Non-relativistic Geometry and the Equivalence Principle

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Abstract

We describe a geometric and symmetry-based formulation of the equivalence principle in non-relativistic physics. It applies both on the classical and quantum levels and states that the Newtonian potential can be eliminated in favor of a curved and time-dependent spatial metric. It is this requirement which forces the gravitational mass to be equal to the inertial mass. We identify the symmetry responsible for the equivalence principle as the remnant of time-reparameterization symmetry of the relativistic theory. We also clarify the transformation properties of the Schrödinger wave-function under arbitrary changes of frame.

I. INTRODUCTION

The Strong Equivalence Principle states that all physical effects of a uniform static gravitational field can be eliminated by going to a uniformly accelerated frame. This implies the universality of free fall (the gravitational acceleration of all bodies in a uniform static gravitational field is the same), which is one of the best tested physical laws (see e.g.[1,2]). The Strong Equivalence principle also motivated A. Einstein to create General Relativity Theory. Within General Relativity Theory, gravity is reinterpreted as a curved geometry of space-time, and a uniform static gravitational field corresponds to a flat space-time. The Strong Equivalence Principle then becomes obvious.

On the other hand, the theoretical status of the Strong Equivalence Principle and the universality of free fall in non-relativistic mechanics is less understood. This is because gravity is described by a non-geometric quantity, the Newtonian potential. There seems to be no obvious reason why the gravitational mass (that is, the strength of the coupling to the Newtonian potential) must be the same as the inertial mass (the coefficient in the kinetic energy). In non-relativistic quantum mechanics even the formulation of the equivalence principle is not completely settled, see[3] and references therein.

In this paper we describe a way to demystify the equivalence principle in non-relativistic physics, both classical and quantum. Our starting point is the observation that the Newtonian potential is not sufficient to describe the interactions of material particles with gravitational waves. Gravitational waves are stretching and squeezing of space and must be described by a time-dependent spatial metric. This is so even when one considers the interaction of gravitational waves with non-relativistic systems, such as atoms. The only simplification in that case is that the wavelength of a typical gravitational wave is multiple orders of magnitude greater than the size of an atom ($\sim 1\text{Å}$), and thus one can use a multipole expansion to describe the emission, absorption, and scattering of gravitational waves. Multipole expansion amounts to Taylor-expanding the metric in spatial coordinates while keeping the full time-dependence. This is analogous to the treatment of the interaction of atoms with electromagnetic radiation. Thus one needs both a time-dependent spatial metric and a Newtonian potential to describe all gravitational effects of interest.

Once the spatial metric is allowed to be curved and time-dependent, there is neither a preferred class of spatial coordinate systems nor a preferred class of frames. The usual formulation of the Strong Equivalence Principle becomes meaningless because the notion of a uniform static gravitational field is coordinate-dependent and frame-dependent. We show, however, that an alternative formulation is now possible: any Newtonian potential can be eliminated (at least locally) by a frame change. In general, the spatial metric will look different in the new frame. The case of the uniform static gravitational field is special in that the frame change needed leaves the usual flat spatial metric unchanged. In general, the new spatial metric is curved and time-dependent, and this accounts for the tidal effects of a non-uniform Newtonian potential. We note that the possibility of eliminating the Newtonian potential in favor of more geometric quantities was noticed a long time ago by K. Kuchař[4], but the connection with the equivalence principle was not made then. Kuchař's paper uses a rather complicated machinery (Newton-Cartan geometry). However, all the main points can be explained without using this machinery, as we show below.

We illustrate the manipulations involved in eliminating the Newtonian potential in several examples. We also explain that the ability to eliminate the Newtonian potential by a change of frame is equivalent to a certain symmetry of the non-relativistic theory (also noticed by Kuchař). It is this symmetry which requires the equality of the gravitational and inertial masses. The origin of the symmetry can be traced back to the invariance of the relativistic theory under reparameterizations of the time coordinate.

