

Parity-Violating Møller Scattering at NNLO: Closed Fermion Loops

Yong Du,^{1,*} Ayres Freitas,^{2,†} Hiren H. Patel,^{3,‡} and Michael J. Ramsey-Musolf^{4,1,5,§}

¹*Amherst Center for Fundamental Interactions, Physics Department,
University of Massachusetts Amherst, Amherst, MA 01003 USA*

²*Pittsburgh Particle-physics Astro-physics & Cosmology Center (PITT-PACC),
Department of Physics & Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA*

³*Department of Physics and Santa Cruz Institute for Particle Physics,
University of California, Santa Cruz, CA 95064, USA*

⁴*Tsung-Dao Lee Institute, and School of Physics and Astronomy,
Shanghai Jiao Tong University, Shanghai 200240, China*

⁵*Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, CA 91125 USA*

A complete, gauge-invariant computation of two loop virtual corrections involving closed fermion loops to the polarized Møller scattering asymmetry is presented. The set of contributions involving two closed fermion loops and the set involving one closed fermion loop are numerically similar in magnitude to the one-loop bosonic corrections and yield an overall correction of 1.3% relative to the tree-level asymmetry. We estimate sizes of remaining two-loop contributions and discuss implications for the upcoming MOLLER experiment.

I. INTRODUCTION

Precision measurements of electroweak processes have played a vital role in the development and testing of the Standard Model of particle physics. With the discovery of the Higgs boson at the CERN Large Hadron Collider, the focus of precision tests now falls squarely on the search for signs of physics beyond the Standard Model (BSM). While a variety of open questions clearly point to the existence of BSM physics, it remains to be determined at what mass scale this physics lives and how it interacts with the known elementary particles of the SM.

A powerful probe in this context is parity-violating electron scattering (PVES). The relevant observable in PVES experiments is the asymmetry A_{LR} in the cross-section when otherwise identical beams of longitudinally-polarized electrons of left (L) and right (R) helicities scatter from a fixed target

$$A_{LR} = \frac{d\sigma_L - d\sigma_R}{d\sigma_L + d\sigma_R}. \quad (1)$$

Historically, the measurement A_{LR} in deep-inelastic electron-deuteron scattering singled-out the Glashow-Weinberg-Salam theory [1–3] of the electroweak interaction from other alternatives and provided the first measurement of the all-important weak mixing angle, θ_W . Improved results were later obtained by a variety of PVES measurements at low energies, along with observations of parity violation in atomic Cesium and neutrino-nucleus deep-inelastic scattering. Parity-violating (PV) Møller scattering provides one of the theoretically cleanest such tests. The first measurement of this asymmetry

was made by the E158 Collaboration at SLAC in the mid-2000’s [4], yielding a confirmation of the predicted running of $\sin^2 \theta_W$ with 6σ significance.

A new, more precise measurement of the PV Møller asymmetry—dubbed MOLLER and approved to run at the Jefferson Lab [5, 6]—aims to determine A_{LR} with 2.4% uncertainty. Assuming only SM contributions, the MOLLER experiment will yield a value of $\sin^2 \theta_W$ with an uncertainty comparable to the earlier determinations in high energy e^+e^- annihilation. Within the Standard Model, this measurement can be interpreted as a precision test of the scale-dependence of $\sin^2 \theta_W$ [7–9]. Its value at $\mu = m_Z$ can be obtained either from fits to high energy electroweak precision observables, while PVES experiments yield $\sin^2 \theta_W$ at a low scale $\mu \ll m_Z$.

More significantly, MOLLER will provide a new probe for BSM physics that could reside at either high or low-mass scales. Examples include 1–10 TeV doubly-charged scalar bosons that are implied by left-right symmetric models for the non-vanishing neutrino masses[10] and a light “dark” Z boson that, under certain conditions, may also account for the observed deviation of the muon anomalous magnetic moment from SM predictions[11–13]. In both examples, the PV Møller asymmetry provides a complementary probe to other tests at low- and high-energies.

