Supplemental material

Gauge-invariant effective potential to two loops

In this supplemental material, we collect details of the perturbative calculation that was used for comparison with the nonperturbative results in the main text. The goal is to compute thermal corrections to the effective potential $V_{\rm eff}$ and extract from it values for T_c, latent heat and the condensates, and we do this at two-loop level. The perturbative expansion of $V_{\rm eff}$ in terms of quartic couplings has a peculiar structure at finite temperature, with fractional powers such as $\lambda^{3/2}$ appearing as a consequence of Debye screening. A consistent inclusion of these effects requires daisy resummation in the high-T approximation [42], but as discussed in the main text, it is easier to work directly in the 3d EFT where these resummations are incorporated automatically (Eq. (2) in the main text). We take this approach, generalizing the calculation of [46] to include a background field for the triplet.

Parameters of the EFT are related to those in the full theory by matching relations presented in [36]. Here we simplify the notation by dropping the overline from the EFT parameters. In what follows, all parameters are assumed to be those of the 3d theory and therefore temperature dependent. For completeness we also include the $U(1)_Y$ hypercharge field B_i, so the covariant derivatives read

$$D_i\phi = (\partial_i - \frac{1}{2}ig\sigma_a A_i^a - \frac{1}{2}ig' B_i)\phi, \quad D_i\Sigma^a = \left(\partial_i\Sigma^a + ig\epsilon^{abc}A_i^b\Sigma^c\right). \tag{1}$$

For comparison with the nonperturbative results we have set $g' = 0$, as the $U(1)_Y$ field is not included in our lattice simulations.

We parametrize the scalars as

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ v + \phi_3 + i\phi_4\end{pmatrix}, \quad \vec{\Sigma} = \begin{pmatrix}\Sigma_1 \\ \Sigma_2 \\ x + \Sigma_3\end{pmatrix}, \tag{2}$$

where v and x are real background fields. The Euclidean Lagrangian becomes

$$\mathcal{L}_{\rm 3d} = V_{\rm tree}(v,x) + \mathcal{L}^{(2)}_{\rm 3d} + \mathcal{L}^{(I)}_{\rm 3d}, \tag{3}$$

$$V_{\rm tree}(v,x) = \frac{1}{2}\mu_\phi^2 v^2 + \frac{1}{2}\mu_\Sigma^2 x^2 + \frac{1}{4}\lambda v^4 + \frac{1}{4}b_4 x^4 + \frac{1}{4}a_2 v^2 x^2. \tag{4}$$

Here $\mathcal{L}^{(2)}_{\rm 3d}$ and $\mathcal{L}^{(I)}_{\rm 3d}$ contain quadratic and interaction terms respectively. Terms linear in ϕ_i or Σ_i do not contribute to the effective potential, which is defined (at a finite volume $\mathcal{V}$) through

$$\begin{aligned}\exp\left[-\frac{\mathcal{V}}{\hbar}V_{\rm eff}(v,x)\right] &= \int D\phi\ \exp\left[-\frac{S_{\rm 3d}}{\hbar}\right] \\ &= \exp\left[-\frac{\mathcal{V}}{\hbar}V_{\rm tree}(v,x)\right]\int D\phi\ \exp\left[-\frac{1}{\hbar}\int d^3x\ \mathcal{L}^{(2)}_{\rm 3d}\right]\left\langle \exp\left[-\frac{1}{\hbar}\int d^3x\ \mathcal{L}^{(I)}_{\rm 3d}\right]\right\rangle. \end{aligned}\tag{5}$$

The symbolic measure $D\phi$ denotes functional integration over all dynamical fields, and the expectation value is to be calculated perturbatively.

