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Scattering of a Plane Acoustical Wave by a Spherical Obstacle

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The long-time behavior of the scattered field produced by a plane acoustical pulse striking a spherical obstacle is investigated. Incident pulses are taken that represent a unit step in potential (or equivalently a delta function pulse in pressure), an arbitrary potential pulse of finite duration, and an arbitrary pressure pulse of finite duration. Both hard and soft spheres are considered. In addition, a pulse consisting of a unit step in velocity impinging on a hard sphere is examined. In each case, the time rate of decay to the steady state is established. This is seen to be controlled by the zeroes of certain naturally occurring special polynomials that arise because of the spherical geometry and are independent of the shape of the incident pulse.

INTRODUCTION

OUR principal interest in this investigation is the behavior for long time of the scattered field produced by a plane acoustical pulse striking a spherical obstacle. Much of the previous work has been concerned with short-time effects, that is, the behavior of the scattered field shortly behind the incident front. Chen,[1] however, has considered the long-time aspects of the corresponding two-dimensional problem, and an earlier paper of Baron and Matthews[2] was concerned with a similar problem for a two-dimensional elastic medium. In addition to these specific examples, Lax, Morawetz, and Phillips[3–5] have developed a number of advanced results concerning the general behavior of problems, of which ours is a particular simple example.

We propose here to consider several elementary examples in which the dominant long-time behavior can be examined through simple calculations. The medium is the classic acoustic one in which the velocity potential $u(\mathbf{r},t)$ satisfies the scalar wave equation $\Delta u=(1/c^2)\partial^2u/\partial t^2$. The scattering obstacle is a sphere of radius a, which, in some cases, is considered soft ($u=0$ on the surface) and, in others, hard ($\partial u/\partial r=0$ on the surface). The resulting three-dimensional problem yields at once to separation of variables and the corresponding functions are perhaps the simplest to handle for such problems.

Several types of incident plane pulses are considered: a unit step in potential, an arbitrary pulse of potential of finite duration, and an arbitrary pulse of pressure of finite duration. A unit step in pressure is also studied for the case of the hard sphere. In each case, we are primarily concerned with the long-time behavior. Utilizing a recent result of Habetler[6] on the behavior of the zeroes of the polynomials associated with the spherical Hankel functions, it will be possible to pick out the dominant mode in the portion of the scattered potential that decays exponentially in time.

Part I. Plane Potential Wave

I. FORMULATION OF THE PROBLEM

In this section, we consider the problem of the scattering of a plane pulse by a sphere of radius a (see Fig. 1). This solution will also serve as a Green's function for determining the scattering of an arbitrarily shaped pulse. The total scalar field $u(\mathbf{r},t)$ can be

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[3] P. D. Lax and R. S. Phillips, "The Wave Equation in Exterior Domains," Bull. Am. Math. Soc. 68, 47–49 (1962).

[4] P. D. Lax, C. S. Morawetz, and R. S. Phillips, "The Exponential Decay of Solutions of the Wave Equation in the Exterior of a Star-Shaped Obstacle," Bull. Am. Math. Soc. 68, 593–595 (1962).

[5] P. D. Lax and R. S. Phillips, "Scattering Theory," Bull. Am. Math. Soc. 70, 130–142 (1964).

[6] G. Habetler, "On Bessel Polynomials" (to be published).

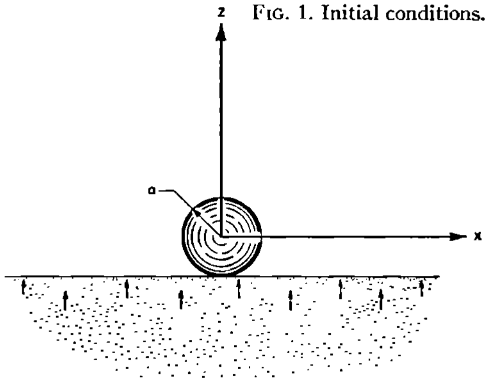

FIG. 1. Initial conditions.

written as

$$u(\mathbf{r},t)=u_i(\mathbf{r},t)+u_s(\mathbf{r},t), \qquad (1)$$

where u_i is the incident field and u_s is the scattered field. The incident field is given as

$$u_i(\mathbf{r},t)=H(t-(z+a)/c), \qquad (2)$$

where $H(x)$ is the usual Heaviside function defined as 1 for $x>0$ and 0 for $x<0$. Thus, at time $t=0$ the front of the pulse just strikes the sphere. Our problem is to determine the scattered field $u_s(\mathbf{r},t)$. Since the total field satisfies the scalar wave equation, the scattered field will also be a solution of

$$\Delta u_s(\mathbf{r},t)=(1/c^2)\partial^2 u_s/\partial t^2 \quad \text{for} \quad r>a, \quad t>0. \qquad (3)$$

