_T(r, \theta, z, t)$$
$$= q(r) + \tilde{q}(r) \exp i[m\theta + kz + \omega t] \quad (4.1)$$
$$= q(r) + \tilde{q}^0 f(r), \quad (4.2)$$

where $\tilde{q}^0 = $ (const) exp $i\,[m\theta + kz + \omega t]$.

We perform a first-order analysis by neglecting products of two or more perturbation quantities.

In the outer nonconducting medium, hydrodynamic and electromagnetic effects are uncoupled. The medium is assumed to be compressible. If it has a uniform pressure and density distribution and is at rest in the equilibrium state, we can derive an acoustic wave equation in $\tilde{p}$ by using (2.1), (2.2), and (2.3). This equation reduces to the modified Bessel equation when

$$\partial_\theta \rightarrow im\theta; \qquad \partial_z \rightarrow ikz; \qquad \partial_t \rightarrow i\omega.$$

Thus,

$$\tilde{p} = \tilde{p}^0 K_m(\xi_g r). \quad (4.3)$$

K_m is the modified Bessel function of the second kind and

$$\xi_g^{\,2} = k^2 - \omega^2/c_g^{\,2} = k^2 - \omega^2 e_o/\gamma P_o \quad (4.4)$$

Note that

$$K_m(iz) = -(i\pi/2)e^{-m\pi i/2}H_m^{\,(2)}(z) \quad (4.5)$$
$$\rightarrow (\pi/2z)^{\frac{1}{2}}e^{-iz},$$

where $H_m^{\,(2)}$ is the Hankel function of the second kind. The latter expression holds for large z and corresponds to outward going radial waves. The perturbed velocity field is

$$\tilde{\mathbf{V}}_0 = \frac{\tilde{p}^0}{\rho_0\omega r}\{[i\xi_g r K_m'], [-mK_m], [-kK_m]\}, \quad (4.6)$$

where the arguments of K_m are understood to be $\xi_g r$.

Maxwell's equations in the outer medium can be combined to yield wave equations in $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$. With similar assumptions, the z component of these equations reduces to modified Bessel's equations, and therefore

$$\tilde{E}_z = \tilde{E}_z^{\,0} K_m(\xi_0 r); \qquad \tilde{B}_z = \tilde{B}_z^{\,0} K_m(\xi_0 r), \quad (4.7)$$

where

$$\xi_0^{\,2} = k^2 - \omega^2/c^2. \quad (4.8)$$

If $\omega/c = 0$, we can determine the perturbation in the outer magnetic field, $\tilde{\mathbf{B}}_0$, from (2.7) as

$$\tilde{\mathbf{B}}_0 = (\tilde{B}^0/kr)\{-ikrK_m'(kr),\, mK_m(kr),\, krK_m(kr)\}. \quad (4.9)$$

V. MATCHING BOUNDARY CONDITIONS—THE DISPERSION RELATION

The dispersion relation is obtained by substituting solutions in the plasma region (subscript p; these will be given in Sec. VI) and outer region (subscript 0) into the stress "dynamic" boundary conditions. By using the solutions obtained in Sec. IV, we will obtain a relation which depends only on plasma variables.

If the equilibrium plus perturbation quantities are substituted into the r component of the pressure continuity equation (2.17) we obtain

$$\mathcal{P} = \pi_p - \pi_0 + \tilde{\pi}_p - \tilde{\pi}_0 = 0. \quad (5.1)$$

$\mathcal{P}$, the normalized hydromagnetic pressure *difference*, is zero across the boundary as the boundary moves. We characterize the surface of discontinuity, f, by having $\mathcal{P}$ satisfy the same differential equation as does f, namely (2.19). By substituting $\mathcal{P}$ for f in this equation and combining, we obtain

$$i\bar{\omega}(\tilde{\pi}_p - \tilde{\pi}_0) + \tilde{V}_{rp}\,\partial_r(\pi_p - \pi_0) = 0, \quad (5.2)$$

where

$$\bar{\omega} = \omega + \omega_p, \qquad \omega_p = \Lambda\omega_A m', \quad (5.3)$$

$$\omega_A = V_A/R, \qquad m' = m + Xh, \qquad X = kR. \quad (5.4)$$

The normal-vector differential equation (2.20) is solved using $\mathbf{n} = (\mp 1, 0, 0)$ in equilibrium, and we obtain

$$\mathbf{n} = (\tilde{V}_r/\bar{\omega}R)(0, m, X). \quad (5.5)$$

Thus, the velocity continuity condition (2.16) yields

$$-\omega\tilde{V}_{rp} = -\bar{\omega}\tilde{V}_{r0}, \quad (5.6)$$

and the flux continuity condition yields, after normalizing,

$$\tilde{B}_{r0} = \tilde{B}_{rp} + \delta(\tilde{V}_{rp}/\bar{u})(\mu_0\rho_p)^{\frac{1}{2}}, \quad (5.7)$$

where

$$\delta = m(b_\theta - 1) + Xh(b_z - 1), \quad (5.8)$$

$$u = \omega/\omega_A, \qquad \bar{u} = \bar{\omega}/\omega_A. \quad (5.9)$$

The hydromagnetic pressure in the outer medium, $\pi_{0T} = \pi_0 + \tilde{\pi}_0$, is constructed from the basic definition (2.13) by using (3.2) and (3.3). Thus,

$$\pi_0 = p_0/\rho_p + (V_A^{\,2}/2)[(b_\theta R/r)^2 + (b_z h)^2], \quad (5.10)$$

$$\tilde{\pi}_0 = \tilde{p}_0/\rho_p + V_A(\mu_0\rho_p)^{-\frac{1}{2}}[b_\theta\tilde{B}_{\theta 0} + b_z h\tilde{B}_{z0}]. \tag{5.11}$$