As discussed in [44], the value of $V_{\rm eff}$ in its minimum is guaranteed, by Nielsen identities, to be gauge invariant order-by-order in the loop-counting parameter $\hbar$. Expanding the potential and its minima as

$$V_{\rm eff} = V_0 + \hbar V_1 + \hbar^2 V_2, \quad v_{\rm min} = v_0 + \hbar v_1 + \hbar^2 v_2, \quad x_{\rm min} = x_0 + \hbar x_1 + \hbar^2 x_2 \tag{6}$$

and generalizing the analysis of [43,44] to the case of two background fields gives the "$\hbar$ expansion"

$$V_{\rm eff}(v_{\rm min}, x_{\rm min}) = V_0(v_0,x_0) + \hbar V_1(v_0,x_0) + \hbar^2\left[V_2(v_0,x_0) - \frac{1}{2}v_1^2\frac{\partial^2 V_0}{\partial v^2} - \frac{1}{2}x_1^2\frac{\partial^2 V_0}{\partial x^2} - v_1 x_1\frac{\partial^2 V_0}{\partial v\partial x}\right] + \mathcal{O}(\hbar^3), \tag{7}$$

$$v_1 = \left[\left(\frac{\partial^2 V_0}{\partial v\partial x}\right)^2 - \left(\frac{\partial^2 V_0}{\partial v^2}\right)\left(\frac{\partial^2 V_0}{\partial x^2}\right)\right]^{-1}\left[\left(\frac{\partial^2 V_0}{\partial x^2}\right)\left(\frac{\partial V_1}{\partial v}\right) - \left(\frac{\partial^2 V_0}{\partial v\partial x}\right)\left(\frac{\partial V_1}{\partial x}\right)\right], \tag{8}$$

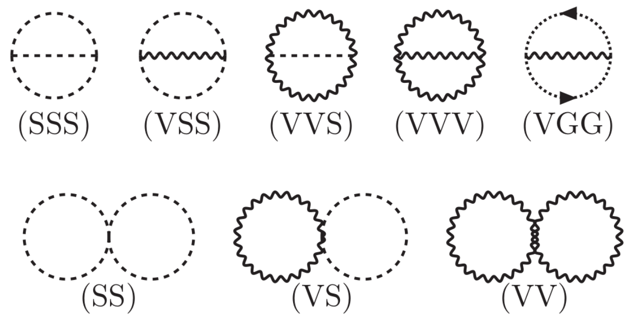

Figure 4: Diagram topologies that enter the calculation of two-loop effective potential. Dashed lines denote scalars (S), wavy lines denote vector bosons (V) and dotted lines refer to ghost fields (G).

$$x_1 = \left[\left(\frac{\partial^2 V_0}{\partial v \partial x} \right)^2 - \left(\frac{\partial^2 V_0}{\partial v^2} \right) \left(\frac{\partial^2 V_0}{\partial x^2} \right) \right]^{-1} \left[\left(\frac{\partial^2 V_0}{\partial v^2} \right) \left(\frac{\partial V_1}{\partial x} \right) - \left(\frac{\partial^2 V_0}{\partial v \partial x} \right) \left(\frac{\partial V_1}{\partial v} \right) \right]. \tag{9}$$

All derivatives are to be evaluated at the tree-level minimum (v_0, x_0). Note that corrections to the VEVs contribute only at $\mathcal{O}(\hbar^2)$. This form of $V_{\text{eff}}(v_{\min}, x_{\min})$ is gauge invariant, and we shall calculate it in Landau gauge $\xi = 0$. With this choice, ghost fields remain massless after symmetry breaking and decouple from Goldstone modes.

From Eq. (5) we obtain $V_0(v,x) = V_{\text{tree}}(v,x)$, while the $\mathcal{O}(\hbar)$ part can be calculated by diagonalizing the quadratic Lagrangian in momentum space. In the $\mathcal{V} \to \infty$ limit, the result is the familiar Coleman-Weinberg correction in $d = 3 - 2\epsilon$ Euclidean dimensions:

$$V_1(v,x) = 2(d-1) J\left(m_W^2\right) + (d-1) J\left(m_Z^2\right) + 3J\left(m_1^2\right) + 2J\left(m_2^2\right) + J\left(m_+^2\right) + J\left(m_-^2\right). \tag{10}$$