In addition,

$$u_s(\mathbf{r},t)\equiv 0, \quad \text{for} \quad t\leq 0 \quad \text{and} \quad r>a. \qquad (4)$$

Since the surface of the sphere is taken to be soft,

$$u_s(\mathbf{a},t)=-u_i(\mathbf{a},t) \quad \text{for} \quad t>0. \qquad (5)$$

From the symmetry of the problem, u_s is independent of the azimuthal angle ϕ. The problem, stated above, can be solved in the standard fashion by transform and separation of variables methods. The solution thus obtained is

$$u_s(\mathbf{r},t)=\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)I_n(r,t), \qquad (6)$$

where

$$I_n(r,t)=\frac{1}{2\pi i}\int_{i\eta-\infty}^{i\eta+\infty}\frac{e^{-i[t-(a/c)]\omega}j_n(a\omega/c)h_n^{(1)}(r\omega/c)}{\omega h_n^{(1)}(a\omega/c)}d\omega,$$

$$(n=0,1,2,\cdots). \qquad (7)$$

Here, $P_n(x)$ is the Legendre polynomial of degree n, and $j_n(x)$ and $h_n^{(1)}(x)$ are, respectively, the spherical Bessel function and Hankel function of the first kind of order n.

II. EVALUATION OF THE INTEGRALS $I_n(r,t)$

In order to determine properties of the scattered field Eq. 6, we must evaluate the integrals Eq. 7. This can be accomplished by closing the contour with a semicircle in the appropriate half-plane and using Cauchy's residue theorem. Now $j_n(z)$ is an entire function and $h_n^{(1)}(z)$ is meromorphic, with a single pole located at the origin. Furthermore, $h_n^{(1)}(z)$ has exactly n simple zeros located in the lower half of the z plane. Hence, the integrand of I_n has n simple poles at the zeros of $h_n^{(1)}(a\omega/c)$. Furthermore, in the neighborhood of $z=0$ (Ref. 7),

$$j_n(z)\sim k_1 z^n, \quad h_n^{(1)}(z)\sim k_1 z^n+k_2 z^{-n-1}, \qquad (8)$$

where k_1 and k_2 are constants. Thus, in the neighborhood of $\omega=0$, the integrand in Eq. 7 behaves like

$$k_3[e^{-i(t-a/c)\omega}]\omega^{n-1}, \qquad (9)$$

where again k_3 is a constant. Thus the integrand of I_0 has a simple pole at $\omega=0$, and the integrand of $I_n (n\geq 1)$ has no singularity at $\omega=0$.

Making use of the fact that

$$j_n(z)=\tfrac{1}{2}[h_n^{(1)}(z)+h_n^{(2)}(z)],$$

where $h_n^{(2)}(z)$ is the spherical Hankel function of the second kind, we may write

$$I_n(r,t)=\frac{1}{2\pi i}\int_{i\eta-\infty}^{i\eta+\infty}e^{-i[t-(a/c)]\omega}\frac{h_n^{(1)}(r\omega/c)}{2\omega}d\omega$$

$$+\frac{1}{2\pi i}\int_{-\infty+i\eta}^{\infty+i\eta}\frac{e^{-i[t-(a/c)]\omega}h_n^{(2)}(a\omega/c)h_n^{(1)}(r\omega/c)}{2\omega h_n^{(1)}(a\omega/c)}d\omega. \qquad (10)$$

Using the asymptotic formulas for $z\rightarrow\omega$,

$$h_n^{(1)}(z)\sim(1/z)e^{i[z-(n+1)(\pi/2)]}, \quad -\pi<\arg(z)<2\pi, \qquad (11)$$

$$h_n^{(2)}(z)\sim(1/z)e^{-i[z-(n+1)(\pi/2)]}, \quad -2\pi<\arg(z)<\pi,$$

it is seen that as $\omega\rightarrow\infty$ the integrand of the first integral in Eq. 10 is asymptotic to

$$c[e^{-i[t-(a/c)-(r/c)]\omega}e^{-i[(n+1)(\pi/2)]}]/2r\omega^2, \qquad (12)$$

and the integrand of the second integral is asymptotic to

$$c[e^{-i[t+(a/c)-(r/c)]\omega}e^{i[(n+1)(\pi/2)]}]/2r\omega^2. \qquad (13)$$

We therefore conclude that the first integral in Eq. 10 can be evaluated by closing the contour in the lower half-plane if $r<ct-a$ or by closing in the upper half-plane if $r>ct-a$. For the second integral in Eq. 10, we close in the lower half-plane if $r<ct+a$ or in the upper half-plane if $r>ct+a$. Since $ct-a<ct+a$,

$$I_n(r,t)=\begin{cases}\sum \text{(RESIDUES FROM BOTH INTEGRALS IN Eq. 10)} & \text{if } r<ct-a,\\ \sum \text{(RESIDUES FROM SECOND INTEGRAL IN Eq. 10)} & \text{if } ct-a<r<ct+a,\\ 0 & \text{if } r>ct+a.\end{cases} \qquad (14)$$

[7] A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Co., Inc., New York, 1953), Vol. 2.