By using (4.9), the θ and z components of $\tilde{\mathbf{B}}_0$ can be expressed in terms of the r component. Similarly, $\tilde{p}_0$ can be expressed in terms of $\tilde{V}_{r0}$ (and hence $\tilde{V}_{rp}$) by using (4.3) and (4.6), and the boundary relation (5.6). By using all these relations, we obtain

$$\begin{aligned}\tilde{\pi}_0|_{r=R} &= i\tilde{V}_{rp}(V_A/\bar{u})\{-(\rho_0/\rho_p)u^2\mathcal{K}_m^{-1}(\xi_0 r)\\ &\quad + \delta\mathcal{K}_m^{-1}(X)[mb_\theta + Xhb_z]\}\\ &\quad + i\tilde{B}_{rp}V_A(\mu_0\rho_p)^{-\frac{1}{2}}\{\mathcal{K}_m^{-1}(X)[mb_\theta + Xhb_z]\}.\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}\mathcal{K}_m(z) &= zK_m'(z)/K_m(z);\\ \mathcal{K}_m(iz) &= zH_m^{(2)\prime}(z)/H_m^{(2)}(z).\end{aligned} \tag{5.13}$$

If (5.10) and (3.5) are substituted into $\partial_r(\pi_p - \pi_0)$, we obtain

$$\partial_r(\pi_p - \pi_0) = \omega_A^2 R[\Lambda^2 - (1 - b_\theta^2)]. \tag{5.14}$$

Now, if (5.12) and (5.14) are substituted into (5.2), and the result is divided by $\tilde{V}_{rp}(\omega_A^2 R)$, we obtain the dispersion relation

$$\begin{aligned}&[\tilde{\pi}_p/\tilde{V}_{rp}]i\bar{u}V_A^{-1} + (\Lambda^2 - 1 + b_\theta^2)\\ &\quad - (\rho_0/\rho_p)u^2\mathcal{K}_m^{-1}(\xi_0 R) + [mb_\theta + Xhb_z]\\ &\quad \cdot[\delta + \bar{u}(\mu_0\rho_p)^{-\frac{1}{2}}(\tilde{B}_{rp}/\tilde{V}_{rp})]\mathcal{K}_m^{-1}(X) = 0.\end{aligned} \tag{5.15}$$

It is instructive to point out the physical significance of each term in Eq. (5.15): (1) perturbation of hydromagnetic pressure; (2) convective boundary effect; that is, the boundary moves into a region of a different equilibrium hydromagnetic pressure; (3) perturbation of the kinetic pressure in the outer region; (4) perturbation of the magnetic pressure in the outer region.

VI. CHARACTERISTIC PLASMA DIFFERENTIAL EQUATION AND PLASMA VARIABLES

If we substitute the equilibrium plus perturbation quantities into the symmetric Eqs. (2.9) and (2.10), we obtain six linear algebraic equations in the six unknowns $[\tilde{Q}$:

$$[a][\tilde{Q} = [\tilde{\pi} \tag{6.1}$$

or

$$\begin{bmatrix} i\bar{u}_- & -u_{-\theta} & 0 & 0 & -u_{+\theta} & 0\\ u_{-\theta} & i\bar{u}_- & 0 & u_{+\theta} & 0 & 0\\ 0 & 0 & i\bar{u}_- & 0 & 0 & 0\\ 0 & -u_{-\theta} & 0 & i\bar{u}_+ & -u_{+\theta} & 0\\ u_{-\theta} & 0 & 0 & u_{+\theta} & i\bar{u}_+ & 0\\ 0 & 0 & 0 & 0 & 0 & i\bar{u}_+ \end{bmatrix}\begin{bmatrix}\tilde{Q}_{+r}\\ \tilde{Q}_{+\theta}\\ \tilde{Q}_{+z}\\ \tilde{Q}_{-r}\\ \tilde{Q}_{-\theta}\\ \tilde{Q}_{-z}\end{bmatrix} = \frac{-i}{\omega_A}\begin{bmatrix}-i\partial_r\tilde{\pi}\\ (m/r)\tilde{\pi}\\ k\tilde{\pi}\\ -i\partial_r\tilde{\pi}\\ (m/r)\tilde{\pi}\\ k\tilde{\pi}\end{bmatrix}, \tag{6.2}$$

where

$$u_{\pm\theta} = mQ_{\pm\theta}/\omega_A R = m(\Lambda \pm 1), \tag{6.3}$$

$$u_{\pm z} = kQ_{\pm z}/\omega_A = Xh(\Lambda \pm 1), \tag{6.4}$$

$$u_{\pm p} = u_{\pm\theta} + u_{\pm z} = m'(\Lambda \pm 1), \tag{6.5}$$

$$u_p = \Lambda m', \tag{6.6}$$

$$\bar{u}_\pm = u + u_{\pm p}, \qquad \bar{u} = u + u_p. \tag{6.7}$$

The simplicity of Eqs. (6.3)–(6.7) follows from the assumptions made in describing the equilibrium state, (3.2) and (3.3), and from the assumed nature of the perturbation solutions, (4.1).