Here the integral

$$J(m^2) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln\left(p^2 + m^2\right) = -\frac{(m^2)^{3/2}}{12\pi} + \mathcal{O}(\epsilon) \tag{11}$$

is finite in 3d, and the field-dependent masses read

$$\begin{aligned} m_W^2 &= \frac{1}{4} g^2 v^2 + g^2 x^2, \quad m_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2, \\ m_1^2 &= \mu_\phi^2 + \lambda v^2 + \frac{1}{2} a_2 x^2, \quad m_2^2 = \mu_\Sigma^2 + b_4 x^2 + \frac{1}{2} a_2 v^2, \\ m_\pm^2 &= \frac{1}{2} \left(m_3^2 + m_4^2 \pm \sqrt{(m_3^2 - m_4^2)^2 + 4a_2^2 v^2 x^2} \right) \end{aligned} \tag{12}$$

with

$$m_3^2 = \mu_\phi^2 + 3\lambda v^2 + \frac{1}{2} a_2 x^2, \quad m_4^2 = \mu_\Sigma^2 + 3b_4 x^2 + \frac{1}{2} a_2 v^2. \tag{13}$$

The $\mathcal{O}(\hbar^2)$ correction consists of the 2-loop potential evaluated at a tree-level minimum, as well as 1-loop corrections to locations of the minima. The latter is obtained from Eqs. (8)-(10), while the former requires computation of one-particle-irreducible vacuum diagrams depicted in Fig. 4. Including the minus sign from $\exp\left[-\frac{1}{\hbar}\int d^3x\, \mathcal{L}_{3\text{d}}^{(I)}\right]$ in the diagrammatic vertex rules, V_2 is given by minus the sum of diagrams in Fig. 4.

The calculation of $V_2(v,x)$ at general field values is somewhat complicated as one needs to introduce a field-dependent mixing angle for the neutral scalars. A simpler approach is to perform the computation directly at a tree-level minimum (v_0, x_0), which is all that is needed for the $\mathcal{O}(\hbar^2)$ correction. In the case of ΣSM, there is then no mixing between the mass eigenstates of ϕ and Σ as guaranteed by the Z_2 symmetry. Consequently, the masses $m_\pm^2$ in Eq. (12) reduce to m_3^2 and m_4^2. Below we present results for the different diagram topologies at two loops, expressed in terms of master integrals, but emphasize that these results are not applicable if v_0 and x_0 are simultaneously non-vanishing. Contributions from the Σ field are collected in curly brackets.

$$\begin{aligned}(\text{SSS}) &= 3\lambda^2 v_0^2 \mathcal{D}_{SSS}(m_3,m_1,m_1) + 3\lambda^2 v_0^2 \mathcal{D}_{SSS}(m_3,m_3,m_3) \\ &+ \bigg\{ \frac{3}{4} a_2^2 x_0^2 \mathcal{D}_{SSS}(m_1,m_1,m_4) + \frac{1}{2} a_2^2 v_0^2 \mathcal{D}_{SSS}(m_2,m_2,m_3) + \frac{1}{4} a_2^2 x_0^2 \mathcal{D}_{SSS}(m_3,m_3,m_4) \\ &+ \frac{1}{4} a_2^2 v_0^2 \mathcal{D}_{SSS}(m_4,m_4,m_3) + 2b_4^2 x_0^2 \mathcal{D}_{SSS}(m_2,m_2,m_4) + 3b_4^2 x_0^2 \mathcal{D}_{SSS}(m_4,m_4,m_4) \bigg\}_{\Sigma SM}, \end{aligned} \tag{14}$$

$$\begin{aligned}(\text{VSS}) &= \frac{1}{4} g^2 \mathcal{D}_{VSS}(m_1,m_1,m_W) + \frac{1}{4} g^2 \mathcal{D}_{VSS}(m_3,m_1,m_W) + \frac{1}{8}(g^2+g'^2)\mathcal{D}_{VSS}(m_3,m_1,m_Z) \\ &+ \frac{1}{8}\frac{(g^2-g'^2)^2}{g^2+g'^2}\mathcal{D}_{VSS}(m_1,m_1,m_Z) + \frac{1}{2}\frac{g^2 g'^2}{g^2+g'^2}\mathcal{D}_{VSS}(m_1,m_1,0) \\ &+ \bigg\{ g^2 \mathcal{D}_{VSS}(m_4,m_2,m_W) + \frac{1}{2}\frac{g^4}{g^2+g'^2}\mathcal{D}_{VSS}(m_2,m_2,m_Z) \\ &+ \frac{1}{2}\frac{g^2 g'^2}{g^2+g'^2}\mathcal{D}_{VSS}(m_2,m_2,0) \bigg\}_{\Sigma SM}, \end{aligned} \tag{15}$$