The unique potential of the PV Møller scattering follows from two features: the purely leptonic character of the process and a fortuitous suppression of the leading-order (LO) asymmetry by $1 - 4\sin^2 \theta_W$ ($\sin^2 \theta_W$ is numerically close to $1/4$). Specifically, the theoretical prediction for the PV Møller asymmetry can be written as [14]

$$A_{LR} = \frac{G_\mu Q^2}{\sqrt{2}\pi\alpha} \frac{1-y}{1+y^4+(1-y)^4} (1 - 4\sin^2 \theta_W + \Delta Q_W^e) \quad (2)$$

where $y = Q^2/s$, and ΔQ_W^e accounts for radiative corrections.

Some terms (SM and possibly BSM) entering through

*Electronic address: yongdu@umass.edu

†Electronic address: afreitas@pitt.edu

‡Electronic address: hpatel6@ucsc.edu

§Electronic address: mjrm@physics.umass.edu

ΔQ_W^e do not carry the factor $1 - 4\sin^2\theta_W$ and thus their relative impact is enhanced. Importantly, the NLO electroweak corrections, whose relative impact should be nominally $\mathcal{O}(\alpha) \sim 0.01$ are roughly 40% in magnitude[15]. These corrections are dominated by contributions from closed-fermion loops that enter the running of $\sin^2\theta_W$. The WW and γZ boxes also produce sizeable corrections. Given this enhanced NLO sensitivity, it is important to determine the magnitude of NNLO SM corrections if one wishes to interpret correctly a 2.4% measurement of A_{LR} in terms of BSM physics. Partial results at the NNLO level have been presented in Refs. [16–19]. Furthermore, second-order QED effects have been studied in the context of electron-proton scattering [20], which shares many features with electron-electron scattering.

In what follows, we report on a computation of all NNLO contributions involving closed fermion loops. This subset of the complete NNLO electroweak corrections is gauge-invariant and, thus, constitutes a well-defined contribution to the asymmetry. Since closed fermion loops dominate the NLO corrections and since they entail a sum over all colors and flavors of SM fermions, we expect them to generate the leading effect at NNLO. We find a resulting 1.3% correction to the LO asymmetry, again significantly larger than one might expect based on $\alpha/4\pi$ counting. As we discuss below, we expect the contributions from the remaining NNLO corrections to be smaller in magnitude. We thus anticipate the overall uncertainty in the SM prediction for A_{LR} lies below the planned experimental uncertainty.

II. METHOD

We calculate the left-right asymmetry by expanding eq. (1) up to two-loop order. Non vanishing contributions to A_{LR} arise from the interference of a purely electromagnetic amplitude with the PV component of the weak neutral current amplitude arising from Z -exchange, with the electromagnetic contributions dominating the denominator in eq. (1). For these building blocks, the NNLO corrections to $d\sigma$ stem from two-loop matrix elements contracted with Born amplitudes, as well as the interference of two one-loop matrix elements. The two-loop matrix elements receive contributions from genuine two-loop self-energy, vertex and box diagrams, and from one-particle reducible two-loop diagrams (see Fig. 1 for examples).

When counting the numbers of fermion loops, we do so at the level of the final observable A_{LR} . This means that we include contributions from two-loop diagrams with at least one closed fermion loop, as well as products of a one-loop diagram with fermion loop with another one-loop diagram. However, for consistency we exclude products of one-loop diagrams without closed fermion loops that could arise from interference terms obtained by expanding the denominator of Eq. (1) to two-loop order.

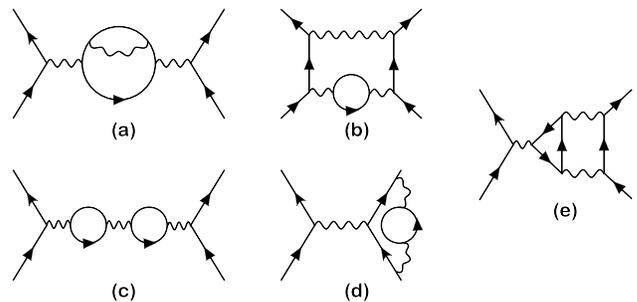


FIG. 1: Examples of two-loop Feynman diagrams with at least one closed fermion loop.