We have previously seen that the integrand in I_0 has a simple pole at $\omega=0$, whereas that in I_n for $n\geq 1$ has no pole at the origin, although each of the integrands taken separately has a pole of order $n+2$ at the origin. Let $\omega_{n,m}$ denote the mth zero of $h_n^{(1)}(a\omega/c)$ for $n=1, 2, 3, \cdots$. The integrals can then be written as

$$I_n(r,t)=\begin{cases} R_n(r,t) & \text{if } r<ct-a, \\ R_n(r,t)+S_n(r,t) & \text{if } ct-a<r<ct+a, \\ 0 & \text{if } r>ct+a, \end{cases} \tag{15}$$

where, for $n\geq 1$,

$$\begin{aligned} R_n(r,t)&=-\sum_{m=1}^{n}\frac{e^{i[t-(a/c)]\omega_{n,m}}h_n^{(2)}(a\omega_{n,m}/c)h_n^{(1)}(r\omega_{n,m}/c)}{2\omega_{n,m}[(d/d\omega)h_n^{(1)}(a\omega/c)]_{\omega=\omega_{n,m}}}, \\ S_n(r,t)&=\frac{-1}{(n+1)!}\frac{d^{n+1}}{d\omega^{n+1}}\left[\frac{\omega^{n+2}e^{-i[t-(a/c)]\omega}h_n^{(2)}(a\omega/c)h_n^{(1)}(r\omega/a)}{2\omega h_n^{(1)}(a\omega/c)}\right]_{\omega=0}. \end{aligned} \tag{16}$$

In addition, since $j_0(z)=(\sin z)/z$ and $h_0^{(1)}(z)=(-ie^{iz})/z$, we obtain

$$I_0(r,t)=\begin{cases} -a/r & \text{if } r<ct-a, \\ (r-a-ct)/2r & \text{if } ct-a<r<ct+a, \\ 0 & \text{if } r>ct+a. \end{cases} \tag{17}$$

Furthermore,

$$I_1(r,t)=\begin{cases} -(a/ir)(1-a/r)e^{-(1-r/a)}e^{-ct/a} & \text{if } r<ct-a, \\ -(a/ir)(1-a/r)e^{-(1-r/a)}e^{-ct/a}+[r^2-(a-ct)^2]/4ir & \text{if } ct-a<r<ct+a, \\ 0 & \text{if } r>ct+a. \end{cases} \tag{18}$$

III. PROPERTIES OF THE SCATTERED FIELD

Using the results just obtained in Sec. II, we find, for $ct-a<r<ct+a$,

$$u_s(\mathbf{r},t)=\frac{r-a-ct}{2r}+\sum_{n=1}^{\infty}(2n+1)i^nP_n(\cos\theta)\times[R_n(r,t)+S_n(r,t)]. \tag{19}$$

Note that the region $ct-a<r<ct+a$ contains a portion that lies in front of the advancing incident wave (see Fig. 2). Furthermore, each term in this series yields a nonzero contribution in the portion of the region ahead of the incident wavefront. On physical grounds, however, it is clear that the series must sum to zero at least in this region. It is this fact that accounts for the well-known result that, in the geometric shadow, series solutions of our type converge slowly, for many terms must be retained to yield a mathematically negligible field ahead of the incident pulse. The extent of the scattered wave can be determined by methods indicated by Friedlander[8] or Zauderer,[9] for example. Thus, the series need be evaluated only in that part of the field where the scattered field does not vanish.

We now examine the scattered field at a point long after the incident pulse has passed the sphere; this is what we mean by large time. More precisely, we mean that for fixed r, t is sufficiently large so that $r<ct+a$ or, equivalently, $(r-a)/c<t$. In this case, Eq. 6 becomes

$$u_s(\mathbf{r},t)=-(a/r)-\sum_{n=1}^{\infty}(2n+1)i^nP_n(\cos\theta)\times\sum_{m=1}^{n}\frac{e^{-i[t-(a/c)]\omega_{n,m}}h_n^{(2)}(a\omega_{n,m}/c)h_n^{(1)}(r\omega_{n,m}/c)}{2\omega_{n,m}[(d/d\omega)h_n^{(1)}(a\omega/c)]_{\omega=\omega_{n,m}}}. \tag{20}$$

