Journal Pre-proof

Fusion systems with $U_3(3)$ J-components

Michael Aschbacher

PII: S0021-8693(21)00298-2
DOI: https://doi.org/10.1016/j.jalgebra.2021.06.006
Reference: YJABR 18152

To appear in: *Journal of Algebra*

Received date: 27 September 2020

FUSION SYSTEMS WITH $U_3(3)$ J-COMPONENTS

Michael Aschbacher

California Institute of Technology

Abstract. We determine the 2-fusion systems of J-component type in which the centralizer of some fully centralized involution has a maximal J-component that is the 2-fusion system of $U_3(3)$.

The results in this paper are part of a program to, first, classify a large subclass of the class of simple 2-fusion systems of component type, and then, second, to use the theorem on fusion systems to simplify the proof of the theorem classifying the finite simple groups. See [A4] and [A5] for a description of the program; there is also a some discussion of the program below.

Let p be a prime and S a finite p-group. A *fusion system* on S is a category $\mathcal{F}$ whose objects are the subgroups of S and, for subgroups P, Q of S, the set $\hom_{\mathcal{F}}(P, Q)$ of morphisms from P to Q is a set of injective group homomorphisms from P to Q, and that set satisfies two weak axioms. The standard example is the fusion system $\mathcal{F}_S(G)$ for G a finite group and $S \in Syl_p(G)$, whose morphisms are those induced via conjugation in G. A fusion system is *saturated* if it satisfies two more axioms easily seen to hold in the standard example using Sylow's Theorem. See [AKO] for notation, terminology, and basic definitions and results on fusion systems.

Let $\mathcal{F}$ be a saturated fusion system on a finite 2-group S. Proceeding by analogy with finite groups, one can define the notion of a *normal subsystem* of $\mathcal{F}$, which can then be used to define the notions of *simple* and *quasisimple* systems, *subnormal subsystems* of $\mathcal{F}$, and the set $\mathrm{Comp}(\mathcal{F})$ of *components* of $\mathcal{F}$. For t an involution in S the *centralizer* $C_{\mathcal{F}}(t)$ of t in $\mathcal{F}$ is defined, and if t is *fully centralized* (ie. $|C_S(t)| \geq |C_S(x)|$ for each conjugate x of t) then $C_{\mathcal{F}}(t)$ is saturated, so we can define $\mathrm{Comp}(C_{\mathcal{F}}(t))$.

This work was partially supported by DMS NSF-1265587 and DMS NSF-1601063

Define $\mathfrak{C}(\mathcal{F})$ to be the set of *components of centralizers of involutions* in $\mathcal{F}$; that is $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ if there exists some involution $t \in S$ and a conjugate $(\bar{t}, \bar{\mathcal{C}})$ of $(t, \mathcal{C})$ such that $\bar{t}$ is fully centralized and $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{F}}(\bar{t}))$; we write $\mathcal{I}(\mathcal{C})$ for the set of such involutions t. We say that $\mathcal{F}$ is of *component type* if $\mathfrak{C}(\mathcal{F})$ is nonempty.

Let $\mathcal{K}$ be the class of "known" simple 2-fusion systems, and $\tilde{\mathcal{K}}$ the class of "known" quasisimple 2-fusion systems: those whose central factor system is in $\mathcal{K}$. In our attempt to classify the simple systems of component type, we proceed inductively, and hence assume each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$.

Now $\mathcal{K}$ consists of the 2-fusion systems of the known simple groups that are not Goldschmidt groups, together with the exotic Benson-Solomon systems $\mathcal{F}_{Sol}(q)$. We partition $\tilde{\mathcal{K}}$ into two subclasses $\mathcal{K}_{sub}$ and $\mathcal{K}_{non}$; see Definition 1.1 for the definitions of these subclasses. Our system $\mathcal{F}$ is of *subintrinsic component type* if some member of $\mathfrak{C}(\mathcal{F})$ is in $\mathcal{K}_{sub}$, while $\mathcal{F}$ is of *J-component type* if $\mathfrak{C}(\mathcal{F})$ is contained in $\mathcal{K}_{non}$ and $\mathfrak{C}_J(\mathcal{F})$ is nonempty. Here $\mathcal{J}$ denotes the set of involutions j of S with $m_2(C_S(j)) = m_2(S)$, and for $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, we set $\mathcal{J}(\mathcal{C}) = \mathcal{J} \cap \mathcal{I}(\mathcal{C})$ and $\mathfrak{C}_J(\mathcal{F}) = \{\mathcal{C} \in \mathfrak{C}(\mathcal{F}) : \mathcal{J}(\mathcal{C}) \neq \varnothing\}$.