The determinant of the above system of equations is

$$D_a = (\bar{u}_-\bar{u}_+)^2(\Omega^2 - 1), \tag{6.8}$$

where

$$\Omega = \frac{\Lambda + 1}{u + m'(\Lambda + 1)} + \frac{\Lambda - 1}{u + m'(\Lambda - 1)} \tag{6.9}$$

or

$$\Omega = \frac{2[\Lambda u + m'(\Lambda^2 - 1)]}{\bar{u}_+\bar{u}_-} \tag{6.10}$$

Ω^{-1} is equivalent to a normalized Doppler wavelength. $D_a \neq 0$ the solution of (6.2) can be written as[14]

$$\tilde{Q}] = -\frac{1}{\bar{u}_-\omega_A(1 - \Omega^2)}\begin{bmatrix} -i[\partial_r\tilde{\pi} + (m/r)\Omega\tilde{\pi}]\\ [\Omega\,\partial_r\tilde{\pi} + (m/r)\tilde{\pi}]\\ (1 - \Omega^2)k\tilde{\pi}\\ -\beta i[\partial_r\tilde{\pi} + (m/r)\Omega\tilde{\pi}]\\ \beta[\Omega\,\partial_r\tilde{\pi} + (m/r)\tilde{\pi}]\\ \beta(1 - \Omega^2)k\tilde{\pi}\end{bmatrix}, \tag{6.11}$$

where

$$\beta = \bar{u}_-/\bar{u}_+. \tag{6.12}$$

[14] When $\Omega = \pm 1$, $D_a = 0$ and $[a]^{-1}$, and therefore $\tilde{Q}]$, is undetermined by the foregoing procedure. This problem is considered in detail in Appendix 1, for, as will be shown $m = 0$, $\Omega = \pm 1$ and $m > 0$, $\Omega = -1$ *seem* to be solutions of the dispersion relation. Actually, they *are not solutions.*

Note that

$$\tilde{Q}_{-i} = \beta\tilde{Q}_{+i}. \tag{6.13}$$

If the first three terms of (6.11) are substituted into (2.11), we obtain the modified Bessel equation in $\tilde{\pi}$:

$$\partial_{rr}{}^2\tilde{\pi} + (1/r)\,\partial_r\tilde{\pi} - [m^2/r^2 + \xi_p{}^2]\tilde{\pi} = 0, \tag{6.14}$$

where

$$\xi_p{}^2 = k^2(1 - \Omega^2). \tag{6.15}$$

Thus the solution of (6.14) is

$$\tilde{\pi} = \tilde{\pi}^0 I_m(\xi_p r), \tag{6.16}$$

and

$$\tilde{V}_r = [\tfrac{1}{2}(1 + \beta)i/\bar{u}_-\omega_A(1 - \Omega^2)] \cdot [\partial_r\tilde{\pi} + (m/r)\Omega\tilde{\pi}], \tag{6.17}$$

$$\tilde{B}_r = [(1 - \beta)/(1 + \beta)]\tilde{V}_r(\mu_0\rho_p)^{\frac{1}{2}}. \tag{6.18}$$

By substituting these results into (5.15) and setting $r = R$, we obtain

$$-2(1 - \Omega^2)\bar{u}[\bar{u}_-\bar{u}_+/(\bar{u}_- + \bar{u}_+)] = \{\mathcal{I}_m(\xi_p R) + m\Omega\}D, \tag{6.19}$$

where the modified Bessel function ratio is defined by

$$\mathcal{I}_m(z) = zI_m'(z)/I_m(z) = [xJ_m'(x)/J_m(x)]_{x=-iz}, \tag{6.20}$$

and

$$D = \Lambda^2 - 1 + b_\theta{}^2 - (\rho_0/\rho_p)u^2\mathcal{K}_m{}^{-1}(\xi_g R) + (mb_\theta + Xhb_z)^2\mathcal{K}_m{}^{-1}(X), \tag{6.21}$$

where we have used

$$\bar{u} = \tfrac{1}{2}(\bar{u}_+)(1 + \beta), \tag{6.22}$$

$$\tfrac{1}{2}(\bar{u}_+ - \bar{u}_-) = \tfrac{1}{2}(\bar{u}_+)(1 - \beta) = m'. \tag{6.23}$$

If we use the $u\Omega$ transformation (6.10), we obtain

$$F = \Omega\, D[\mathcal{I}_m(\xi_p R) + m\Omega] + 2(1 - \Omega^2)[\Lambda u + m'(\Lambda^2 - 1)] = 0. \tag{6.24}$$

The form of this equation is characteristic of the dispersion relations for hydromagnetic stability problems.[15] It involves the sum of transcendental functions, which are characteristic of the geometry ($\mathcal{I}_m$ and $\mathcal{K}_m$), and rational fractions in the eigenvalues u. The former have as arguments different rational fractions in u. Both these rational fractions have coefficients which are dependent on the properties (e.g. compressibility, etc.) and equilibrium configuration of the plasma and outer medium.

[15] See Eq. 30 of reference cited in footnote 1.

VII. PROPERTIES OF THE $u\Omega$ TRANSFORMATION

The properties of the plasma are most conveniently described in terms of the variable Ω, which is related to the normalized eigenfrequency, u, through the transformation given in (6.9). The properties of this transformation will now be considered, since they determine the behavior of the zeros of the dispersion relation.

Rearrangement of (6.9) as a quadratic in u yields

$$u^2 + u[2\Lambda m'(1 - y)] + 2m'^2(\Lambda^2 - 1)(\tfrac{1}{2} - y) = 0, \tag{7.1}$$

where

$$y = 1/m'\Omega. \tag{7.2}$$

Thus we see that the u plane is a two-sheeted Riemann surface which transforms into the Ω plane. The singularities in the transformation occur at

$$u = -m'(\Lambda \pm 1), \tag{7.3}$$

and the branch points are at ($d\Omega/du = 0$):

$$u = (m'/\Lambda)[(1 - \Lambda^2) \pm (1 - \Lambda^2)^{\frac{1}{2}}]. \tag{7.4}$$

If (7.1) is solved for u, we obtain

$$-u/m' = \{\Lambda(1 - y) \pm (1 - 2y + \Lambda^2y^2)^{\frac{1}{2}}\}. \tag{7.5}$$

Note that u can be complex (unstable modes) for *y real* if

$$1 - (1 - \Lambda^2)^{\frac{1}{2}} < y < 1 + (1 - \Lambda^2)^{\frac{1}{2}}. \tag{7.6}$$

Hence, if $|\Lambda| \geq 1$, no unstable modes arise which correspond to *real* values of Ω. The bounds on $m'\Omega$ are given in the stability diagram of Fig. 2. Thus the smaller m', the larger the length of the positive real Ω axis which corresponds to unstable modes. For example, if $m = 0$, a small m' means a small Xh, or physically the case of a large wavelength and a weak longitudinal magnetic field.