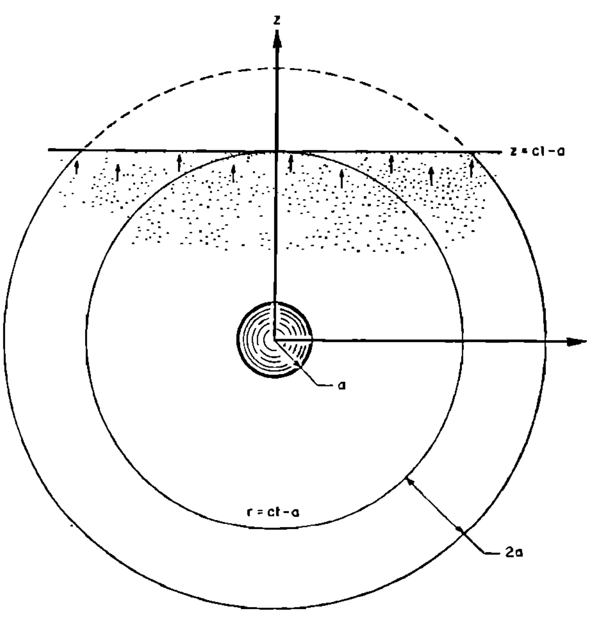

FIG. 2. Regions of integral contributions.

[8] F. G. Friedlander, *Sound Pulses* (Cambridge University Press, New York, 1958).
[9] E. Zauderer, "Domains of Dependence for Mixed Problems for Wave Equations," Arch. Rat. Mech. Anal. 15, 69–78 (1964).

Since every $\omega_{n,m}$ has a negative imaginary part, the sum in Eq. 20 decays exponentially, and, for large time, the total field $u(\mathbf{r},t)$ is asymptotic to $1-a/r$. This is physically reasonable since $1-a/r$ is the solution of the static problem $\Delta u=0$ in $r>a$ and $u=0$ on $r=a$, and this is the steady-state problem corresponding to our original problem, Eqs. 3–5.

It has been shown[6] that the zero $z=-i$ of $h_1^{(1)}(z)$ lies closer to the real axis than the zeros of any other $h_n^{(1)}(z)$. Thus, the $n=1$ term corresponds to the slowest rate of exponential decay. Consequently, we have that as $t\rightarrow\infty$,

$$u_s(\mathbf{r},t)+(a/r)\sim g(r,\theta)e^{-ct/a}, \tag{21}$$

where

$$g(r,\theta)=(3a/r)[1-(a/r)]e^{-[1-(r/a)]}\cos\theta. \tag{22}$$

The asymptotic formula Eq. 21 gives the exact rate of approach to the steady state for the scattering of a plane pulse by a sphere.

IV. SOLUTION FOR A HARD SPHERE

The solution of the problem of a plane pulse impinging on a hard sphere is obtained in a manner completely analogous to that of the soft sphere. If we denote the solution by $u^*(\mathbf{r},t)=u_i^*(\mathbf{r},t)+u_s^*(\mathbf{r},t)$, $u_i^*(\mathbf{r},t)=H(t-(z+a)/c)$, and u_s^* satisfies Eqs. 3 and 4. Equation 5 is replaced by

$$\partial u_s^*(\mathbf{a},t)/\partial r=-\partial u_i^*(\mathbf{a},t)/\partial r,\quad t>0. \tag{23}$$

The solution is given by

$$u_s^*(\mathbf{r},t)=\sum_{n=0}^{\infty}(2n+1)i^nP_n(\cos\theta)I_n^*(r,t), \tag{24}$$

where

$$I_n^*(r,t)=\begin{cases} R_n^*(r,t) & \text{if } r<ct-a,\\ R_n^*(r,t)+S_n^*(r,t) & \text{if } ct-a<r<ct+a,\\ 0 & \text{if } r>ct+a.\end{cases} \tag{25}$$

If we use the notation

$$h_n'(a\omega/c)\equiv(d/dz)h_n(z)\big|_{z=a\omega/c} \tag{26}$$

and let $\omega_{n,m}^*$ $(m=1,\cdots,n+1)$ denote the roots of

$$h_n^{(1)\prime}(a\omega/c)=0, \tag{27}$$

we can show that

$$R_n^*(r,t)=-\sum_{m=1}^{n+1}\frac{e^{-i[t-(a/c)]\omega_{n,m}^*}h_n^{(2)\prime}(a\omega_{n,m}^*/c)h_n^{(1)}(r\omega_{n,m}^*/c)}{2\omega_{n,m}^*[(d/d\omega)h_n^{(1)\prime}(a\omega/c)]_{\omega=\omega_{n,m}^*}}, \tag{28}$$

and

$$S_n^*(r,t)=\frac{-1}{(n+1)!}\frac{d^{n+1}}{d\omega^{n+1}}\left[\frac{\omega^{n+2}e^{-i[t-(a/c)]\omega}h_n^{(2)\prime}(a\omega/c)h_n^{(1)}(r\omega/c)}{2\omega h_n^{(1)\prime}(a\omega/c)}\right]_{\omega=0} \tag{29}$$

