Supplementary material for "Does filling-dependent band renormalization aid pairing in twisted bilayer graphene?"

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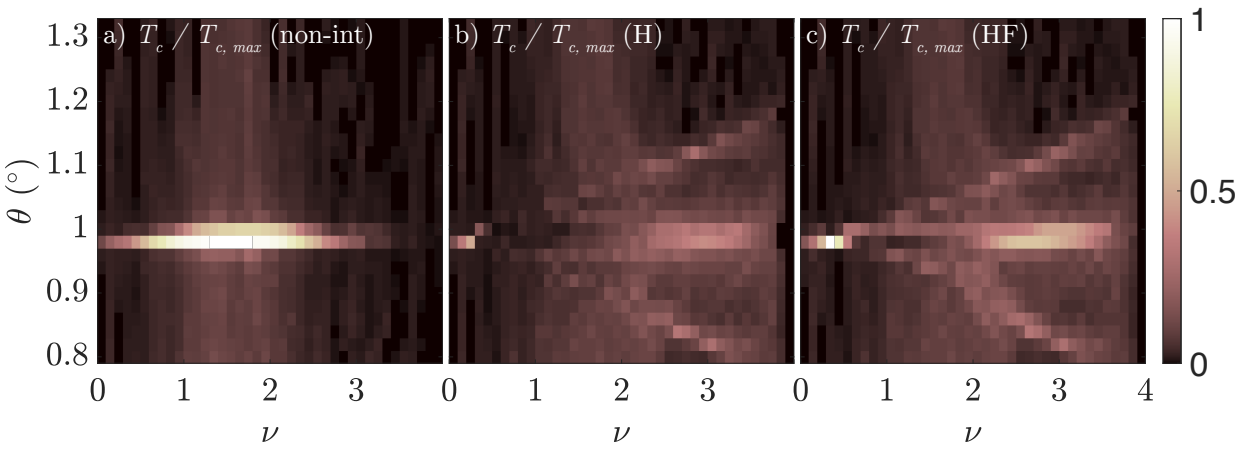
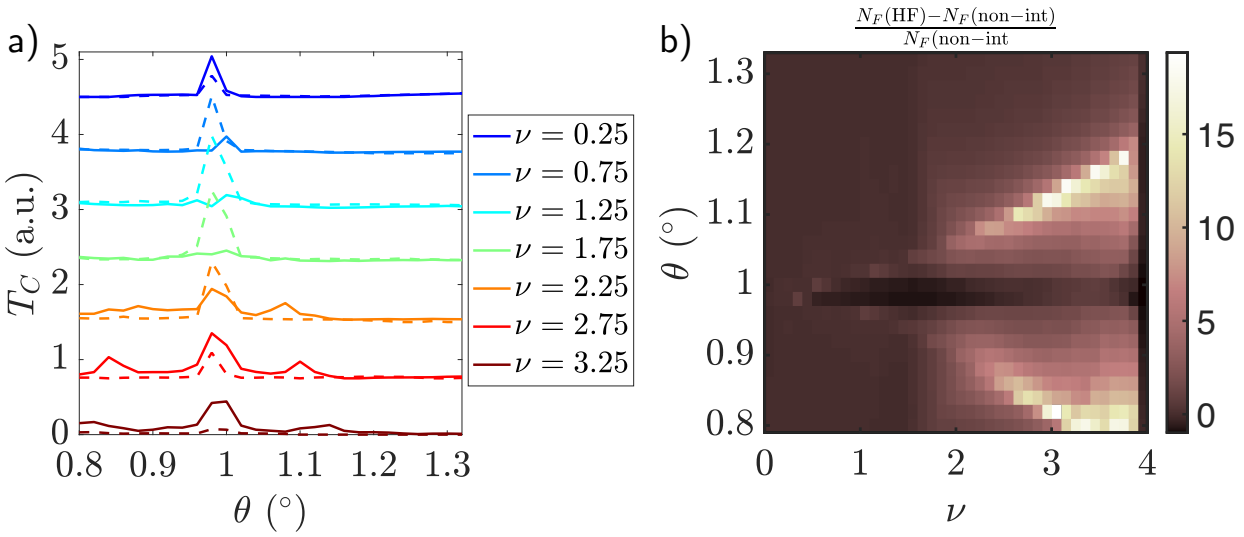
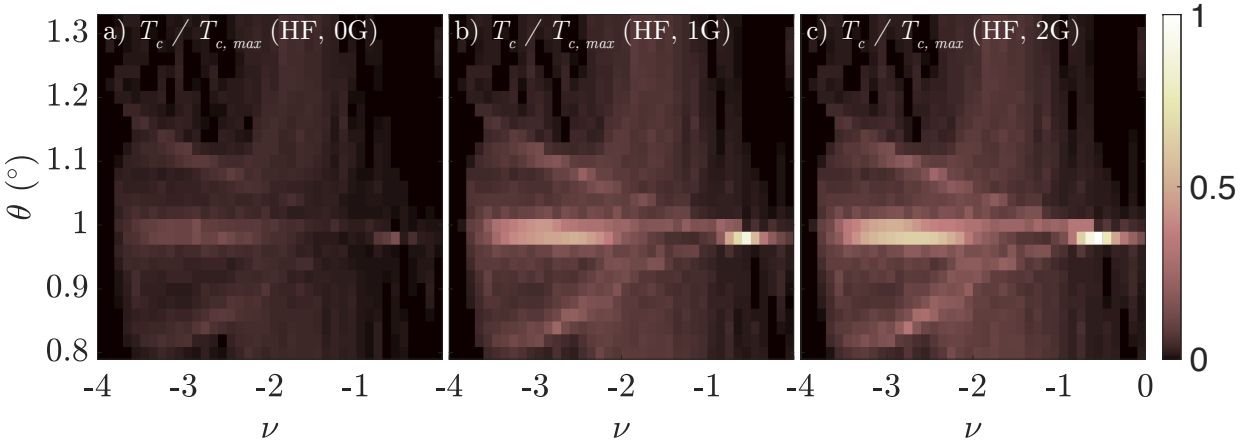
Supplementary Figures