It should be noted that taking $m' = 0$ or $\Lambda = \pm 1$ reduces (6.9) to a single-valued transformation.

If $\Lambda = 0$, each real value of Ω corresponds to values of u which are symmetrically distributed about the origin. For $|\Lambda| > 0$ (and $\Lambda \neq \pm 1$) the symmetry (or degeneracy) is removed. For $\Lambda > 0$ (< 0) the values of u are displaced toward the negative (positive) region.

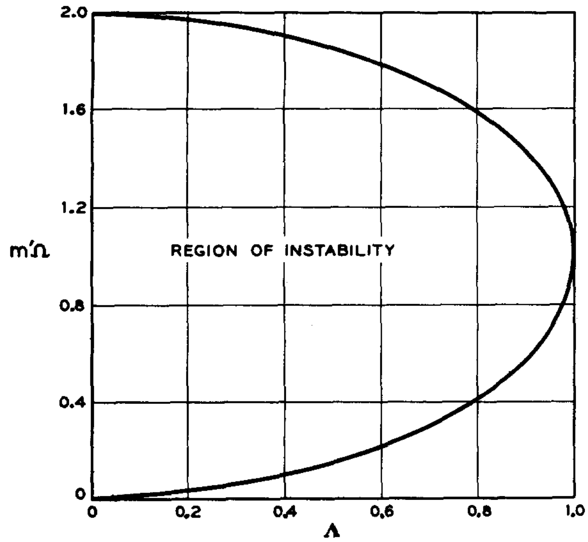


FIG. 2. The stability regions for incompressible Flow.

VIII. PROPERTIES OF THE DISPERSION RELATION

A. Negative Parameters

If Λ is replaced by $-\Lambda$ and u by $-u$ in both (6.9) and (6.24), then these equations are unaffected. Hence, the result of reversing the direction of flow does not affect the magnitude of the roots of the dispersion relation, but only interchanges the role of growing and decaying modes.

If $X \to -X$ and $h \to -h$, the zeros of the dispersion relation are unaffected. Thus, if the longitudinal field is reversed, the direction of propagation of the constant phase surfaces is reversed.

B. The Ω Plane

Since u appears explicitly in (6.24) only twice, it is advantageous to seek the zeros of F in the Ω plane. The normalized eigenfrequencies are obtained by applying the $u\Omega$ transformation to these results. As Ω increases,

$$\begin{aligned} \mathcal{J}_m(\xi_p R) &\to X\Omega J_m'(X\Omega)/J_m(X\Omega) \\ &\to -X\Omega \tan(X\Omega - m\pi/2 - \pi/4). \end{aligned} \tag{8.1}$$

Thus we will have an infinite number of roots corresponding to large Ω and these will correspond to eigenfrequencies clustered around the singularities of the $u\Omega$ transformation (7.3).

C. Behavior of the Dispersion Relation with a Λ Variation

For convenience we will assume that the outer gas is removed, $\rho_0 = 0$. Thus u no longer appears in D and appears only once in (6.24) as the term Λu. Other investigators have shown that when $\Lambda = 0$ (no flow) there are unstable (nonoscillatory) modes which correspond to real values of Ω. These modes correspond to imaginary values of u. When Λ is small but finite, u appears explicitly in the dispersion relation and Ω must become *complex* in order to satisfy (6.24) in the same neighborhood of variables and parameters. Thus, *with flow present we have overstable modes.*

If Λ is made large and X assumed small, the dispersion relation simplifies so that analytical investigations are permitted. In Appendix 2 we demonstrate that for $\beta_p = 2.0$, $b_z = 1.0$,[16] X small ($= 0.1$) that:

(1) The dispersion relation is homogeneous in Λ^2, if only the terms of highest power in Λ are considered. Thus for large Λ the eigenvalues in the Ω plane converge to a fixed point. The calculations (for $m = 1$ and $X = 0.1$) yield $\Omega = 1.96258 \pm i0.289068$, while the computer study (Sec. VIII, D) yields $\Omega = 1.96437 \pm i0.282453$ when $m = 1.0$, $X = 0.1$, and $\Lambda = 15.0$.

(2) If only terms of $\mathcal{O}(\Lambda^2)$ are included, the dispersion relation is independent of h. Thus, no matter how strong is the longitudinal magnetic field (h), the system will be unstable if Λ is made sufficiently large and it will have the same Ω eigenvalue.

If terms of $\mathcal{O}(\Lambda^2, \Lambda)$ are included, D has a term which involves h/Λ. If this term is made sufficiently small, the system will be unstable. Appendix 2 shows that the condition for *instability* is

$$\frac{h}{\Lambda} < \frac{(1/a)(2m-1) - m^2}{2mX} \tag{8.2}$$

$$1/a = |\mathcal{K}_m(X)| \geq 1.0 \quad \text{for} \quad |m| \geq 1. \tag{8.3}$$

For $m = 1$, $X = 0.1$, this is $(h/\Lambda) < 0.12315$. Thus for a given wave number X, stability is achieved by increasing the longitudinal magnetic field, that is h. This stable condition is removed by increasing Λ until condition (8.2) is satisfied.