$$\begin{aligned}(\text{VVS}) &= \frac{1}{8} g^4 v_0^2 \mathcal{D}_{VVS}(m_3,m_W,m_W) + \frac{1}{16}(g^2+g'^2)^2 v_0^2 \mathcal{D}_{VVS}(m_3,m_Z,m_Z) \\ &+ \frac{1}{4}\frac{g^4 g'^2 v_0^2}{g^2+g'^2}\mathcal{D}_{VVS}(m_1,m_W,0) + \frac{1}{4}\frac{g^2 g'^4 v_0^2}{g^2+g'^2}\mathcal{D}_{VVS}(m_1,m_W,m_Z) \\ &+ \bigg\{ 2g^4 x_0^2 \mathcal{D}_{VVS}(m_4,m_W,m_W) + \frac{g^6 x_0^2}{g^2+g'^2}\mathcal{D}_{VVS}(m_2,m_W,m_Z) \\ &+ \frac{g^4 g'^2 x_0^2}{g^2+g'^2}\mathcal{D}_{VVS}(m_2,m_W,0) \bigg\}_{\Sigma SM}, \end{aligned} \tag{16}$$

$$(\text{VVV}) = \frac{1}{2}\frac{g^4}{g^2+g'^2}\mathcal{D}_{VVV}(m_W,m_W,m_Z) + \frac{1}{2}\frac{g^2 g'^2}{g^2+g'^2}\mathcal{D}_{VVV}(m_W,m_W,0), \tag{17}$$

$$(\text{VGG}) = -2g^2 \mathcal{D}_{VGG}(m_W) - \frac{g^4}{g^2+g'^2}\mathcal{D}_{VGG}(m_Z), \tag{18}$$

$$\begin{aligned}(\text{SS}) &= -\frac{15}{4}\lambda\Big(I_1^3(m_1)\Big)^2 - \frac{3}{2}\lambda I_1(m_1)I_1(m_3) - \frac{3}{4}\lambda\Big(I_1(m_3)\Big)^2 \\ &+ \bigg\{ -\frac{3}{2}a_2 I_1(m_1)I_1(m_2) - 2b_4\Big(I_1(m_2)\Big)^2 - \frac{1}{2}a_2 I_1(m_2)I_1(m_3) - \frac{3}{4}a_2 I_1(m_1)I_1(m_4) \\ &- b_4 I_1(m_2)I_1(m_4) - \frac{1}{4}a_2 I_1(m_3)I_1(m_4) - \frac{3}{4}b_4\Big(I_1(m_4)\Big)^2 \bigg\}_{\Sigma SM}, \end{aligned} \tag{19}$$

$$\begin{aligned}(\text{VS}) &= -\frac{3}{4}(d-1)g^2 I_1(m_1)I_1(m_W) - \frac{1}{4}(d-1)\frac{(g^2-g'^2)^2}{g^2+g'^2}I_1(m_1)I_1(m_Z) \\ &- \frac{1}{4}(d-1)g^2 I_1(m_3)I_1(m_W) - \frac{1}{8}(d-1)(g^2+g'^2)I_1(m_1)I_1(m_Z) \\ &- \frac{1}{8}(d-1)(g^2+g'^2)I_1(m_3)I_1(m_Z) \end{aligned}$$

$$+\left\{-(d-1)g^2 I_1(m_2)I_1(m_W)-(d-1)g^2 I_1(m_4)I_1(m_W)\right.$$
$$\left.-(d-1)\frac{g^4}{g^2+g'^2}I_1(m_2)I_1(m_Z)\right\}_{\Sigma SM}, \tag{20}$$

$$(\mathrm{VV})=-\frac{1}{2}g^2\mathcal{D}_{VV}(m_W,m_W)-\frac{g^4}{g^2+g'^2}\mathcal{D}_{VV}(m_W,m_Z). \tag{21}$$