Logarithmically enhanced contributions from real photon emission cancel out in the ratio (1) and thus it is not necessary to compute them. Similarly, all infrared (IR) singularities in the virtual loop contributions also cancel. However, this cancellation is only achieved when combining all terms that contribute to A_{LR} , but individual loop diagrams are IR divergent. We use a small photon mass m_γ and electron mass m_e to regulate the soft and collinear divergencies appearing at intermediate steps, respectively.

In addition, there are ultraviolet (UV) divergences, for which we employ dimensional regularization. The UV divergencies are eliminated by appropriate renormalization conditions. We employ a renormalization scheme similar to Ref. [15], where the on-shell (OS) scheme is used for the electromagnetic coupling and the Z -boson, Higgs boson and fermion masses. For the weak mixing angle $\sin^2\theta_W$ we use the $\overline{\text{MS}}$ renormalization scheme to make contact with descriptions of $\sin^2\theta_W$ as a running parameter in the literature. Specifically, we use the $\overline{\text{MS}}$ scheme in the full SM, without any decoupled degrees of freedom, which ensures that $\sin^2\theta_W(\mu)_{\overline{\text{MS}}}$ is gauge invariant. By default, the scale choice $\mu = m_Z$ is used in the following. Expressions for the on-shell counterterms can be found in Ref. [21].

To guarantee the cancellation of UV divergencies, one must impose the relation $\sin^2\theta_W = 1 - m_W^2/m_Z^2$, where m_W and m_Z are the renormalized gauge boson masses in any given renormalization scheme (not necessarily the OS scheme). This implies that one cannot choose an independent renormalization condition for m_W , but instead the m_W counterterm is restricted to

$$\delta m_W^2 = (1 - s_W^2)\delta m_{Z,\text{OS}}^2 - m_Z^2\delta s_{W,\overline{\text{MS}}}^2, \quad (3)$$

where $s_W^2 \equiv \sin^2\theta_W(\mu)_{\overline{\text{MS}}}$. The renormalized mass, $m_{W,\text{ren}}$, defined in this fashion differs from the OS mass, $m_{W,\text{OS}}$, and an additional finite correction would be needed to relate the two. However, given that m_W is never used as an input or output in our calculation, this correction is never explicitly needed in our case.

When performing calculations in dimensional regularization, one has to be careful about the treatment of γ_5 . In $d \neq 4$ dimensions, the anticommutation

rule $\{\gamma^\mu, \gamma_5\}$ is incompatible with the trace identity $\text{tr}\{\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \gamma_5\} = -4i\varepsilon^{\alpha\beta\gamma\delta}$. Contributions from such traces arise from vertex diagrams with a triangle sub-loop, see Fig. 1 (e), and from box diagrams. However, in both of these cases, contributions stemming from ε -tensors are UV-finite (after including the sub-loop counterterms for the box graphs) and thus can be computed in 4 dimensions without ambiguity¹.

Throughout the calculation, we exploit the hierarchy of scales $m_e^2 \ll Q^2 \ll m_{\text{weak}}^2$, where $m_{\text{weak}} \sim m_W, m_Z, m_H, m_t$. In practice, this means that the mass of the external electrons is set to zero everywhere except where it is needed to regularize collinear singularities. Furthermore, we perform a large-mass expansion for $m_{\text{weak}}^2 \gg Q^2$, up to order m_{weak}^{-2} , which is the leading order needed for parity violating effects [14, 15]. This expansion is based on the method of regions [24–26], and in many cases it leads to products of one-loop integrals and two-loop vacuum integrals, which are analytically known [27–30]. The only exception are vertex and box diagrams with a light-fermion loop inside a photon or photon-Z propagator (see Fig. 1 (b)). Here “light fermion” refers to any SM fermion except the top quark.