Since the roots $\omega_{n,m}^*$ all lie in the lower half-plane, the scattered field $u_s(\mathbf{r},t)$ will decay exponentially for large values of time. The leading terms can be found from the facts that

$$\begin{aligned} R_0^*(r,t)&=(a/r)e^{-(ct-r+a)/a},\\ S_0^*(r,t)&=(2r)^{-1}(r+a-ct),\\ R_1^*(r,t)&=\frac{a}{ir}e^{-(ct-r+a)/a}\left[-\sin\left(\frac{ct-r+a}{a}\right)+\left(1-\frac{a}{r}\right)\cos\left(\frac{ct-r+a}{a}\right)\right]\\ S_1^*(r,t)&=(4ir^2)^{-1}[r^2-(a-ct)^2].\end{aligned} \tag{30}$$

Thus for $t\rightarrow\infty$, the behavior of the scattered field is given by

$$u_s^*(\mathbf{r},t)\sim(a/r)e^{-(ct-r+a)/a}\left\{1+3\left[-\sin\left(\frac{ct-r+a}{a}\right)+\left(1-\frac{a}{r}\right)\cos\left(\frac{ct-r+a}{a}\right)\right]\cos\theta\right\}, \tag{31}$$

Part II. Arbitrary Pulse Shape

I. ARBITRARY PULSE FOR SOFT SPHERE

We now consider the scattering by a soft sphere of a plane pulse of arbitrary shape and finite duration. More precisely, we determine the solution of the following: let $v(\mathbf{r},t)=v_i(\mathbf{r},t)+v_s(\mathbf{r},t)$, where, for an arbitrary function F,

$$\Delta v=(1/c^2)\partial^2 v/\partial t^2, \tag{32}$$

$$v_i(\mathbf{r},t)=\begin{cases} F(t-(z+a)/c) & \text{if}\quad (z+a)/c<t<(z+\alpha a)/c\quad(\alpha>1),\\ 0 \text{ OTHERWISE},\end{cases} \tag{33}$$

$$v_s(\mathbf{a},t)=-v_i(\mathbf{a},t)\quad\text{for}\quad t>0, \tag{34}$$

and

$$v_s(\mathbf{r},t)\equiv0\qquad\text{for}\quad t\leq0. \tag{35}$$

The scattered field $v_s(\mathbf{r},t)$ will also satisfy the scalar wave equation by virtue of Eqs. 32 and 33.

The solution can be written at once as a Duhamel integral in terms of the step plane-wave solution $u(\mathbf{r},t)$ given in the previous sections. The desired function is

$$v(\mathbf{r},t)=\int_{-\infty}^{\infty}F(\tau)(\partial/\partial t)u(\mathbf{r},t-\tau)d\tau. \qquad (36)$$

Since $u_i(\mathbf{r},t)$ is the Heaviside step function,

$$v_i(\mathbf{r},t)=\int_{-\infty}^{\infty}F(\tau)(\partial/\partial t)u_i(\mathbf{r},t-\tau)d\tau, \qquad (37)$$

and, by subtraction,

$$v_s(\mathbf{r},t)=\int_{-\infty}^{\infty}F(\tau)(\partial/\partial t)u_s(\mathbf{r},t-\tau)d\tau. \qquad (38)$$

Now $F(\tau)\neq 0$ only for $0<\tau<(\alpha-1)a/c$ and $u_s(\mathbf{r},t-\tau)\equiv 0$ for $\tau\geq t$. Hence the range of integration in the last integral can certainly be reduced to

$$v_s(\mathbf{r},t)=\int_{0}^{t}F(\tau)(\partial/\partial t)u_s(\mathbf{r},t-\tau)d\tau. \qquad (39)$$

Furthermore, if $t>(\alpha-1)a/c$, the upper limit in Eq. 39 can be replaced by $(\alpha-1)a/c$. It can easily be seen from the properties of $u_s(\mathbf{r},t)$ that Eq. 39 yields the desired solution.

We now examine the properties of Eq. 39 more closely and, in particular, study the long-time behavior. Referring to the results found for $u_s(\mathbf{r},t)$, we see that

$$v_s(\mathbf{r},t)=\sum_{n=0}^{\infty}(2n+1)i^nP_n(\cos\theta)K_n, \qquad (40)$$

where

$$K_n=\int_0^t F(\tau)\partial/\partial tI_n(r,t-\tau)d\tau. \qquad (41)$$

Now

$$I_n(r,t-\tau)=\begin{cases} R_n(r,t-\tau) & \text{for } 0\leq\tau\leq t-(r+a)/c,\\ R_n(r,t-\tau)+S_n(r,t-\tau) & \text{for } t-(r+a)/c<\tau<t-(r-a)/c,\\ 0 & \text{for } \tau>t-(r-a)/c.\end{cases} \qquad (42)$$

Consequently,

$$K_n=\int_0^{t-(r+a)/c}F(\tau)(\partial/\partial t)R_n(r,t-\tau)d\tau$$
$$+\int_{t-(r+a)/c}^{t-(r-a)/c}F(\tau)[(\partial/\partial t)R_n(r,t-\tau)$$
$$+(\partial/\partial t)S_n(r,t-\tau)]d\tau. \qquad (43)$$

Furthermore, if $t<(r-a)/c$,

$$K_n=0;$$

if $(r-a)/c<t<(r+a)/c$, then

$$K_n=\int_0^{t-(-a)/c}F(\tau)[(\partial/\partial t)R_n(r,t-\tau)$$
$$+(\partial/\partial t)S_n(r,t-\tau)]d\tau; \qquad (44)$$

and, finally, if $t>(r+a)/c$, all the terms of Eq. 43 will enter.