The *odd* saturated 2-fusion systems are those of subintrinsic component type or J-component type. Our program seeks to classify the odd systems $\mathcal{F}$ for which each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$. In [A7], we determined the systems $\mathcal{F}$ of subintrinsic component type with $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, for all but a finite number of choices of $\mathcal{C}$ in $\mathcal{K}_{sub}$.

Our Main Theorem treats those systems $\mathcal{F}$ of J-component type such that some maximal member of $\mathfrak{C}_J(\mathcal{F})$ is the 2-fusion system of $U_3(3)$.

Main Theorem. *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S such that $\mathcal{F}$ is of J-component type. Assume $\mathcal{C} \in \mathfrak{C}_J(\mathcal{F})$ with $\mathcal{C}$ the 2-fusion system of $K \cong U_3(3)$. Assume in addition that there exists no $\mathcal{C}' \in \mathfrak{C}_J(\mathcal{F})$ such that $\mathcal{C}'$ is the 2-fusion system of $U_4(3)$. Let $j \in \mathcal{J}(\mathcal{C})$. Then either*

(1) $\mathcal{C}$ is a component of $\mathcal{F}$, or

(2) $\mathcal{F}$ is the 2-fusion system of $D_0 \leq Aut(D)$ with $D \cong U_4(3)$, $D_0/D \cong E_4$, and j induces an outer automorphism on D. In particular $j \notin \mathfrak{foc}(\mathcal{F})$.

In our program, we are primarily interested in almost simple systems $\mathcal{F}$ with $\mathcal{F} = O^2(\mathcal{F})$, so we are content with either of the two conclusions in the Main Theorem. The Main Theorem is used in the treatment of systems $\mathcal{F}$ in which $\mathfrak{C}_J(\mathcal{F})$ contains the 2-fusion system of $L_2(9)$ as a maximal member.

The proof of the Main Theorem appears at the end of section 6.

Section 1. Preliminaries

Basic notation, terminology, definitions, and results on fusion systems can be found in [AKO]. Our basic reference on finite groups is [FGT].

Throughout this section we assume $\mathcal{F}$ is a saturated fusion system on a 2-group S.

The notion of a *tame realization* of a fusion system by a group is defined and discussed in section 3.3 of [AO]. By Theorems 3.5 and 3.6 in [AO], the members of $\mathcal{K}$, other than the exotic Benson-Solomon systems, are tamely realized by some known simple group, and hence by Theorem 2.20 in [AO], each of its coverings is tamely realized by a known quasisimple group.

Recall from section 1.2 in [A6] that if $\mathcal{F}$ is a saturated fusion system on a p-group S, then for $P \leq S$, $\mathfrak{A}(P)$ consists of those $\alpha \in \hom_{\mathcal{F}}(N_S(P), S)$ such that $P\alpha \in \mathcal{F}^f$, and that various properties of this notation are listed in 1.2.5 in [A6].

Definition 1.1. Define $\mathcal{K}_{int}$ to consist of the *intrinsic* members of $\tilde{\mathcal{K}}$; that is $\mathcal{C} \in \mathcal{K}_{int}$ if $Z(\mathcal{C}) \neq 1$. The class $\mathcal{K}_{sub}$ consists of the *subintrinsic* members of $\tilde{\mathcal{K}}$; those $\mathcal{C}$ such that there is $\mathcal{E} \in \mathfrak{C}(\mathcal{C})$ with $\mathcal{I}(\mathcal{E}) \cap Z(\mathcal{E}) \neq \varnothing$. Finally $\mathcal{K}_{non}$ consists of those members of $\mathcal{K}$ not in $\mathcal{K}_{sub}$. The members of $\mathcal{K}_{non}$ are tamely realized by members of the class $\mathcal{K}_{ev}$ of simple groups appearing in 6.4.1 in [A5], or better by members of the smaller class $\mathcal{K}_{no}$ described in 3.1 in [A8].

(1.2) *Suppose t is a fully centralized involution in $\mathcal{F}$ and $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}}(t))$. Then there is $\mathcal{D} \in \mathrm{Comp}(\mathcal{F})$ such that one of the following holds:*

(1) $\mathcal{C} = \mathcal{D}$.

(2) $\mathcal{D} \neq \mathcal{D}^t$ and $\mathcal{C} = E(C_{\mathcal{D}\mathcal{D}^t}(t))$ is diagonally embedded in $\mathcal{D}\mathcal{D}^t$, so that $\mathcal{C}$ is a morphic image of $\mathcal{D}$.