We evaluate these two-loop integrals using the numerical dispersion integral technique [31] (see also Refs. [32, 33]). Since only the transverse part of the sub-loop self-energy $\Sigma_{\mu\nu}(k^2)$ contributes, we decompose it as

$$\Sigma_{\mu\nu} = [g_{\mu\nu}k^2 - k_\mu k_\nu] \Pi_{\text{T}}(k^2), \quad (4)$$

$$\Pi_{\text{T}}(k^2) = c_\varepsilon + \frac{k^2}{\pi} \int_0^\infty d\sigma \frac{\text{Im}\{\Pi_{\text{T}}(\sigma)\}}{\sigma(\sigma - k^2 - i0)}. \quad (5)$$

The contribution of a fermion with mass m_f is given by

$$c_\varepsilon = \frac{N_c g_1 g_2}{12\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{m_f^2} \right), \quad (6)$$

$$\begin{aligned} \text{Im}\{\Pi_{\text{T}}(\sigma)\} &= \frac{N_c g_1 g_2}{12\pi} \left(1 + \frac{2m_f^2}{\sigma} \right) \\ &\times \sqrt{1 - \frac{4m_f^2}{\sigma}} \Theta(\sigma - 4m_f^2), \end{aligned} \quad (7)$$

where $1/\varepsilon = 2/(4-d)$, $\Theta(x)$ is the Heaviside step function, and $N_c = 1$ (3) for leptons (quarks). The couplings are $g_1 g_2 = e^2 Q_f^2$ and $g_1 g_2 = \frac{e^2 Q_f (2s_W^2 Q_f - I_{3f})}{2s_W c_W}$ for the photon and photon-Z self-energy, respectively. Inserting these expressions into the outer loop leads to integrals of the form

$$\begin{aligned} &\int \frac{d^d k}{i\pi^{d/2}} \frac{N(k)}{\prod_i [(k+p_i)^2 - m_i^2 + i0]} \\ &\times \left[c_\varepsilon - \frac{1}{\pi} \int \frac{d\sigma}{\sigma} \text{Im}\{\Pi_{\text{T}}(\sigma)\} \frac{k^2}{k^2 - \sigma + i0} \right]. \end{aligned} \quad (8)$$

Here p_i are sets of external momenta, as they appear in a given vertex or box diagram, and $N(k)$ accounts for dot products $(k^2, k \cdot p_i)$ and \not{k} in the numerator. The k -integral in Eq. (8) is a conventional one-loop integral, which can be performed analytically and reduced to basic scalar one-loop functions using the standard Passarino-Veltman method. The remaining σ integral, which is UV-finite, is easily evaluated numerically with high precision. It is interesting to note that the σ integrals involving $\log m_\gamma^2$ and $\log m_e^2$ may be performed analytically so that the cancellation of IR singularities in the full result can be checked algebraically.

These dispersion integrals are not well-defined for light quarks ($f = u, d, s$) in the inner loop, since the dominant contribution to the integral arises from region where $k^2 \sim m_f^2$, where hadronization effects become important. In fact, the same problem already occurs at the one-loop level for the self-energy contribution to the γ -Z self-energy in the t - and u -channel [15, 34], due to the fact that $Q^2 < \Lambda_{\text{QCD}}^2$.

The non-perturbative hadronic corrections can be approximately accounted for by using dressed quark masses. A well-motivated quark mass definition for this purpose are the threshold masses derived in Refs. [8, 9]. In our calculation we use these quark masses in all places where mass-dependent terms remain after expanding in large m_{weak}^2 . However, for consistency, we exclude two-loop self-energy diagrams involving only quark and photon propagators in the loops, such as Fig. 1 (a) with a photon inside the loop, since QED effects are already subsumed in the non-perturbative hadron dynamics. In addition, following Ref. [15], we also set $Q^2 \rightarrow 0$ in the t - and u -channel self-energies, since the differences $\Pi_{\text{T}}^{\gamma\gamma}(t) - \Pi_{\text{T}}^{\gamma\gamma}(0)$ and $\Pi_{\text{T}}^{\gamma Z}(t) - \Pi_{\text{T}}^{\gamma Z}(0)$ are estimated to be negligibly small [15] (similar for t replaced by u). We leave a more detailed study of hadronic effects for future work.

As shown in Eq. (2), A_{LR} is commonly normalized in terms of the Fermi constant G_μ , which is related to SM parameters according to

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi\alpha}{2s_W^2 c_W^2 m_Z^2} (1 + \Delta r), \quad (9)$$

where Δr includes radiative corrections. The required two-loop contributions to Δr with one or two closed fermion loops have been taken from Refs. [21, 35] (see also Ref. [36]).