Our formulation of the equivalence principle makes sense in both classical and quantum mechanics. In the quantum case one has to use a covariant formulation of the Schrödinger equation[4,5]. We provide a streamlined derivation of this equation which uses a bare minimum of differential geometry and discuss the transformation properties of the wave-function under arbitrary frame changes.

II. NON-RELATIVISTIC GEOMETRY

In non-relativistic physics, clocks can be synchronized instantaneously and thus the notion of global time is well-defined. Let the global time coordinate be called t. If the space is curved, there is no preferred set of spatial coordinates, and thus physics should be formulated in a form which is invariant under arbitrary changes of coordinates of the form

$$x^k \mapsto \tilde{x}^k = \tilde{x}^k(\mathbf{x}, t), \quad k = 1, 2, 3. \tag{1}$$

Since $\tilde{x}^k$ is allowed to depend on t, in general this is not just a coordinate change, but a frame change. Partial derivatives with respect to time and space coordinates transform as follows:

$$\frac{\partial}{\partial \tilde{x}^k} = \left(A^{-1}\right)^j_k \frac{\partial}{\partial x^j}, \quad \left.\frac{\partial}{\partial t}\right|_{\tilde{x}} = \left.\frac{\partial}{\partial t}\right|_x - (A^{-1})^k_j B^j \frac{\partial}{\partial x^k}, \tag{2}$$

where

$$A^j_k = \frac{\partial \tilde{x}^j}{\partial x^k}, \quad B^j = \frac{\partial \tilde{x}^j}{\partial t}. \tag{3}$$

Since the time-derivative transforms non-covariantly, to write covariant actions and equations of motion one needs to introduce a suitable connection. In this context, a connection is a vector field $N^j(\mathbf{x}, t)$ which under a frame change (1) transforms as follows:

$$\tilde{N}^j = A^j_k N^k - B^j. \tag{4}$$

In the mathematical literature such an object defines what is known as an Ehresmann connection[6]. In general, an Ehresmann gives a notion of parallel transport for a fiber bundle $\pi : E \to S$ with a typical fiber F. Namely, given a path on S connecting $s_0 \in S$ and $s_1 \in S$, an Ehresmann connection provides a diffeomorphism (smooth 1-1 idenfitication) of the fibers over s_0 and s_1. This is the only reasonable notion of parallel transport when the fiber does not have any additional structure beyond that of a smooth manifold. Connections on principal G-bundles and affine connections on vector bundles can be viewed as special cases of a general Ehresmann connection.

To specify an Ehresmann connection, one can specify a 1-form $\boldsymbol{\omega}$ on E with values in the vertical sub-bundle $V = \ker(d\pi) \subset TE$ such that for any section v of V one has $\boldsymbol{\omega}(v) = v$. Then the complementary horizontal sub-bundle H of TE is defined by the condition $\boldsymbol{\omega}(h) = 0$, where h is any section of $H \subset TE$. The horizontal sub-bundle has the property that $d\pi : TE \to TS$ becomes a bundle isomorphism when restricted to H. Thus every vector field on the base S can be lifted in a unique way to a horizontal vector field on E.

In our case, S is the real line parameterized by the time coordinate t, while F is a spatial slice with local coordinates x^k, so the 1-form $\boldsymbol{\omega}$ locally takes the form

$$\boldsymbol{\omega} = \partial_i \otimes dx^i + N^k \partial_k \otimes dt. \tag{5}$$

In other words, $\boldsymbol{\omega}$ is locally encoded in a vector field $N^k(\mathbf{x}, t)$. The invariance of $\boldsymbol{\omega}$ under a change of frame (1) produces the transformation law (4).

Given a function f on space-time, one can define its covariant derivative $D_t f$ with respect to the time coordinate t by lifting the vector field $\partial/\partial t$ to a horizontal vector field and taking the derivative of f along this vector field. In coordinates, this gives

$$D_t f = \partial_t f - N^j \partial_j f. \tag{6}$$

The covariant derivative looks the same in all frames.