(3) t acts nontrivially on $\mathcal{D}$ and $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{D}}(t))$.

Proof. This is a special case of 10.11.3 in [A3].

Notation 1.3. (The *bar setup* and *bar notation*) Assume t and a are commuting involutions in S with $t \in \mathcal{F}^f$ and $a \in \mathcal{F}_t^f$. Let $\alpha \in \mathfrak{A}(a)$ and for $U \subseteq C_S(a)$, set $\bar{U} = U\alpha$. Set $\bar{\mathcal{F}} = C_{\mathcal{F}}(\bar{a})$ and for $\mathcal{C} \leq C_{\mathcal{F}_t}(a)$ set $\bar{\mathcal{C}} = \mathcal{C}\alpha^*$.

(1.4) *Assume the bar setup of 1.3. Then*

(1) $\bar{t} \in \bar{\mathcal{F}}^f$ and $\alpha : C_{\mathcal{F}_t}(a) \to C_{\bar{\mathcal{F}}}(\bar{t})$ is an isomorphism of fusion systems.

(2) If $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}_t}(a))$ then $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\bar{\mathcal{F}}}(\bar{t}))$.

Proof. Part (1) follows from 2.2 in [A2]. Then (1) implies (2).

(1.5) *Assume the bar setup of 1.3 with* $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}_t}(a))$*. Then one of the following holds:*

(1) $\bar{\mathcal{C}} \in \mathrm{Comp}(\bar{\mathcal{F}})$*.*

(2) There exists a component $\mathcal{D}$ *of* $\bar{\mathcal{F}}$ *such that* $\bar{t}$ *is nontrivial on* $\mathcal{D}$ *and* $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{D}}(\bar{t}))$*.* ∎

(3) There exists a component $\mathcal{D}$ *of* $\bar{\mathcal{F}}$ *such that* $\mathcal{D} \neq \mathcal{D}^{\bar{t}}$ *and* $\bar{\mathcal{C}} = E(C_{\mathcal{D}\mathcal{D}^{\bar{t}}}(\bar{t}))$ *is a morphic image of* $\mathcal{D}$*.*

Proof. See the proof of 6.1.11 in [A5], or use 1.2 and 1.4.

If case (1) or (2) of 1.5 holds with $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ and $t \in \mathcal{I}(\mathcal{C})$, and if $(t_1, a_1, \mathcal{C}_1)$ is a conjugate of $(t, a, \mathcal{C})$, then we say that $\mathcal{D}$ is a *pump up* of $\mathcal{C}_1$, and in case (2) we say that $\mathcal{D}$ is a *proper pump up* of $\mathcal{C}_1$. If $\mathcal{F}$ is of J-component type with $\mathcal{C} \in \mathfrak{C}_J(\mathcal{F})$ and $m(C_S(\langle t, a\rangle)) = m(S)$, then we term the pump up $\mathcal{D}$ a *J-pump up*. Moreover $\mathcal{C}$ is said to be *maximal* in $\mathfrak{C}_J(\mathcal{F})$ if $\mathcal{C}$ has no proper J-pump ups.

Notation 1.6. Let $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ and $t \in \mathcal{I}(\mathcal{C})$. Define $\mathcal{X}(\mathcal{C})$ and $\tilde{\mathcal{X}}(\mathcal{C})$ as in 6.1.2 in [A5], and let $\alpha \in \mathfrak{A}(t)$. As $t \in \mathcal{I}(\mathcal{C})$, we have $\mathcal{C}\alpha^* \in \mathrm{Comp}(\mathcal{F}_{t\alpha})$, so $Q_{t\alpha} = C_S(t\alpha) \cap C_S(\mathcal{C}\alpha^*)$ is defined in 2.2.1 in [A5]. Now $Q_t = (C_S(t)\alpha \cap Q_{t\alpha})\alpha^{-1}$ is defined in 6.1.15 of [A5]. By 6.1.15.2 in [A5], Q_t is independent of the choice of α.

(1.7) *Assume* $\mathcal{F}$ *is of J-component type and* $Z = \Omega_1(Z(S))$ *is of order 2. Then* $C_{\mathcal{F}}(Z)$ *is constrained.*

Proof. Assume $\mathcal{Z} = C_{\mathcal{F}}(Z)$ is not constrained. Then by II.11.1 in [AKO], $\mathcal{E} = E(\mathcal{Z}) \neq 1$. Now the Sylow group E of $\mathcal{E}$ is normal in S, so as $|Z| = 2$ we conclude that $Z \leq E$. Therefore $Z(\mathcal{E}) \neq 1$, so $Z(\mathcal{D}) \neq 1$ for some $\mathcal{D} \in \mathrm{Comp}(\mathcal{Z})$. However as $\mathcal{F}$ is of J-component type, each member of $\mathfrak{C}(\mathcal{F})$ is simple, a contradiction.