The calculation has been carried out with extensive use of computer algebra tools. Diagrams and amplitudes were generated with FEYNARTS [37]. For the Lorentz and Dirac algebra, we employed PACKAGE-X [38] and cross-checked against private code written in MATHEMATICA. The large-mass expansion was implemented in-house in two independently developed MATHEMATICA programs. Two-loop integrals with non-trivial numerator structures have been reduced to simple scalar integrals using FIRE 5 [39] and using private code based on Ref. [40, 41]. For basic one-loop integrals and two-

¹ A similar argument holds for a set of useful identities for 4-fermion scattering matrix elements [22, 23].

loop vacuum integrals, analytical formulae are available [27–30]. We have numerically checked our implementation of the one-loop formulae against the COLLIER library [42]. The numerical dispersion integrals for two-loop vertex and box integrals have been implemented in C and MATHEMATICA.

Each building block of the final result has been computed in two independent setups within our collaboration and cross-checked against each other. We have confirmed cancellation of UV and IR divergencies in the full result by verifying that the coefficients of the $1/\varepsilon$, $\log m_e^2$ and $\log m_\gamma^2$ terms vanish algebraically. Furthermore, as an intermediate step, we have reproduced the one-loop result of Ref. [15] and found exact agreement with the analytical formulae given there.

III. RESULTS

To evaluate the numerical impact of the closed fermion-loop NNLO corrections to A_{LR} , we used the following input parameters:

$$\begin{aligned}
m_Z &= 91.1876 \text{ GeV}, & s_W^2 &= 0.2314, \\
m_H &= 125.1 \text{ GeV}, & m_t &= 173.0 \text{ GeV}, \\
m_\tau &= 1.777 \text{ GeV}, & m_b &= 3.99 \text{ GeV}, \\
m_\mu &= 105.7 \text{ MeV}, & m_c &= 1.185 \text{ GeV}, \\
m_e &= 0.511 \text{ MeV}, & m_s &= 0.342_{-0.053}^{+0.048} \text{ GeV}, \\
& & m_{u,d} &= 0.246_{-0.057}^{+0.054} \text{ GeV}, \\
\Delta\alpha &= 0.02761_{\text{had.}} + 0.0314976_{\text{lep.}}, & & (10)
\end{aligned}$$

at the representative kinematic point

$$\sqrt{s} = 11 \text{ MeV}, \quad y = 0.4. \quad (11)$$

Here $\Delta\alpha$ accounts for the renormalization group running of the fine structure constant between scales $\mu = 0$ and $\mu = m_Z$, and enters our calculation through the OS charge renormalization. The first number reflects the hadronic contribution to $\Delta\alpha$, which is obtained from $e^+e^- \rightarrow \text{hadrons}$ data (see Refs. [43–45] for recent evaluations), while the second number is the perturbatively calculable leptonic contribution [46?].

As explained above, the light fermion masses m_f , $f \neq t$ enter in loop integrals with a fermionic photon or γ - Z self-energy subloop. The values for the light quark masses are taken from Ref. [9]. There is a strong anti-correlation between the reported uncertainties of m_s and $m_{u,d}$. We will assume them to be 100% anti-correlated for the results which we present below.

With these inputs we obtain numerical results for the asymmetry (2) as shown in Table I. The first row corresponds to the tree-level contribution, and the remaining rows $\Delta Q_{W(L,n_f)}^e$ are the radiative corrections with L loops and n_f closed fermion loops. No resummation of logarithms has been carried out. In particular, the electroweak logarithms, which conventionally define the

Quantity	Contribution ($\times 10^{-3}$)
$1 - 4 \sin^2 \theta_W$	+74.4
$\Delta Q_{W(1,1)}^e$	-29.0
$\Delta Q_{W(1,0)}^e$	+ 3.1
$\Delta Q_{W(2,2)}^e$	- 0.18 $_{-0.0040}^{+0.0024}$
$\Delta Q_{W(2,1)}^e$	+ 1.18 $_{-0.010}^{+0.015}$
$\Delta Q_{W(2,0)}^e$	± 0.13 (estimate)