D. Computer Study

Quantitative information on the variation of the eigenvalues of (6.24) with Λ was found by using an iterative procedure in the complex Ω plane based on Newton's method.[17] The function, f, studied was

[16] These numbers were used in the computer study described below.

[17] The computation procedures designed for use on the MOD. 205 Datatron are outlined in Appendix 6 of the author's thesis, (Department of Physics, California Institute of Technology, 1959).

TABLE I. A survey of the physical situations studied.

Fig.	m	h	X	Λ(abscissa)	Ordinate
3(a)	0	0.1	0.1, 1.0, 3.0	$0 < \Lambda < 1.0$	Im $(u/\sqrt{X})$
3(b)	0	0.1	0.1, 1.0, 3.0	$0 < \Lambda < 1.0$	Re $(u/\sqrt{X})$
3(c)	+1.0	0.1	0.1, 1.0, 3.0	$0 < \Lambda \leq 1.6$	Im $(u/\sqrt{X})$
3(d)	+1.0	0.1	0.1, 1.0, 3.0	$0 < \Lambda \leq 1.6$	Re $(-u/\sqrt{X})$
3(e)	+1.0	1.0	0.1, 1.0	$0 < \Lambda \leq 1.5$	Re $(\pm u/\sqrt{X})$

explicitly a function of Ω. This function was obtained by solving (6.24) for $u(\rho_0 = 0 \text{ in } D)$:

$$u = -m'g/\Lambda g_1, \tag{8.4}$$

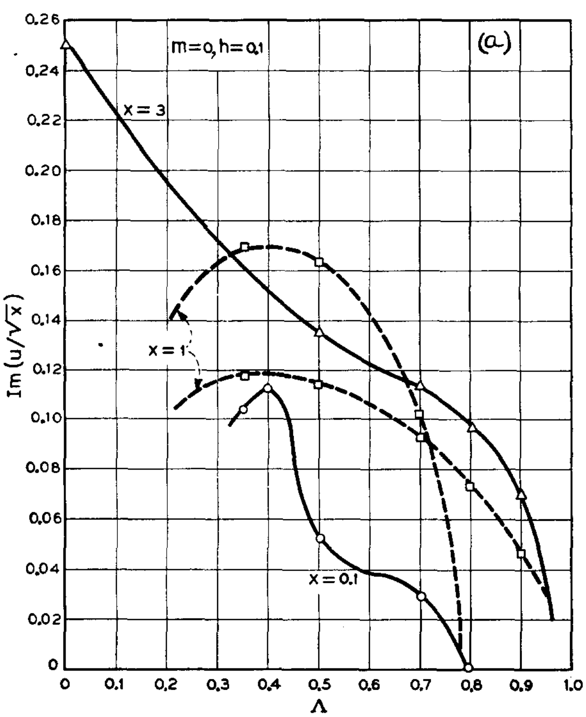
where

$$g = Dg_2 + (\Lambda^2 - 1)g_1,$$
$$g_1 = m'(1 - \Omega^2), \quad g_2 = \tfrac{1}{2}[\Omega \mathcal{J}_m(\xi_p R) + m\Omega^2]. \tag{8.5}$$

After substituting in (7.1) and rearranging, we obtain

$$f = g^2 - g[2\Lambda^2 g_1(1 - y)] + 2(\Lambda g_1)^2(\Lambda^2 - 1)(\tfrac{1}{2} - y) = 0. \tag{8.6}$$

Two parameters were fixed: $b_z = 1.0$; $\beta_p = 2.0$. The cases studied are summarized in the following table, where reference is given to the appropriate figure numbers. No attempt was made to find all the modes—only the dominant ones were sought.

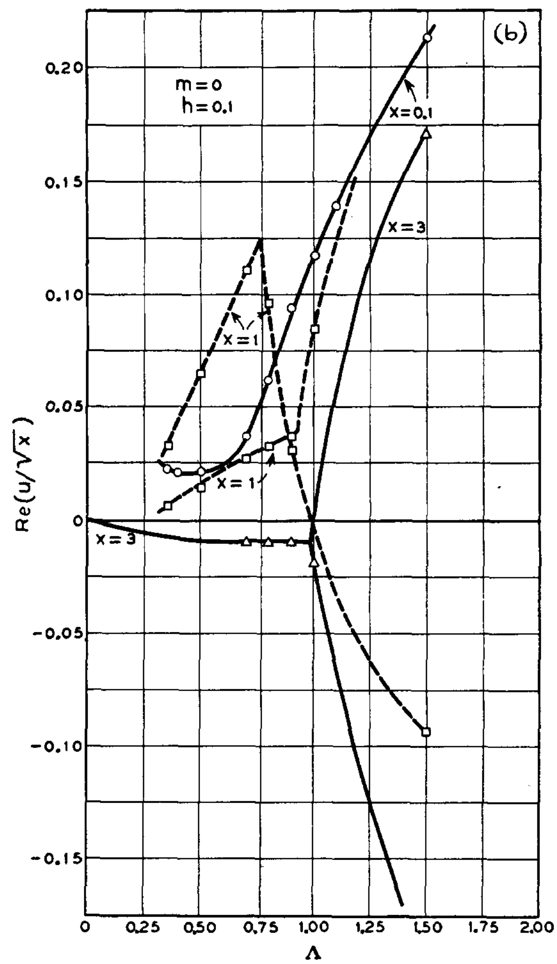


(a)