(1.8) *Assume* $\mathcal{F}$ *is of J-component type with* $m(S) \leq 5$ *and* $\mathcal{C} \in \mathfrak{C}_J(\mathcal{F})$ *is the 2-fusion system of* $U_3(3)$*. Assume* $\mathcal{D}$ *is a proper pump up of* $\mathcal{C}$*. Then* $\mathcal{D}$ *is the 2-fusion system of* $U_4(3)$*.*

Proof. As $\mathcal{D}$ is a proper pump up of $\mathcal{C}$ we may choose a bar setup t, a with $t \in \mathcal{I}(\mathcal{C})$, $a \in Q_t$, and $\mathcal{D}$ a component of $\bar{\mathcal{F}}$ satisfying 1.5.2. As $\mathcal{F}$ is of J-component type, $\mathcal{D}$ is tamely realized by some L in the class $\mathcal{K}_{no}$ of simple groups described in 3.1 of [A8], as discussed in 1.1. As $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{D}}(\bar{t}))$, $C_L(\bar{t})$ has a component K which is the 2-fusion system of $L_3^{\epsilon}(q)$, and such that K has wreathed Sylow 2-subgroups of order 32. If $m(L) \leq 3$ then L appears in 6.5.2 in [A5]; then by inspection of the lists of involution centralizers of L in [GLS3], no such centralizer has such a component K. Therefore as $m(S) \leq 5$ it follows that $m(L) = 4$. We inspect the list in 3.1 of [A8] for groups L of 2-rank 4 with the desired centralizer, and conclude that L is $U_4(3)$.

Section 2. Three lemmas on 2-groups

In this section G is a finite 2-group.

(2.1) *Assume $\mathbf{Z}_4 \cong V \leq G$ with $C_G(V) = V$. Then G is isomorphic to $\mathbf{Z}_4$ or D_8, or G is quaternion or semidihedral.*

Proof. If $G = V$ the lemma holds, so assume otherwise; then $V < H = N_G(V)$. As $V = C_G(V)$ and $Aut(V)$ is generated by the inversion map, it follows that H is D_8 or Q_8. Thus if $G = H$ then the lemma holds, so we may assume H is proper in G and hence also in $N_G(H)$. But if $H \cong D_8$ then V is characteristic in H, so $V \trianglelefteq N_G(H)$, a contradiction. Therefore $H \cong Q_8$.

As $V = C_G(V)$, $Z = \Omega_1(V) = Z(G)$. Set $G^* = G/Z$. Then $C_{G^*}(V^*) = H^* \cong E_4$ so by Suzuki's Lemma (cf. Exercise 8.6 in [FGT]), G^* is dihedral or semidihedral. Let Y be the preimage in G of $Z(G^*)$ and X the preimage of the cyclic subgroup of G^* of index 2. As $H \cong Q_8$, Y is cyclic, so as $Y \leq X$, it follows that X is cyclic of index 2 in G. Then as G has a Q_8-subgroup it follows from 23.4 in [FGT] that G is quaternion or semidihedral.

(2.2) *Assume a is an involution in G such that $C_G(a) = \langle a \rangle \times V$, where $V \cong \mathbf{Z}_4$. Assume $m(G) > 2$. Then*

(1) $m(G) = 3$.

(2) $G = \langle a, b, v \rangle$ where b is an involution such that $D = \langle a, b \rangle \cong D_{2^{n+1}}$ for some $n \geq 3$, $V = \langle v \rangle$, $j = v^2$ generates $Z(D)$, and b inverts V.

(3) Let d be an element of order 4 in D. Then $U = \langle dv, j \rangle$ is the unique normal 4-subgroup of G and $C_G(b) = U\langle b \rangle \cong E_8$.