The loop integrals are defined, in dimensional regularization with $\overline{\mathrm{MS}}$ scale Λ, as

$$\int_p \equiv \Big(\frac{e^\gamma\Lambda^2}{4\pi}\Big)^\epsilon \int \frac{d^dp}{(2\pi)^d}, \tag{22}$$

$$I_\alpha(m)\equiv\int_p \frac{1}{(p^2+m^2)^\alpha}=\Big(\frac{e^\gamma\Lambda^2}{4\pi}\Big)^\epsilon\frac{(m^2)^{\frac{d}{2}-\alpha}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}, \tag{23}$$

$$\mathcal{D}_{VV}(m_1,m_2)\equiv\int_{p,k}\frac{\delta_{ir}\delta_{js}+\delta_{ij}\delta_{rs}-2\delta_{is}\delta_{jr}}{(p^2+m_1^2)(k^2+m_2^2)}\Big(\delta_{ij}-\frac{p_ip_j}{p^2}\Big)\Big(\delta_{rs}-\frac{k_rk_s}{k^2}\Big)$$
$$=\frac{(d-1)^3}{d}I_1(m_1)I_1(m_2), \tag{24}$$

$$\mathcal{D}_{SSS}(m_1,m_2,m_3)\equiv\int_{p,k}\frac{1}{(p^2+m_1^2)(k^2+m_2^2)((p+k)^2+m_3^2)}$$
$$=\frac{1}{16\pi^2}\bigg(\frac{1}{4\epsilon}+\frac{1}{2}+\ln\Big(\frac{\Lambda}{m_1+m_2+m_3}\Big)\bigg)+\mathcal{O}(\epsilon), \tag{25}$$

$$\mathcal{D}_{VGG}(m)\equiv\int_{p,k}\frac{k_i(p+k)_j}{(p^2+m^2)(p+k)^2k^2}\Big(\delta_{ij}-\frac{p_ip_j}{p^2}\Big)=\frac{1}{4}m^2\mathcal{D}_{SSS}(m,0,0), \tag{26}$$

$$\mathcal{D}_{VSS}(m_1,m_2,m_3)\equiv\int_{p,k}\frac{(2p+k)_i(2p+k)_j}{(p^2+m_1^2)(k^2+m_3^2)((p+k)^2+m_2^2)}\Big(\delta_{ij}-\frac{k_ik_j}{k^2}\Big), \tag{27}$$

$$\mathcal{D}_{VVS}(m_1,m_2,m_3)\equiv\int_{p,k}\frac{\delta_{ik}\delta_{jl}}{(p^2+m_2^2)(k^2+m_3^2)((p+k)^2+m_1^2)}\Big(\delta_{ij}-\frac{p_ip_j}{p^2}\Big)\Big(\delta_{kl}-\frac{k_kk_l}{k^2}\Big), \tag{28}$$

$$\mathcal{D}_{VVV}(m_1,m_2,m_3)\equiv\int_{p,k}\frac{1}{(p^2+m_1^2)(k^2+m_2^2)((p+k)^2+m_3^2)}\Big(\delta_{ij}-\frac{p_ip_j}{p^2}\Big)\Big(\delta_{kl}-\frac{k_kk_l}{k^2}\Big)\Big(\delta_{rs}-\frac{(p+k)_r(p+k)_s}{(p+k)^2}\Big)$$
$$\times\Big((2k+p)_i\delta_{rk}-(2k+p)_r\delta_{ik}+(k-p)_k\delta_{ir}\Big)\Big((k-p)_l\delta_{sj}-(2k+p)_s\delta_{lj}+(2p+k)_j\delta_{ls}\Big). \tag{29}$$

Some special cases of the vector "sunset" integrals have been calculated previously in [46]. In the presence of the hypercharge gauge field, the following generalizations are needed.