Supplementary Figure 1. **Role of Hartree and Fock corrections on the pairing temperature.** T_c obtained from the linearized Eliashberg equation (Eq. 12) as a function of ν and θ for (a) non-interacting, (b) Hartree-only corrected bandstructure, and (c) Hartree-Fock corrected bandstructure. The values of T_c are normalized in (a), (b) and (c) relative to the highest pairing temperature $T_{C,max}$ in (a) (see discussion in the main text).



Supplementary Figure 2. **A comparison of the pairing tendencies in the non-interacting and HF modified system.** (a) constant filling linecuts of the Fig.2a,b for non-interacting (dashed) and HF-modified (solid) system. Curves are vertically separated for clarity. (b) Relative change of the Fermi surface density of states in the two models. Note how (a,b) demonstrated broadening of the twist angle range over which pairing tendencies can be seen.



Supplementary Figure 3. **Contribution of phonon umklapp processes in enhancing pairing tendencies.** T_c obtained from the linearized Eliashberg equation (Eq. 12) as a function of ν and θ for (a) 0G, (b) 1G, and (c) 2G phonon umklapp processes for a Hartree-Fock corrected bandstructure. The values of T_c are normalized in (a), (b) and (c) relative to the highest pairing temperature $T_{C,max} = 0.0294$ meV of Supplemental Figure 1a (see discussion in text).

Supplementary Discussion

Comparison of Hartree vs. Hartree-Fock and role of electron-phonon umklapp

Here we present additional results for the calculation of T_c as a function of filling and twist angle; we focus specifically on (i) a comparison of a non-interacting, Hartree-only and Hartree-Fock modified bandstructure on pairing tendencies, (ii) the role played by Umklapp processes, and (iii) the shape of the T_c dome for negative fillings $\nu < 0$.

In Supplemental Figure 1 we show the pairing dome profile as a function of twist angle and filling for three different "extents" of incorporating Coulomb interactions. Supplemental Figure 1a shows a pairing profile for a non-interacting bandstructure and wavefunctions which, as discussed in the main text, is sharply peaked near the magic angle and falls precipitously away from it. As expected of a weak-coupling calculation, the dome is peaked at the location of the vHs of the non-interacting bandstructure that remains fixed for all twist angles. When Hartree-only (Supplemental Figure 1b) and Hartree-Fock (Supplemental Figure 1c) corrections are introduced into the SC calculation, we find the appearance of a qualitatively similar-looking three-prong feature stemming from band-flattening physics analysed in the main text. The Hartree-only correction however presents an overall lower maximum T_c near the magic angle as compared to the Hartree-Fock results. This simply is a consequence of the Hartree-only bandstructure featuring a large $\bar{\Gamma}$ point inversion that increases overall bandwidth at the magic angle beyond that of non-interacting or Hartree-Fock (See also Supplemental Figure 2 for further comparison of non-interacting and Hartree-Fock results.). In Supplemental Figure 1, 2, as in the main text, we normalize the pairing temperature T_c by its maximum value for the non-interacting bandstructure of $T_{c,max} = 0.0294$ meV.

In Supplemental Figure 3 we study the effect of including umklapp phonon processes on the pairing temperature. As we argued in Ref.[1], in TBG due to the small size of the effective MBZ, scattering processes that involve an exchange of multiples of reciprocal lattice vectors (umklapp processes) are favourable. We denote a computation which incorporates an exchange of up to m reciprocal lattice vectors G_1^M, G_2^M as an "mG" calculation. Due to the nature of the moiré potential that couples both graphene layers to each other, to construct a low energy effective theory it is necessary to use several graphene states that are coupled

by these reciprocal lattice vectors G. The specific number of such states imposes a critical value of m umklapp processes that can be exchanged above which there is no significant increase in pairing temperature - we refer interested reader to our earlier work Ref.[1]. We stress that again the highest m above which no further increase in pairing temperature (typically $mG = 3G$) occurs is well below any cut-off imposed by the finite truncation of the continuum Hamiltonian (which would correspond to $mG = 15G$ as specified in the Methods section of the main text. Indeed this physics finds representation in our results of Supplemental Figure 3a-c, where inclusion of higher number of umklapp processes enhances pairing temperature. We also note in passing that this effect becomes less pronounced the larger is the twist angle as the MBZ size increases, thereby suppressing umklapp processes. Finally, note that Supplemental Figure 3a-c is evaluated for negative fillings to provide an explicit demonstration of behavior that is qualitatively similar to $\nu > 0$ (i.e. we do not find discernible qualitative differences between results of Fig.2b and Supplemental Figure 1c beyond some small quantitative differences stemming from particle-hole asymmetry that is present in the TBG Hamiltonian).

Supplementary References

[1] Lewandowski, C., Chowdhury, D. & Ruhman, J. Pairing in magic-angle twisted bilayer graphene: Role of phonon and plasmon umklapp. *Phys. Rev. B* **103**, 235401 (2021).