Proof. We appeal to 2.1 in [A1]. By that lemma, G has a subgroup G_0 of index at most 2 with $G_0 = D_0 * V$ where $D_0 = \langle a^G \rangle \cong D_{2^n}$ for some $n \geq 3$ and $G^* = G/V$ is dihedral or semidihedral. As $m(G_0) = 2$ it follows that $|G : G_0| = 2$ and there exists an involution

$b \in G - G_0$ with $m(C_{G_0}(b)) = 2$. As G^* is dihedral or semidihedral, it follows that $D_0^* = G_0^* = \langle a^*, a^{b*}\rangle$, so as $D_0 = \langle a^G\rangle$ we conclude that $D_0 = \langle a, a^b\rangle$, and then it follows that $D = \langle a, b\rangle \cong D_{2^{n+1}}$. As $G^* = D^*$ we have $G = DV$. As a centralizes V, so does $D_0 = \langle a^G\rangle$. Let $Z(D) = \langle j\rangle$, $V = \langle v\rangle$, and d of order 4 in D. As $G_0 = D_0 * V$, we have $j = v^2$.

As $C_{G_0^*}(b^*) = \langle d^*\rangle$, we have $C_{G_0}(b) \le V\langle d\rangle = W \cong \mathbf{Z}_4 \times \mathbf{Z}_2$, with $U = \langle dv, j\rangle$ the unique 4-subgroup of W and the unique normal 4-subgroup of G. Therefore b centralizes U, so as b inverts d it also inverts V. The proof is complete.

(2.3) *Let $H \trianglelefteq G$. Then either*

(1) there exists a 4-subgroup of H normal in G, or

(2) H is cyclic, nonabelian dihedral, quaternion, or semidihedral.

Proof. Assume that (1) fails; then H has no noncyclic characteristic abelian subgroups; that is H is of *symplectic type*. Hence by 23.9 in [FGT], $H = R * E$ where $E = 1$ or E is extraspecial, and R is cyclic, or R is dihedral, quaternion, or semidihedral of order at least 16. In particular if $E = 1$ then (2) holds, so we may assume $E \neq 1$.

Assume R is cyclic; then $R = Z(H) \trianglelefteq G$. Suppose first that $|R| > 2$ and set $G^* = G/R$ and let $e \in E$ such that $1 \neq e^* \in Z(G^*)$. Then $V = R\langle e\rangle \trianglelefteq G$ is isomorphic to $R \times \mathbf{Z}_2$, so $E_4 \cong \Omega_1(V) \trianglelefteq G$. Therefore we may assume $|R| = 2$, so that $H = E$ is extraspecial. If H is of width 1 then (2) holds, so assume the width of H is at least 2. Then the number of nonsingular points in the orthogonal space H^* is odd, so G acts on the preimage U of some such point and $U \cong E_4$ so (1) holds.

Therefore we may assume R is noncyclic. Then by construction $|R| \ge 16$. Let X be the cyclic subgroup of R of index 2 and $X_0 = \Phi(X)$. Then $X_0 = \Phi(H)$, so $XE = C_H(X_0) \trianglelefteq G$. Now the previous reduction applied to XE completes the proof.

Section 3. $\mathcal{C}$ is J-terminal

In this section we assume the following hypothesis:

Hypothesis 3.1. (1) $\mathcal{F}$ is a fusion system of J-component type and $\mathcal{C} \in \mathfrak{C}_J(\mathcal{F})$ is the 2-fusion system of $K = K(\mathcal{C}) \cong U_3(3)$.

(2) There exist no J-components of $\mathcal{C}'$ of $\mathcal{F}$ with $K(\mathcal{C}') \cong U_4(3)$.

Remark 3.2. (1) $\mathcal{C}$ is maximal in $\mathfrak{C}_J(\mathcal{F})$. This follows from 6.6.6 in [A5] and 3.1.2.

(2) Let S be Sylow in $\mathcal{F}$, $t \in \mathcal{J}(\mathcal{C})$, T Sylow in $\mathcal{C}$, and set $Q_0 = C_S(T)$. Recall the definition of $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}(\mathcal{C})$ and Q_t in 1.6.

(3) If $Q_t \neq \langle t \rangle$, then applying 6.6.9.1 in [A5] to Q_t in the role of X, there exists $U \leq Q_t$ of order 4 normal in $C_S(t) \cap N_S(T)$ (and hence $J(C_S(t))$-invariant by 6.6.4 in [A5]) with $U^\# \subseteq \tilde{\mathcal{X}}$.