$$\mathcal{D}_{VSS}(m_1,m_2,m_3)=\frac{1}{m_3^2}\bigg((-m_1^2+m_2^2+m_3^2)I_1(m_2)I_1(m_3)$$
$$+\Big(-m_3^2I_1(m_2)+(m_1^2-m_2^2+m_3^2)I_1(m_3)\Big)I_1(m_1)-(m_1^2-m_2^2)^2\mathcal{D}_{SSS}(m_1,m_2,0)$$
$$+(m_1-m_2-m_3)(m_1+m_2-m_3)(m_1-m_2+m_3)(m_1+m_2+m_3)\mathcal{D}_{SSS}(m_1,m_2,m_3)\bigg), \tag{30}$$

$$\mathcal{D}_{VSS}(m_1,m_2,0) = -(d-1)\Big((m_1^2+m_2^2)\mathcal{D}_{SSS}(m_1,m_2,0) + I_1(m_1)I_1(m_2)\Big), \tag{31}$$

$$\begin{aligned}\mathcal{D}_{VVS}(m_1,m_2,m_3) = \frac{1}{4m_2^2m_3^2}\Big(&-m_3^2I_1(m_1)I_1(m_2) + \Big(-m_2^2I_1(m_1)\\ &+(-m_1^2+m_2^2+m_3^2)I_1(m_2)\Big)I_1(m_3) + m_1^4\mathcal{D}_{SSS}(m_1,0,0)\\ &-(m_2^2-m_1^2)^2\mathcal{D}_{SSS}(m_1,m_2,0) - (m_3^2-m_1^2)^2\mathcal{D}_{SSS}(m_1,m_3,0)\\ &+\Big((m_2^2-m_1^2)^2 + [-2m_1^2+(4d-6)m_2^2]m_3^2 + m_3^4\Big)\mathcal{D}_{SSS}(m_1,m_2,m_3)\Big),\end{aligned} \tag{32}$$

$$\mathcal{D}_{VVS}(m_1,m_2,0) = -\frac{d-1}{4m_2^2}\Big((m_1^2-3m_2^2)\mathcal{D}_{SSS}(m_1,m_2,0) - m_1^2\mathcal{D}_{SSS}(m_1,0,0) + I_1(m_1)I_1(m_2)\Big), \tag{33}$$

$$\mathcal{D}_{VVS}(m,0,0) = \frac{d(d-1)}{4}\mathcal{D}_{SSS}(m,0,0), \tag{34}$$

$$\mathcal{D}_{VVS}(0,m,0) = \frac{3d(d-1)}{4}\mathcal{D}_{SSS}(m,0,0), \tag{35}$$

$$\begin{aligned}\mathcal{D}_{VVV}(m_1,m_1,m_2) = &-\frac{dm_1^4-(5d-4)m_1^2m_2^2-d(4d-7)m_2^4}{2dm_1^2m_2^2}I_1(m_1)I_1(m_2)\\ &+\frac{4(3d^2-4d-1)m_1^4-2d(4d-7)m_1^2m_2^2-dm_2^4}{4dm_1^4}\Big(I_1(m_1)\Big)^2\\ &-\frac{(m_1^2-m_2^2)^2\Big(m_1^4+2(2d-3)m_1^2m_2^2+m_2^4\Big)}{2m_1^4m_2^2}\mathcal{D}_{SSS}(m_1,m_2,0)\\ &-\frac{(4m_1^2-m_2^2)\Big(4(d-1)m_1^4+4(2d-3)m_1^2m_2^2+m_2^4\Big)}{4m_1^4}\mathcal{D}_{SSS}(m_1,m_1,m_2)\\ &+\frac{m_2^6}{4m_1^4}\mathcal{D}_{SSS}(m_2,0,0)+\frac{m_1^4}{2m_2^2}\mathcal{D}_{SSS}(m_1,0,0),\end{aligned} \tag{36}$$

$$\mathcal{D}_{VVV}(m,m,0) = \frac{5d^3-19d^2+15d+3}{(d-3)d}\Big(I_1(m)\Big)^2 - \frac{(3d-5)}{2}m^2\mathcal{D}_{SSS}(m,0,0). \tag{37}$$

Many of the expressions above utilize integration-by-part techniques developed in [47,48] (for thermal sum-integrals, see [49]). We are grateful to Philipp Schicho for providing particularly simple expressions for the special cases of D_{VSS}, D_{VVS} and D_{VVV}.