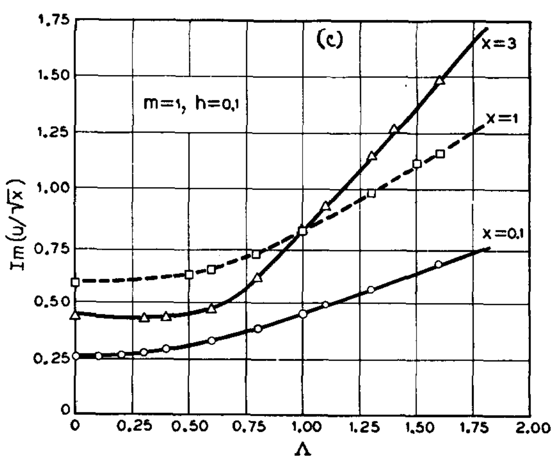
A study of Fig. 3 permits the following conclusions:

(1) $m = 0, h = 0.1$

(a) Overstable modes exist in each case. For small wave numbers ($X = 0.1, 1.0$) these arise (a pair in the latter case) with $\Lambda > 0$, then build up



(b)



(c)

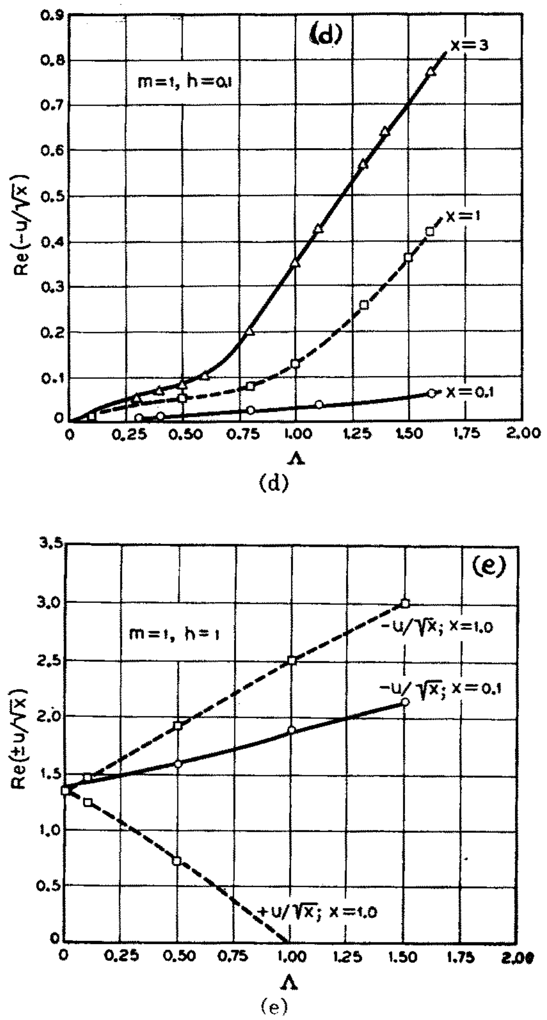


FIG. 3. Normalized eigenvalue variation with flow. Real or imaginary $(u/\sqrt{X})$ vs Λ.

to a maximum, and vanish with increasing Λ. With $X = 3.0$ one mode is present at $\Lambda = 0$ as a pure divergence. These modes show the characteristic of increasing magnitude with increasing wave numbers.

(b) The magnitude of the oscillation frequency of the modes increases with increasing Λ.

(2) $m = 0, h = 1.0$

(a) For the region $0 < \Lambda < 1.5$ the overstable modes have been removed by the strong field.

(3) $m = 1, h = 0.1, 1.0$

(a) The magnitude of the oscillation frequencies and growth rates increase monotonically with Λ and X when $h = 0.1$.

(b) When $h = 1.0$, the instabilities are removed in the region $0 < \Lambda < 1.5$ for $X = 0.1$ and 1.0.

IX. COMPARISONS

It is easily demonstrated that the dispersion relation (6.24) reduces to that given by Kruskal and Tuck (see reference 1, Eq. 30) when $c_s \to \infty$ (incompressibility); $\omega/c \to 0$ (displacement current vanishes); and $\Lambda = 0$. Roberts[3] also treated twisted magnetic fields in the infinite cylindrical geometry assuming $\Lambda = 0$ and incompressibility. The characteristic differential equation of the plasma (6.14) reduces to his equation (Eq. 39 with the assumption given in Eqs. 47 and 48, Model 1). However, it is difficult to compare results, as he used the simple boundary condition (his Eq. 52)

$$\tilde{\pi}_p = 0. \tag{9.1}$$

It should be noted that Roberts' dispersion relation [his Eq. (53)] has stable solutions for very large and very small wave numbers X. This does not agree with the result of Kruskal and Tuck. They demonstrate that for $m = 1$, very small wave number perturbations cannot be stabilized in an infinite cylindrical geometry.

Chrandrasekhar[18] first investigated the flow problem for the case of $\Lambda = 1$. His conclusion of stability was based on a calculation which used the hydromagnetic equations of motion and the simplified boundary condition (his Eq. 14)

$$\tilde{\mathbf{Q}}_- = 0. \tag{9.2}$$

This simplification permitted him to express ω^2 as the ratio of two positive definite integrals.

In a recent investigation Trehan[5] also used (9.1) as his dispersion relation [his Eq. (36)]. We will now demonstrate that this imposes restrictions on the physical variables. Equation (9.1) implicitly requires that the braced quantity in (5.15) or the quantity D in (6.24) be $= 0$. Thus if $\rho_0 = 0$, (6.21) implies

$$D = 0 = 2\Lambda^2 - h^2(b_z^2 - 1) - (mb_\theta + Xhb_z)^2 a, \tag{9.3}$$

where a is given by (8.3) and b_θ^2 is given by (3.8) with $\beta_p = 2.0$ and $\beta_0 = 0$, as

$$b_\theta^2 = \Lambda^2 + 1 - h^2(b_z^2 - 1). \tag{9.4}$$

We will demonstrate that (9.1) and (9.4) *imply*: a specific variation of b_z and b_θ with Λ^2; and restricted regions of m and X where (9.1) is applicable.