We recall, on the other hand, that $F(\tau)=0$ if $\tau\geq(\alpha-1)a/c$. Let us now consider the solution for large values of t; that is, we consider r fixed and t sufficiently large so that $t-(r+a)/c\geq(\alpha-1)a/c$. In other words, t and r are such that $t\geq(r+a)/c$. This means that the elapsed time exceeds the time that it would take the tail of the wave to reach a distance r from the center of the sphere traveling along a normal to the wavefront. In this case, Eq. 43 reduces to

$$K_n=\int_0^{(\alpha-1)a/c}F(\tau)(\partial/\partial t)R_n(r,t-\tau)d\tau, \qquad (45)$$

or

$$K_n=\sum_{m=1}^{n}\frac{ie^{i[t-(a/c)]\omega_{n,m}}h_n^{(2)}(a\omega_{n,m}/c)h_n^{(1)}(r\omega_{n,m}/c)}{2[(d/d\omega)h_n^{(1)}(a\omega/c)]_{\omega=\omega_{n,m}}}\times\int_0^{(\alpha-1)a/c}F(\tau)e^{i\tau\omega_{n,m}}d\tau. \qquad (46)$$

Equation 46 shows that the arbitrarily shaped pulse has the same exponential decay as the infinite step pulse. In addition, due to the finite duration of the pulse, there is no nonvanishing static solution. It can also be seen that the leading terms for the K_n are

$$K_0=0$$

and

$$K_1=(c/ir)[1-(a/r)]e^{-(ct-r+a)/a}\times\int_0^{(\alpha-1)a/c}F(\tau)e^{c\tau/a}d\tau. \qquad (47)$$

Therefore, for $t \to \infty$,

$$v_s(\mathbf{r},t) \sim (3c/r)[1-(a/r)]e^{-(ct-r+a)/a}\cos\theta \times \int_0^{(\alpha-1)a/c} F(\tau)e^{c\tau/a}d\tau. \quad (48)$$

II. ARBITRARY PULSE FOR A HARD SPHERE

The scattering of an arbitrary pulse of finite duration by a hard sphere can be found in exactly the same manner as that for the soft sphere, with the rôle of $u(\mathbf{r},t)$ being replaced by $u^*(\mathbf{r},t)$. We denote the solution by $v^*(\mathbf{r},t)=v_i^*(\mathbf{r},t)+v_s^*(\mathbf{r},t)$. Equations 32–35 govern this problem, except that Eq. 34 is replaced by

$$\partial v_s^*(\mathbf{a},t)/\partial r = -\partial v_i^*(\mathbf{a},t)/\partial r \quad \text{for} \quad t>0. \quad (49)$$

The solution is given by

$$v_s^* = \sum_{n=0}^{\infty}(2n+1)i^nP_n(\cos\theta)K_n^*, \quad (50)$$

where

$$K_n^* = \int_0^t F(\tau)(\partial/\partial t)I_n^*(r,\, t-\tau)d\tau. \quad (51)$$

As in the previous example, if $t \geq (r+\alpha a)/c$,

$$K_n^* = \sum_{m=1}^{n+1} \frac{ie^{-i[t-(a/c)]\omega_{n,m}^*}h_n^{(2)\prime}(a\omega_{n,m}^*/c)h_n^{(1)}(r\omega_{n,m}^*/c)}{2[(d/d\omega)h_n^{(1)\prime}(a\omega/c)]_{\omega=\omega_{n,m}^*}} \times \int_0^{(\alpha-1)a/c} F(\tau)e^{i\tau\omega_{n,m}^*}d\tau. \quad (52)$$

Again, we see that the scattered wave has the same exponential decay as the step wave. The leading term for $t \to \infty$ is

$$v_s^*(\mathbf{r},t) \sim (c/r)e^{-(ct-r+a)/a}\left\{-\int_0^{(\alpha-1)a/c} F(\tau)e^{c\tau/a}d\tau + 3\cos\theta\int_0^{(\alpha-1)a/c} F(\tau)e^{c\tau/a}\left[\frac{a}{r}\sin\left(\frac{ct-c\tau-r+a}{a}\right) - \left(2-\frac{a}{r}\right)\cos\left(\frac{ct-c\tau-r+a}{a}\right)\right]d\tau\right\}. \quad (53)$$