The two-loop diagrams are UV divergent, but are regulated (apart from the vacuum divergence) by mass counterterms in the tree-level part (4). These are given by

$$\delta\mu_\phi^2 = -\frac{1}{16\pi^2}\frac{1}{4\epsilon}\left(\frac{39}{16}g^4 - \frac{5}{16}g'^4 - \frac{9}{8}g^2g'^2 + 3\lambda(3g^2+g'^2) - 12\lambda^2 - \frac{3}{2}a_2^2 + 6a_2g^2\right) \tag{38}$$

$$\delta\mu_\Sigma^2 = -\frac{1}{16\pi^2}\frac{1}{4\epsilon}\Big(-g^4 + a_2(3g^2+g'^2) + 20b_4g^2 - 2a_2^2 - 10b_4^2\Big), \tag{39}$$

which were also obtained independently in Ref. [36]. Due to super-renormalizability, there are no further corrections to the counterterms at higher loop orders. Apart from contributions from the triplet and the hypercharge field, the two-loop expressions in Eqs. (14)-(21) agree with those given in [46] for an SU(2)-Higgs theory.

To study the phase structure, we evaluate $V_{\rm eff}(v_0,x_0)$ separately in the three phases,

$$V_{\rm eff}^{\rm symm}(T) = V_{\rm eff}(0,0), \qquad V_{\rm eff}^{\phi}(T) = V_{\rm eff}(\sqrt{-\mu_\phi^2/\lambda},0), \qquad V_{\rm eff}^{\Sigma}(T) = V_{\rm eff}(0,\sqrt{-\mu_\Sigma^2/b_4}), \tag{40}$$

and varying the temperature (which is now encapsuled in the 3d parameters). Not all of the above minima exist simultaneously at a given temperature; this needs to be checked separately. The condition for T_c is that the value of $V_{\rm eff}$ in any two minima is degenerate, *e.g.* $V_{\rm eff}^{\Sigma}(T_c) = V_{\rm eff}^{\phi}(T_c)$ for $\Sigma \to \phi$ transitions. Latent heat is calculated from the 3d potential (which has units GeV^3) as

$$L = -T^2 \left[\left(\frac{\partial V_{\rm eff}}{\partial T} \right)_{\text{high-T phase}} - \left(\frac{\partial V_{\rm eff}}{\partial T} \right)_{\text{low-T phase}} \right]. \tag{41}$$

Finally, the scalar condensates are given by [62]

$$\langle \phi^\dagger \phi \rangle = \frac{\partial V_{\rm eff}}{\partial \mu_\phi^2}, \qquad \frac{1}{2} \langle \Sigma^a \Sigma^a \rangle = \frac{\partial V_{\rm eff}}{\partial \mu_\Sigma^2}. \tag{42}$$

As discussed in the main text and in Refs. [43,50], the two-loop potential constructed here is not useful for studying thermodynamic properties near the critical temperature for transitions out of the symmetric phase $(v_0, x_0) = (0, 0)$. The issue lies in Eq. (6), which assumes that the true minimum is related to the tree-level one through small perturbations. This assumption breaks down at temperatures close to $O \to \phi$ or $O \to \Sigma$ transitions, for which the tree-level condition for T_c is that the thermally-corrected mass parameter vanishes, $\mu_\phi^2(T_c) = 0$ or $\mu_\Sigma^2(T_c) = 0$. Given that the high-T expansion parameter is $\sim g^2 T \times (\text{mass scale})^{-1}$, perturbation theory is unreliable near T_c. In particular, there is an explicit divergence at two-loop order due to the vanishing scalar mass [43]. This problem does not arise for transitions between two broken phases $(\Sigma \to \phi)$ as the tree-level masses need not vanish for such transitions.

One may hope to regulate the problem by giving up on the expansion of $(v_{\rm min}, x_{\rm min})$ altogether and solve for the minimum of $V_{\rm eff}$ "exactly", as is frequently done in the literature. This automatically incorporates higher-order corrections of the VEVs into the potential. The downside is that these corrections also include uncancelled gauge dependence, and estimating the effects of this residual gauge dependence on final results is not a well-defined endeavor. Even if the resulting potential is free of spurious IR divergences, there is still no guarantee that the perturbative description near T_c is reliable.