(3.3) *Suppose V is a subgroup of Q_t of order at least 4 with $V^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C})$, and $a \in C_{Q_t}(V)$ is an involution. Then $a \in \mathcal{I}(\mathcal{C})$.*

Proof. Assume otherwise. Replacing $(t, \mathcal{C})$ by $(t\beta, \mathcal{C}\beta^*)$ for some $\beta \in \mathfrak{A}(t)$, we may assume t is fully centralized. Similarly we may assume $a \in \mathcal{F}_t^f$. Hence we are in the bar setup of 1.3, and we adopt the bar notation of 1.3; in particular $\alpha \in \mathfrak{A}(a)$. By the existence of V and 6.6.9.2 in [A5], $\bar{\mathcal{C}} = \mathcal{C}\alpha^*$ is properly contained in a component $\mathcal{D}$ of $\bar{\mathcal{F}} = \bar{\mathcal{F}}_{\bar{a}}$ and $L = K(\mathcal{D}) \cong U_4(3)$ tamely realizes $\mathcal{D}$, such that $\mathbf{Z}_4 \cong \bar{V}$ induces a faithful group of outer automorphisms on L with $C_{Aut(L)}(K) = Aut_{\bar{V}}(L)$. Hence $V \cong \mathbf{Z}_4$.

Next $m(L) = 4$, so $m(S_{\bar{a}}) \geq 5$, and then by 3.1.2, $m(S) > m(S_{\bar{a}}) \geq 5$. Set $Q = Q_t$. As $m(Aut(K)) = 3$ and $m(S_t) = m(S)$, $m(Q) > 2$. Therefore by 2.3, we can choose U as in 3.2.3 with $U \cong E_4$. Set $P = C_Q(U)$. As we showed that $V \cong \mathbf{Z}_4$, applying this reduction to U in the role of V, it follows that $P^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C})$.

If $C_P(a) = \langle t \rangle$ then Q is dihedral or semidihedral, contradicting $m(Q) > 2$. Therefore $|C_P(a)| > 2$, so we can choose $V = C_P(a)$, and hence $C_P(a) \cong \mathbf{Z}_4$. Hence by 2.2, $m(Q) = 3$. Then as $m(S_t) > 5$ and $m(Aut(K)) = 3$, it follows that $m(S) = 6$ and there exists $A \in \mathcal{A}(S_t))$ with $A \cong E_{64}$, $B = A \cap Q \cong E_8$, and $m(Aut_A(K)) = 3$. As $m(K) = 2$, $F = A \cap QK \cong E_{32}$; let $c \in A - F$. As $5 \leq m(S_{\bar{a}}) < m(S) = 6$, we have $m(S_{\bar{a}}) = 5$.

By 2.2, $Q = \langle a, b, v \rangle$, where b is an involution, $D = \langle a, b \rangle \cong D_{2^{n+1}}$ with $n \geq 3$, $V = \langle v \rangle$ centralizes a, and b inverts V. Let d be of order 4 in D; then by 2.2, $U = \langle dv, t \rangle$ is the normal 4-subgroup of Q, and we may take $B = \langle b, U \rangle$. Write X for the cyclic subgroup of D of index 2 and set $X_0 = \Phi(X)$ and $\bar{Q} = Q/\Phi(Q)$.

Next $\Phi(Q) = X_0$ and $\bar{Q} = \langle \bar{a}, \bar{b}, \bar{v} \rangle \cong E_8$. As c centralizes b and dv, and as $\bar{v} = \overline{dv}$, c centralizes $\langle \bar{b}, \bar{v} \rangle$. As $X_0\langle av \rangle$ is the unique quaternion subgroup of Q of index 2, c centralizes $\overline{av}$, so c centralizes $\bar{Q}$. Therefore c acts on aX_0, so $a^c = a^x$ for some $x \in X$. Hence $xc \in C_{S_t}(a)$ induces conjugation via c on K. Let $V_0 = V\langle xc \rangle$; then $\langle a \rangle V_0 = C_{Q\langle c \rangle}(a)$ with $(xc)^2 = xx^c \in C_{X_0}(a) = \langle t \rangle$, so as xc induces an outer automorphism on K, it follows that $\bar{V}_0$ induces a faithful group of outer automorphisms on L. Then as $Out(L) \cong D_8$, we conclude $L\bar{V}_0 = Aut(L)$. But then as $m(Aut(L)) = 5$, we have $m(S_{\bar{a}}) > 5$, contrary to an earlier remark.

(3.4) *Assume $Q_t \neq \langle t\rangle$ and choose U as in 3.2.3. Then for each involution $a \in C_{Q_t}(U)$, we have $a \in \mathcal{I}(\mathcal{C})$.*

Proof. Apply 3.3 to U in the role of V.

(3.5) $Q_t^{\#} \subseteq \tilde{\mathcal{X}}(\mathcal{C})$.