III. ARBITRARY PRESSURE PULSE

It is sometimes more convenient to specify the incident wave in terms of a pressure pulse rather than a potential pulse. Since the pressure p is related to the static pressure p_0, the density ρ_0, and the potential ϕ by

$$p-p_0=\rho_0(\partial\phi/\partial t), \quad (54)$$

the problem for the soft sphere may be stated as follows. Let

$$\phi(\mathbf{r},t)=\phi_i(\mathbf{r},t)+\phi_s(\mathbf{r},t), \quad (55)$$

where, for an arbitrary function G,

$$\Delta\phi=(1/c^2)\partial^2\phi/\partial t^2,$$
$$\partial\phi_i/\partial t=\begin{cases} G(t-(z+a)/c) & \text{if } (z+a)/c<t<(z+\alpha a)/c, \quad (\alpha>1), \\ 0 & \text{OTHERWISE,}\end{cases} \quad (56)$$
$$\phi_s(\mathbf{r},t)\equiv 0, \quad t\leq 0,$$
$$\phi_i(\mathbf{a},t)=-\phi_s(\mathbf{a},t).$$

The solution is given by

$$\phi(\mathbf{r},t)=\int_{-\infty}^{\infty} G(\tau)u(\mathbf{r},\, t-\tau)d\tau. \quad (57)$$

We again find that

$$\phi_s(\mathbf{r},t)=\sum_{n=0}^{\infty}(2n+1)i^nP_n(\cos\theta)M_n(\mathbf{r},t), \quad (58)$$

where

$$M_n(r,t)=\int_0^t G(\tau)I_n(r,\, t-\tau)d\tau. \quad (59)$$

For t sufficiently large so that $t\geq(r+\alpha a)/c$,

$$M_n=-\sum_{m=1}^{n}\frac{e^{-i[t-(a/c)]\omega_{n,m}}h_n^{(2)}(a\omega_{n,m}/c)h_n^{(1)}(r\omega_{n,m}/c)}{2\omega_{n,m}[(d/d\omega)h_n^{(1)}(a\omega/c)]_{\omega=\omega_{n,m}}} \times \int_0^{(\alpha-1)a/c} G(\tau)e^{i\tau\omega_{n,m}}d\tau. \quad (60)$$

The leading terms for $t\to\infty$ are

$$\phi_s(\mathbf{r},t)\sim\frac{-a}{r}\int_0^{(\alpha-1)a/c}G(\tau)d\tau - \frac{3a}{r}\cos\theta(1-a/r)e^{-(ct+a-r)/a} \times \int_0^{(\alpha-1)a/c}G(\tau)e^{c\tau/a}d\tau. \quad (61)$$

We see that the rates of exponential decay are exactly the same as those for the potential pulse impinging on the soft sphere. In this case, moreover, there is a non-vanishing static term $(-a/r)\int_0^{(\alpha-1)a/c}G(\tau)d\tau$.

The solution, $\phi^*(\mathbf{r},t)$ for the pressure pulse impinging on a hard sphere is found in the same way. For t

$\geq (r+\alpha a)/c$, we find

$$\phi_s^*(\mathbf{r},t) = -\sum_{n=0}^{\infty}(2n+1)i^nP_n(\cos\theta)\sum_{m=1}^{n+1}\frac{e^{-i[t-(a/c)]\omega_{n,m}^*}h_n^{(2)\prime}(a\omega_{n,m}^*/c)h_n^{(1)}(r\omega_{n,m}^*/c)}{2\omega_{n,m}^*[(d/d\omega)h_n^{(1)\prime}(a\omega/c)]_{\omega=\omega_{n,m}^*}}\int_0^{(\alpha-1)a/c}G(\tau)e^{i\tau\omega_{n,m}^*}d\tau. \quad (62)$$

The leading terms as $t\to\infty$ are

$$\phi_s^*(\mathbf{r},t)\sim(a/r)e^{-(ct-r+a)/a}\left\{\int_0^{(\alpha-1)a/c}G(\tau)e^{c\tau/a}d\tau+\frac{3a}{r}\cos\theta\int_0^{(\alpha-1)a/c}G(\tau)\right.$$
$$\left.\times\left[-e^{+c\tau/a}\sin\left(\frac{ct-c\tau-r+a}{a}\right)+\left(1-\frac{a}{r}\right)e^{c\tau/a}\cos\left(\frac{ct-c\tau-r+a}{a}\right)\right]d\tau\right\}. \quad (63)$$

In this case, there is no static term as $t\to\infty$.

Part III. Plane Velocity or Pressure Pulse

I. SOLUTION OF PROBLEM

We now consider an incident pulse that represents a jump in velocity or pressure impinging on a hard sphere of radius a. In principle, this question is essentially an example of the problem last studied. Examining the resulting solution for large values of time has one difficulty. Since the incident pulse is not of finite duration, both R_n^* and S_n^* terms will have to be included in the integrals. The behavior of the latter terms is not easy to estimate. Consequently, we consider the problem afresh.