If we define

$$\alpha = 1/hb_z, \qquad \Delta = b_\theta/hb_z = \alpha b_\theta, \tag{9.5}$$

[18] S. Chandrasekhar, Proc. Natl. Acad. Sci. 42, 273, 1956.

then we can write (9.3) and (9.4) as

$$(2\Lambda^2 + h^2)\alpha^2 - 1 - (m\,\Delta + X)^2 a = 0 \tag{9.6}$$

and

$$\Delta^2 = (\Lambda^2 + 1 + h^2)\alpha^2 - 1. \tag{9.7}$$

By substituting (9.4) into (9.3) and solving the resulting quadratic for Δ, we get

$$\Delta = \frac{mXa \pm \{-\tau^2 + \tau(1 + X^2 a + m^2 a^2) - m^2 a^2\}^{\frac{1}{2}}}{\tau - m^2 a}, \tag{9.8}$$

where

$$\tau = (2\Lambda^2 + h^2)/(\Lambda^2 + h^2 + 1). \tag{9.9}$$

Thus for Λ/h large, τ approaches a constant = 2.0, and Δ is independent of Λ.

By setting $\tau = 2$ in Eq. (9.8), one arrives at the condition between m and X such that Δ is real, namely,

$$2X^2 + m^2 > 2/a. \tag{9.10}$$

Thus, (9.1) *cannot* be used as a dispersion relation for small wave number disturbances. In particular, for $m = \pm 1$, X must be greater than 1.0.

For large Λ (9.6) and (9.7) yield, respectively,

$$\alpha^{-2} = (hb_z)^2 = (\Lambda^2 + 1 + h^2)/(\Delta^2 + 1) \rightarrow (\text{const})\Lambda^2, \tag{9.11}$$

$$b_\theta^{\,2} = \Delta^2/\alpha^2 \rightarrow (\text{const})\Lambda^2. \tag{9.12}$$

Thus, use of (9.1) as a dispersion relation removes a degree of freedom by implicitly coupling the external magnetic fields to the flow. This implicit stabilization is probably the reason Trehan (his Fig. 2) did not observe any instabilities for $m = -1$ and Λ large.

ACKNOWLEDGMENTS

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APPENDIX 1. $\Omega^2 = 1$

If $\Omega^2 = 1$, the inverse of matrix $[a]$ is not defined and one must return to the original equations. Equation (6.13) is evident from an inspection of (6.2) and is not affected by the vanishing of Det $|\,a\,|$. Using this result, the first three equations of (6.2) yield

$$\begin{bmatrix} i\bar{u}_- & -(u_{-\theta} + \beta u_{+\theta}) & 0 \\ u_{-\theta} + \beta u_{+\theta} & i\bar{u}_- & 0 \\ 0 & 0 & i\bar{u}_- \end{bmatrix} \begin{bmatrix} \tilde{Q}_{+r} \\ \tilde{Q}_{+\theta} \\ \tilde{Q}_{+z} \end{bmatrix} = \frac{1}{\omega_A} \begin{bmatrix} -\partial_r\tilde{\pi} \\ -(im/r)\tilde{\pi} \\ -ik\tilde{\pi} \end{bmatrix}. \tag{A1.1}$$

Thus $\tilde{Q}_{+z} = -(k/\omega_A\bar{u})\tilde{\pi}$, and the remaining pair of equations becomes

$$\begin{bmatrix} i & -\Omega \\ \Omega & i \end{bmatrix} \begin{bmatrix} \tilde{Q}_{+r} \\ \tilde{Q}_{+\theta} \end{bmatrix} \begin{matrix} = -(1/\bar{u}_-\omega_A)\,\partial_r\tilde{\pi} & \text{(A1.2)} \\ = -(1/\bar{u}_-\omega_A)(im/r)\tilde{\pi}. & \text{(A1.3)} \end{matrix}$$

If (A1.2) is substituted into $\nabla\cdot\tilde{\mathbf{Q}}_+ = 0$ and the result rearranged, we obtain

$$\partial_r\tilde{Q}_{+r} + \frac{1}{r}(1 - m/\Omega)\tilde{Q}_{+r} = \left(\frac{i}{\bar{u}_-\omega_A}\right)\left[\frac{-m}{r\Omega}\,\partial_r\tilde{\pi} + k^2\tilde{\pi}\right]. \tag{A1.4}$$

Similarly, for (A1.3) the following was obtained:

$$\partial_r\tilde{Q}_{+r} + (1/r)(1 - m\Omega)\tilde{Q}_{+r} = (i/\bar{u}_-\omega_A)[(m^2/r^2) + k^2]\tilde{\pi}. \tag{A1.5}$$

Subtraction of (A1.4) from (A1.5) yields

$$[(1/\Omega) - \Omega]\tilde{Q}_{+r} = (i/\bar{u}_-\omega_A) \cdot [(1/\Omega)\,\partial_r\tilde{\pi} + (m/r)\tilde{\pi}]. \tag{A1.6}$$

After setting $\Omega = \pm 1$ we are left with a first order equation for $\tilde{\pi}$ whose solution is

$$\tilde{\pi} = \tilde{\pi}^0 r^{-(m\,\Omega)}. \tag{A1.7}$$

This solution is valid when $m = \mp 1.0$, $\Omega = \pm 1.0$, etc. *Equation* (A.1.7) *is also the solution which one obtains from* (6.4) *by setting* $\Omega^2 = 1$. This follows since the value of the determinant cancels out of the procedure which determines the plasma differential equation.