TABLE I: Numerical estimates of the calculated contributions to the polarized Møller scattering asymmetry defined in (2) through NNLO using input values in (10) and (11). Subscripted indices on $\Delta Q_{W(L,n_f)}^e$ refer to the loop order L and number of closed loops n_f .

running $\sin^2 \theta_W$, are left explicitly in the one and two loop results. The last two rows $\Delta Q_{W(2,2)}^e$ and $\Delta Q_{W(2,1)}^e$ are obtained using our newly computed NNLO corrections to the asymmetry. The error intervals reflect the hadronic uncertainties due to the threshold quark masses in Eq. (10).

The precision goal for the MOLLER experiment corresponds to a measurement of the weak charge with an uncertainty of $\delta_{\text{exp}} Q_W^e = 1.1 \times 10^{-3}$. The NNLO corrections with closed fermion loops add up to

$$\Delta Q_{W(2,2)}^e + \Delta Q_{W(2,1)}^e = 1.00_{-0.008}^{+0.012} \times 10^{-3}, \quad (12)$$

which is comparable to the experimental target, thus highlighting the importance of accounting for the NNLO corrections. On the other hand, anti-correlation between the hadronic uncertainties in the individual contributions with one and two closed fermion loops leads to a reduced overall hadronic uncertainty.

The resulting hadronic uncertainty from quark loops is negligible compared to the experimental target precision. It is likely that the our estimate based on quark mass errors overestimates this uncertainty, since we cannot account for correlations between the quark masses and the K factors in Ref. [9]². A more detailed analysis of hadronic effects will be given in a future publication.

The correction ΔQ_W^e depends very mildly on y (i.e. on the scattering angle). Varying y in the experimentally relevant range (0.25, 0.75) [6], we find that ΔQ_W^e changes by 0.04×10^{-3} for the NLO corrections, and by 0.01×10^{-3} for the NNLO corrections, both of which are negligible.

Finally, we attempt to estimate size of the currently missing NNLO corrections without closed fermion loops $\Delta Q_{W(2,0)}^e$ (called “bosonic” corrections in the following). For this purpose, we begin by comparing the relative size of the fermion-loop and bosonic correction at

² In fact, when estimating the leading hadronic effects by plugging these quark masses into the NLO correction, one finds an uncertainty that is a factor few larger than the detailed renormalization-group evaluation in Ref. [9].

NLO. From Table I, these are $\Delta Q_{W(1,1)}^e = -0.0290$ and $\Delta Q_{W(1,0)}^e = +0.0031$, respectively. Assuming a similar ratio between the corrections with one closed fermion loop and the bosonic corrections at NNLO, we obtain an estimate of 0.13×10^{-3} for the size of the latter. This would be safely below the experimental target precision.

IV. CONCLUSIONS

To correctly interpret the proposed 2.4% measurement of the parity-violating asymmetry A_{LR} from the MOLLER experiment at the Jefferson Lab in terms of BSM physics, we calculate the NNLO SM contributions to A_{LR} using large-mass expansion and numerical integration of sub-loop dispersion relation. We summarize our results in Table I. We find that the corrections to ΔQ_W^e from diagrams with closed fermion loops are comparable to the experimental target precision. The dependence of ΔQ_W^e on the scattering angle is very mild in the experimentally relevant range and can be ignored for

most practical purposes. Finally, we also consider the impact of the remaining bosonic NNLO corrections and estimate them to be negligible compared to the MOLLER precision goal. However, it is desirable to confirm this with an explicit calculation in the future.

Acknowledgments

The authors would like to thank V.A. Smirnov and J. Erler for useful private communications. YD is grateful to K. Kumar for his financial support at Fermilab and to PITT PACC at the University of Pittsburgh for their hospitality during part of this work. A.F. has been supported in part by the National Science Foundation under grant no. PHY-1820760. Y.D., H.H.P. and MJRM were supported in part under U.S. Department of Energy contract DE-SC0011095. H.H.P. was additionally supported in part by U.S. Department of Energy grant number de-sc0010107.

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