Let us write the solution as $w^*(\mathbf{r},t)=w_i^*(\mathbf{r},t)+w_s^*(\mathbf{r},t)$. We are required to find the solution of the following problem:

$$\begin{aligned} w_i^*(\mathbf{r},t)&=(ct-z-a)H(t-(z+a)/c),\\ \Delta w^*&=(1/c^2)\partial^2w^*/\partial t^2,\quad r>a,\quad t>0,\\ w_s^*(\mathbf{r},t)&\equiv 0,\quad \text{for}\quad t\leq 0,\\ \partial w_s^*(\mathbf{a},t)/\partial r&=-\partial w_i^*(\mathbf{a},t)/\partial r.\end{aligned} \quad (64)$$

Since the velocity field $\mathbf{v}$ is given by $\mathbf{v}=-\nabla w^*$, w_i^* represents a unit jump in velocity propagating as a plane wave in the z direction. Alternatively, w_i^* represents a step wave in pressure $(p-p_0)$ of magnitude c/ρ_0.

It can be shown that the solution to this problem is given by

$$w_s^*(\mathbf{r},t)=\sum_{n=0}^{\infty}(2n+1)i^nP_n(\cos\theta)X_n^*(r,t), \quad (65)$$

where

$$X_n^*(r,t)=\begin{cases}R_n^\dagger(r,t)-(a^3/2r^2)(\cos\theta)\delta_{1n} & \text{if}\quad (r+a)/c<t,\\ R_n^\dagger(r,t)+S_n^\dagger(r,t) & \text{if}\quad (r-a)/c<t<(r+a)/c,\\ 0 & \text{if}\quad t<(r-a)/c,\end{cases} \quad (66)$$

with δ_{1n} being the Kronecker delta. The functions $R_n^\dagger(r,t)$ and $S_n^\dagger(r,t)$ are given by

$$R_n^\dagger(r,t)=-ic\sum_{m=1}^{n+1}\frac{e^{-i[t-(a/c)]\omega_{n,m}^*}h_n^{(2)\prime}(a\omega_{n,m}^*/c)h_n^{(1)}(r\omega_{n,m}^*/c)}{2\omega_{n,m}^*[(d/d\omega)h_n^{(1)\prime}(a\omega/c)]_{\omega=\omega_{n,m}^*}}, \quad (67)$$

and

$$S_n^\dagger(r,t)=\frac{-ic}{(n+2)!}\frac{d^{n+2}}{d\omega^{n+2}}\left[\frac{\omega^{n+3}e^{-i[t-(a/c)]\omega}h_n^{(2)\prime}(a\omega/c)h_n^{(1)}(r\omega/c)}{2\omega^2h_n^{(1)\prime}(a\omega/c)}\right]_{\omega=0}. \quad (68)$$

II. PROPERTIES OF THE SCATTERED FIELD

The remarks of the first paragraph of Sec. III concerning the series in a region ahead of the advancing incident pulse apply equally well here, and, hence, the discussion is not repeated here. Let us now consider the solution for long values of time, in particular for $t>(r+a)/c$. The $S_n^\dagger(r,t)$ terms will then be absent and

$$w_s^*(\mathbf{r},t)=\frac{-a^3\cos\theta}{2r^2}-\sum_{n=1}^{\infty}(2n+1)ci^nP_n(\cos\theta)\sum_{m=1}^{n+1}\frac{e^{-i[t-(a/c)]\omega_{n,m}^*}h_n^{(2)\prime}(a\omega_{n,m}^*/c)h_n^{(1)}(r\omega_{n,m}^*/c)}{2\omega_{n,m}^{*2}[(d/d\omega)h_n^{(1)\prime}(a\omega/c)]_{\omega=\omega_{n,m}^*}}. \quad (69)$$

Since every $\omega_{n,m}^*$ has a negative imaginary part, the sum in Eq. 69 is composed of decaying exponentials. The leading term as $t\to\infty$ is

$$w_s^*(\mathbf{r},t)\sim-(a^3/2r^2)\cos\theta+g(r,\theta)e^{-ct/a}, \qquad (70)$$

where

$$g(r,\theta)=\left\{\frac{a^2}{r}+\frac{3a^3}{2r^2}\cos\theta\left[\sin\left(\frac{ct-r+a}{a}\right)\right.\right.$$
$$\left.\left.-\left(1-\frac{2r}{a}\right)\cos\left(\frac{ct-r+a}{a}\right)\right]\right\}e^{-[1-(r/a)]}. \qquad (71)$$

Let us now recall that the potential about a sphere for steady flow in the z direction with unit velocity of an incompressible, inviscid fluid is

$$-[r+(a^3/2r^2)]\cos\theta. \qquad (72)$$

The first term is the potential of the uniform stream and the second term is that produced by interaction with the sphere. Upon comparing Eqs. 70 and 72, we see that the scattered field of the present problem is asymptotic to the scattered part of the potential corresponding to steady uniform flow about the sphere.

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