Importantly, the connection N^j can always be made to vanish locally by a suitable choice of frame. To find the required frame change, one needs to solve a system of non-linear ordinary differential equations

$$\frac{dx^j(t)}{dt} = N^j(\mathbf{x}(t), t), \quad x^j(0) = \tilde{x}^j. \tag{7}$$

A solution always exists locally in t. Globally a solution may fail to exist because a local solution $x^j(\tilde{\mathbf{x}}, t)$ may escape to infinity in a finite time. One may call a frame where N^j is identically zero an inertial frame. If $N^j = 0$ only in some region of space-time, then one is dealing with a locally inertial frame.

III. CLASSICAL AND QUANTUM PARTICLES IN A GRAVITATIONAL FIELD

A covariant action for a particle of mass m in a Newtonian potential ϕ takes the form

$$S = \frac{m}{2} \int dt \left[\left(\frac{dx^j}{dt} + N^j \right) \left(\frac{dx^k}{dt} + N^k \right) h_{jk} - 2\phi \right]. \tag{8}$$

Here the spatial metric $h_{jk} = h_{kj}$ is positive-definite but otherwise may be an arbitrary matrix function of $\mathbf{x}$ and t. Note that when $\phi = 0$ and $N^j = 0$ (that is, in the absence of the Newtonian potential and in an inertial frame) the action takes the standard form

$$S = \frac{m}{2} \int dt \left[\frac{dx^j}{dt} \frac{dx^k}{dt} h_{jk} \right]. \tag{9}$$

Thus a particle which was at rest at $t = 0$ remains at rest at $t > 0$. This is what distinguishes inertial frames from all other frames.

Our key observation is that the above action is invariant (up to boundary terms) under the following transformation of the connection N^j and the Newtonian potential ϕ:

$$\begin{aligned} N^j &\to N^j + h^{jk} \partial_k F, \\ \phi &\to \phi - D_t F + \frac{1}{2} h^{jk} \partial_j F \partial_k F. \end{aligned} \tag{10}$$

Such transformations depend on an arbitrary function F. This opens possibility to eliminate the Newtonian potential at the expense of modifying the connection N^j. To do this, one has to solve the equation

$$\phi - D_t F + \frac{1}{2} h^{jk} \partial_j F \partial_k F = 0. \tag{11}$$

This equation reduces to the usual Hamilton-Jacobi equation for a particle in a Newtonian potential if we use an inertial frame where $N^j = 0$.

A solution to the Hamilton-Jacobi equation exists locally in t. Suppose we started from a nonzero Newtonian potential ϕ and zero N^j. Having solved the Hamilton-Jacobi equation for F, we can perform the transformations (10) and make $\phi = 0$ at the expense of making N^j non-zero. Then we can eliminate N^j (again locally in t) by solving the equations (7) and performing the corresponding change of coordinates. This changes the metric too. The net result is that we eliminated ϕ at the expense of changing the spatial metric.

The origin of the symmetry (10) can be traced back to time-reparameterization invariance which is present in the relativistic theory but not in the non-relaivistic one. To see this, note that once a global time coordinate has been chosen, a pseudo-Riemannian metric on space-time defines an Ehresmann connection via an ADM parameterization of the metric[7]:

$$ds^2 = -c^2 \left(1 + \frac{2\phi}{c^2}\right) dt^2 + h_{jk}(dx^j + N^j dt)(dx^k + N^k dt). \tag{12}$$

Here c is the speed of light. The identification of the Newtonian potential ϕ in terms of the ADM "lapse" function is standard[7]. It is easy to check that under transformations of spatial coordinates (1) the "shift" vector field N^j transforms as in (4) and thus can be regarded as an Ehresmann connection. We are interested in the transformation of the function ϕ and the vector field N^j under time-reparameterization $t = t' - \frac{F(\mathbf{x},t')}{c^2}$. While it is complicated in general, it simplifies in the limit $c \to \infty$. Keeping only the terms with non-negative powers of c, we find precisely the transformation (10).