Proof. Assume otherwise. As $Q_t^{\#}$ is not contained in $\tilde{\mathcal{X}}(\mathcal{C})$, we have $Q_t \neq \langle t\rangle$. Therefore we may pick U as in 3.2.3. Further there is an involution a in $Q_t - \mathcal{I}(\mathcal{C})$. By 3.2.1, $m(C_S(\langle t, a\rangle)) < m(S)$. By 3.4, $[U, a] \neq 1$. Set $Q_2 = C_{Q_t}(U)$ and $Q_3 = C_{Q_2}(a)$. If $Q_3 \neq \langle t\rangle$ then by 3.4 we may pick $V \leq Q_3$ as in 3.3, contradicting $a \notin \mathcal{I}(\mathcal{C})$ and 3.3. Therefore $Q_3 = \langle t\rangle$, so Q_t is dihedral or semidihedral.

By 3.2.1, $m(C_{S_t}(a)) < m(S_t) = m(S)$. Now $m(C_{Q_tT}(a)) = 4$, so $m(S_t) \geq 5$. But $m(S_t) \leq m(Q_t) + m(Aut(K)) = 2+3 = 5$, so $m(S) = 5$ and $m(C_{S_t}(a)) = 4$. Hence there exists an involution $s \in S_t$ inducing an outer automorphism on K with $m(C_{S_t}(s)) = 5$, so $m(C_{Q_tT}(s)) = 4$. Therefore s centralizes an involution $t \neq b \in Q_t - a^{Q_t}$. Hence $Q_t = \langle a, b\rangle$ is nonabelian dihedral.

Let X be the cyclic subgroup of Q_t of index 2. Then $a^{sx} = a$ for some $x \in X$. Now sx is not an involution, but $(sx)^2 = x^s x \in C_X(a) = \langle t\rangle$, so $(sx)^2 = t$.

As in the proof of 3.3, we may assume $t \in \mathcal{F}^f$ and $a \in \mathcal{F}_t^f$, and adopt the bar notation of 1.3. As $\overline{sx}^2 = \bar{t}$, $\bar{\mathcal{C}}$ is not diagonally embedded in $\mathcal{D}\mathcal{D}^{\bar{t}}$ for some $\mathcal{D} \in \mathrm{Comp}(\bar{\mathcal{F}})$, so $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{D}}(\bar{t}))$ for some component $\mathcal{D}$ of $\bar{\mathcal{F}}$. As $m(S) = 5$, it follows from 1.8 that $K(\mathcal{D})$ is $U_4(3)$, so $m(S_{\bar{a}}) = 5 = m(S)$. But now 3.1.2 supplies a contradiction.

(3.6) *For each $j \in \mathcal{J}(\mathcal{C})$ there exists $Q(j) \in \mathcal{X}(\mathcal{C})$ such that $Q_j \leq Q(j)$, $Q(j)^{\#} \subseteq \tilde{\mathcal{X}}(\mathcal{C})$, and $Q(j)$ contains each $X \in \mathcal{X}(\mathcal{C})$ such that $1 \neq X \cap Q(j)$.*

Proof. This follows from 3.5 and 6.6.11 in [A5].

Theorem 3.7. *Assume Hypothesis 3.1 with $T \in \mathcal{F}^f$. Then either*

(1) $\mathcal{C}$ is a component of $\mathcal{F}$, or

(2) $\mathcal{C}$ is J-terminal in $\mathfrak{C}_J(\mathcal{F})$.

Proof. Assume (1) does not hold. We first claim that $\mathcal{C}^{\perp} = \{\mathcal{C}\}$. (See 6.1.7 in [A5] for the definition of $\mathcal{C}^{\perp}$.) Suppose otherwise; then by 3.6, Hypotheses 7.1.1.2 in [A5] holds. By assumption, $\mathcal{C}$ is not a component of $\mathcal{F}$, so Theorem 7.2.5 in [A5] supplies a contradiction, and this contradiction establishes the claim.

By the claim, 3.5, and the definition of J-terminality in 8.2.1 in [A5], (2) holds, completing the proof of the theorem.

Section 4. $K(\mathcal{C}) \cong U_3(3)$.

In this section we assume the following hypothesis:

Hypothesis 4.1. (1) Hypothesis 3.1 is satisfied with the Sylow group T of $\mathcal{C}$ fully normalized in $\mathcal{F}$.

(2) $\mathcal{C}$ is not a component of $\mathcal{F}$.

Notation 4.2. Set $Z_0 = Z(T)$ and $Z = \Omega_1(Z_0)$. Then $Z = \langle z \rangle$ and $Z_0 = \langle z_0 \rangle$ are cyclic of order 2 and 4, respectively. Set $S_T = N_S(T)$ and $Q_0 = C_S(T)$. For $j \in S$, set $S_j = C_S(j)$ and $\mathcal{F}_j = C_{\mathcal{F}}(j)$.