Substitution of (A1.7) into (A1.5) yields a first order differential equation in $\tilde{Q}_{+r}$ whose solution is

$$\tilde{Q}_{+r} = i\tilde{\pi}^0\,\frac{r^{-m\,\Omega-1}}{2\bar{u}_-\omega_A}\left[-\frac{1}{\Omega} + \frac{(kr)^2}{1 - m\Omega}\right]. \tag{A1.8}$$

From (A1.2) we also determine

$$\tilde{Q}_{+\theta} = \frac{-\tilde{\pi}^0}{\Omega 2\bar{u}_-\omega_A}\left[-\frac{1}{\Omega} + \frac{(kr)^2}{1 - m\Omega} + 2m\Omega\right]. \tag{A1.9}$$

Thus, the quantity substituted into the dispersion relation (5.15) is

$$\frac{\tilde{V}_{rp}}{\tilde{\pi}_p} = \frac{(1+\beta)}{2}\frac{\tilde{Q}_{+rp}}{\tilde{\pi}_p}$$
$$= \frac{i(1+\beta)}{4\bar{u}_{-}\omega_A R}\left[-\frac{1}{\Omega} + \frac{X^2}{m+1}\right]. \quad \text{(A1.10)}$$

We see that $\Omega = -1$ is not a root of the dispersion relation. The same result, (A1.10), is obtained by forming

$$\tilde{Q}_{+r}/\tilde{\pi} = i[\partial_r\tilde{\pi}/\tilde{\pi} + m\Omega/r]/[\bar{u}\omega_A(1-\Omega^2)]$$

from (6.11) and expanding the resulting equation before taking the limit of $\Omega \to -1$.

APPENDIX 2. ASYMPTOTIC ANALYSIS IN Λ

If we retain terms of order Λ^2 and Λ in (7.5) and (6.21) we obtain the respective expressions

$$u = -m'\Lambda(1 - 2/m'\Omega) + \mathcal{O}(1/\Lambda), \quad \text{(A2.1)}$$

$$0 = \Lambda^2 D'$$
$$= \Lambda^2(2 - am^2 - 2amXh/\Lambda) + \mathcal{O}(\Lambda^0), \quad \text{(A2.2)}$$

where a is defined in (8.3). We have taken $b_z = 1.0$ and $\beta_p = 2.0$ so that comparisons to the computer results can be made. If we substitute these into the dispersion relation (6.24) and take Λ large, we obtain the expression

$$F = \Lambda^2\{D'\Omega[\mathcal{J}_m(\xi_p R) + m\Omega]$$
$$+ (4/m'\Omega)(1-\Omega^2)\} = 0. \quad \text{(A2.3)}$$

If the quantity $2amXh/\Lambda$ in (A2.2) is very small, then (A2.3) is homogeneous in the term Λ^2. Thus for large Λ each eigenvalue approaches a fixed point in the complex Ω plane.

We now show that for $m = +1$, X small $(= 0.1)$, that (A2.3) has a complex conjugate pair of modes in the vicinity of the origin of the Ω plane. If we use

$$\mathcal{J}_m(z) = \mathcal{J}_{-m}(z) = |m|$$
$$+ z^2/2(|m|+1) + \cdots, \quad \text{(A2.4)}$$

we can rearrange (A2.3) to the form

$$[\Omega^3(D'\epsilon^2) - \Omega^2 D'(m/4 + \epsilon^2)$$
$$+ \Omega - 1][\Omega + 1] = 0, \quad \text{(A2.5)}$$

where

$$\epsilon^2 = X^2/8(m+1), \quad \text{(A2.6)}$$

and m has been taken as positive to simplify the analysis.

If we take $\epsilon^2 = 0$ and $D' = 2 - am^2$, we are left with a quadratic equation whose solution is

$$\Omega = \frac{2\{-1 \pm [1 - m(2 - am^2)]^{\frac{1}{2}}\}}{m(2 - am^2)}. \quad \text{(A2.7)}$$

For $m = 1$ and $X = 0.1$ [$|\mathcal{K}_1(0.1)| = 1.02463$] this yields

$$\Omega = 1.95305 \pm i0.302565.$$

A better result is obtained if we substitute $\Omega = 2 + \delta$ into (A2.5) and solve the quadratic portion of the resulting cubic equation in δ. If we retain ϵ^2 and again use $m = +1$, $X = 0.1$, we obtain the complex modes

$$\Omega = 1.96258 \pm i0.289068.$$

These results are to be compared with the computer calculated values of:

Λ	Ω
2.0	$1.93841 \pm i0.339691$
3.0	$1.95922 \pm i0.293517$
10.0	$1.96485 \pm i0.280518$
15.0	$1.96437 \pm i0.282453$.

Note that (A2.7) indicates that this complex mode is not present for any other value of $m > 1.0$.

If one includes the term $\mathcal{O}(\Lambda^{-1})$ in D', we can write the condition for *complex* modes (of (A2.5) with $\epsilon^2 = 0$) as

$$1 - 2m + am^2 + 2amXh/\Lambda < 0, \quad \text{(A2.8)}$$

or

$$\frac{h}{\Lambda} < \frac{[(2m-1)/am] - m}{2X} \quad \text{(A2.9)}$$

For $m = +1$ this becomes

$$\frac{h}{\Lambda} < \frac{1/a - 1}{2X}. \quad \text{(A2.10)}$$

Thus for a fixed X, the stronger the longitudinal magnetic field h, the larger must be the flow Λ to obtain complex modes. Note that for Λ very large, h does not appear explicitly in (A2.3) or (A2.5), implying that the same unstable Ω modes will be present regardless of the values of h.