In a sense, the symmetry (10) is how the non-relativistic theory "knows" it arose as a limit of a relativistic one. This makes precise A. Einstein's guess that the Strong Equivalence Principle is explained by the diffeomorphism invariance of the relativistic theory.

In the quantum case, it is easiest to start with a covariantized action for a Schrödinger field Ψ:

$$S_{Schr} = \int d^3x\, dt\, \sqrt{h} \left[\frac{i}{2} \bar{\Psi} \overleftrightarrow{D}_t \Psi - \phi m \bar{\Psi}\Psi - \frac{1}{2m} h^{jk} \partial_j \bar{\Psi} \partial_k \Psi\right], \tag{13}$$

where $h = \det ||h_{jk}||$ and

$$\bar{\Psi}\overleftrightarrow{D}_t\Psi := \bar{\Psi}D_t\Psi - \Psi D_t\bar{\Psi}. \tag{14}$$

The Schrödinger equation derived from this action is

$$iD_t\Psi = -\frac{1}{2m\sqrt{h}}\partial_j\left(\sqrt{h}h^{jk}\partial_k\Psi\right) - m\phi\Psi + \frac{i}{2}\left[\partial_k N^k - D_t(\log\sqrt{h})\right]\Psi. \tag{15}$$

This agrees with the results of[4,5] obtained using Newton-Cartan geometry. It can be checked that the equation (15) is covariant under frame-changes (1) if we postulate that the wave-function Ψ transforms as a scalar. The total probability $\int |\Psi|^2\sqrt{h}d^3x$ is also invariant under these transformations and is independent of time, ensuring a consistent physical interpretation of $|\Psi|^2$ as the probability density. It is also easily checked that the action (13) and the equation (15) are invariant under the transformations (10) if we also transform the wave-function by a phase:

$$\Psi(\mathbf{x}, t) \mapsto e^{iF(\mathbf{x},t)}\Psi(\mathbf{x}, t). \tag{16}$$

Thus in the quantum case it is also true that one can eliminate the Newtonian potential by changing a frame.

It is instructive to compare the above approach to the Newton-Cartan framework introduced in[4,5] (see also[8] for a recent discussion). It describes the Newtonian gravitational field in terms of a spatial metric $\gamma^{\mu\nu}$, a vector field u^μ with the constraint $u^\mu n_\mu = 1$ where n_μ is a nowhere vanishing one form, and a one form A_μ encoding (part of) the Newtonian potential. Such a choice of u^μ is not unique; accordingly one requires invariance under *Milne Boosts*:

$$u^\mu \mapsto u^\mu + k^\mu, \qquad k^\mu n_\mu = 0. \tag{17}$$

This formalism is equivalent to ours through the dictionary:

$$\begin{aligned} N^k &= A^k - u^k, \\ \phi &= A_\mu(2u^\mu - A^\mu). \end{aligned} \tag{18}$$

It can be checked that N^k and ϕ are Milne-invariant. In effect, the Newton-Cartan-Bargmann formalism contains redundant fields as well as an additional gauge symmetry (Milne boosts). Introducing the Ehresmann connection and ϕ eliminates this redundancy as well as the need for Milne boosts.

IV. EXAMPLES

In this section we consider a few examples of elimination of a Newtonian potential. We always start with a flat spatial metric, $d\ell^2 = dx^2 + dy^2 + dz^2$.

A. Uniform gravitational field

Let the Newtonian potential be $\phi = gz$. The Hamilton-Jacobi equation is

$$\partial_t F = \frac{1}{2}(\nabla F)^2 + gz. \tag{19}$$

A particular solution of this PDE is

$$F = gzt + \frac{1}{6}g^2t^3. \tag{20}$$

The corresponding connection N^j is $(0, 0, gt)$. Solving the equation (7), we find that the following frame change eliminates a uniform gravitational field:

$$\tilde{z} = z + \frac{gt^2}{2}. \tag{21}$$

Since this transformation is a t-dependent isometry, the spatial metric is unchanged. This example is discussed in[4].

B. Kepler problem

In radial coordinates (r, θ, φ) the metric takes the form $d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$. Consider the Newtonian potential $\phi = -\frac{G}{r}$. The Hamilton-Jacobi equation is

$$\partial_t F = \frac{1}{2}(\nabla F)^2 - \frac{G}{r}. \tag{22}$$

A particular solution of this PDE is

$$F = \sqrt{8Gr}. \tag{23}$$

The corresponding connection N^j has components $(\sqrt{\frac{2G}{r}}, 0, 0)$. A frame change which eliminates the connection N^k is

$$\tilde{r}(r,t) = \left(r^{3/2} - t\sqrt{\frac{9G}{2}}\right)^{2/3}. \tag{24}$$

In the new frame the metric becomes time-dependent:

$$d\ell^2 = \frac{d\tilde{r}^2}{\left(1 + \frac{t}{\tilde{r}^{3/2}}\sqrt{\frac{9G}{2}}\right)^{2/3}} + \left(1 + \frac{t}{\tilde{r}^{3/2}}\sqrt{\frac{9G}{2}}\right)^{4/3} \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{25}$$

One can show that this is unavoidable: any solution of the Hamilton-Jacobi equation which can be used to eliminate the Newton-Kepler potential $\phi = -G/r$ leads to a time-dependent spatial metric. Note also that the frame-change (24) is only defined locally.

C. Generalized Einstein elevators

It is easy to determine which Newtonian potentials can be eliminated without affecting the flat spatial metric. The most general such potential has the form

$$\phi(\mathbf{x}, t) = \mathbf{x} \cdot \frac{d^2\mathbf{w}(t)}{dt^2} - \frac{1}{2}\left(\frac{d\mathbf{w}(t)}{dt}\right)^2, \tag{26}$$

where $\mathbf{w}(t)$ is an arbitrary vector-valued function of t. This formula describes a uniform but not static gravitational field. The corresponding solution of the Hamilton-Jacobi equation is $F = \mathbf{x} \cdot \frac{d\mathbf{w}}{dt}$, and the frame-change is $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{w}(t)$. This example is also discussed in[4].

V. DISCUSSION

A student first studying General Relativity is often confused by the statement that physics should be formulated in a frame-independent way. Does it mean that one has to go back and re-examine Newtonian physics and non-relativistic quantum mechanics to ensure that they conform to this principle? The above discussion shows that the required modification of non-relativistic physics is fairly straightforward and mostly amounts to replacing time derivatives with covariant time derivatives and allowing for a general spatial metric. The usual equations are recovered by going to an inertial frame where the connection N^j vanishes. However, the covariant approach shows that thanks to the symmetry (10) there is an ambiguity in the definition of N^j and therefore an ambiguity in the definition of the class of inertial frames. It is this ambiguity that underlies the equivalence principle. For several species of particles, the symmetry (10) is present only if the gravitational mass is equal to the inertial mass. This provides a symmetry-based explanation of the equivalence principle in a non-relativistic setting.

The geometric approach also clarifies the transformation properties of the wave-function under frame changes. Many textbooks discuss how the wave-function transforms under Galilean boosts. This transformation law involves a non-obvious phase factor, and it is not easy to see how to generalize it to more general frames, such as uniformly accelerated frames and rotating frames. In our approach, the wave-function transforms as a scalar under arbitrary frame changes. Non-obvious phase factors appear when one performs the transformations (10,16). In certain cases, such as Galilean boosts and uniformly accelerated frames, these transformations can be used to eliminate the connection N^j generated by the frame change. It is these transformations, not frame changes, which generate non-obvious phase factors for the wave-function.

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