(4.3) *(1) For each $j \in \mathcal{J}(\mathcal{C})$ there exists $Q(j) \in \mathcal{X}(\mathcal{C})$ such that $Q_j \leq Q(j)$, $Q(j)^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C})$, and $Q(j)$ contains each $X \in \mathcal{X}(\mathcal{C})$ with $1 \neq X \cap Q(j)$.*

(2) $\mathcal{C}$ is J-terminal in $\mathfrak{C}_J(\mathcal{F})$.

Proof. Part (1) is 3.6 and part (2) follows from 4.1.2 and 3.7.

Observe that if $Q_j \leq X \in \mathcal{X}(\mathcal{C})$ then by 4.3.1, $X \leq Q(j)$, so $Q(j)$ is the unique maximal memver of $\mathcal{X}(\mathcal{C})$ containing Q_j. However in general $Q(j)$ is not the unique maximal member of $\tilde{\mathcal{X}}(\mathcal{C})$. Still in our setup this is the case when $|Q(j)| > 4$ by 4.7.

(4.4) *Let $j \in \mathcal{J}(\mathcal{C})$ and $\alpha \in \mathfrak{A}(j)$. Then*

(1) $\mathcal{C}\alpha^* \trianglelefteq C_{\mathcal{F}}(j\alpha)$.

(2) There exists $\gamma \in \mathfrak{A}(j)$ such that $T\gamma = T$, and for each such γ we have $j\gamma \in \mathcal{J}(\mathcal{C}\gamma^*)$. If $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{C})$ then $\mathcal{C}\gamma^* = \mathcal{C}$.

Proof. By 4.3.2, $\mathcal{C}$ is J-terminal, so $\mathcal{C}^\perp = \{\mathcal{C}\}$ by Definition 8.2.1 in [A5]. Now (1) follows from 4.1 and 6.1.18 in [A5]. Then (2) follows from (1) and the proof of 6.1.14 in [A5].

(4.5) *Let $t \in \mathcal{J}(\mathcal{C})$. Then $C_{Q_0}(t) = Q_t \times Z_0$.*

Proof. By 4.4 there is $\alpha \in \mathfrak{A}(j)$ such that $T\alpha = T$ and $\mathcal{C}\alpha^* \trianglelefteq \mathcal{F}_{t\alpha}$. Then as $C_{Aut(K)}(T) = Z_0$, it follows that the lemma holds for $t\alpha$ in the role of t. Then as $Q_t = (Q_{t\alpha} \cap S_t\alpha)\alpha^{-1}$ and $Z\alpha = Z$, the lemma also holds for t.

(4.6) *Let $j \in \mathcal{J}(\mathcal{C}) \cap \mathcal{F}^f$. Then*

(1) S_j acts on T, Q_j, and $Q(j)$.

(2) Let $P = N_{S_T}(Q(j))$ and t an involution in $Q(j) \cap Z(P)$. Then $t \in \mathcal{J}(\mathcal{C})$ and $Q(j) = Q_t$ and we may take $t \in \mathcal{F}^f$.

(3) Suppose $|Q(j)| > 4$. Then $Q(j) \trianglelefteq Q_0$ and $Q_0 = Q(j) \times Z_0$.

Proof. Let $Q = Q(j)$. By 4.4.1, $\mathcal{C} \trianglelefteq \mathcal{F}_j$, so $T \trianglelefteq S_j$ and $\mathcal{Y} = C_{\mathcal{F}_j}(\mathcal{C}) \trianglelefteq \mathcal{F}_j$, so that also the Sylow group Q_j of $\mathcal{Y}$ is normal in S_j. As $\mathcal{C} \trianglelefteq \mathcal{F}_j$, for $x \in S_j$, x permutes $\mathcal{X}(\mathcal{C})$, so $Q^x \in \mathcal{X}(\mathcal{C})$. Then as $Q_j \leq Q \cap Q^x$, we conclude $Q = Q^x$ from 4.3, completing the proof of (1).

Assume the setup of (2). By (1), $S_j \leq N_{S_T}(Q) = P$, so $S_j \leq C_{S_T}(t)$. Then as $m(S_j) = m(S)$, we have $t \in \mathcal{J}$. By 4.3, $Q^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C})$, so $t \in \mathcal{J}(\mathcal{C})$. As $Q \in \mathcal{X}(\mathcal{C})$ we have $Q \leq Q_t$ by 6.1.16 in [A5]. Now $Q = Q_t$ by 4.3. By 4.4.2 we may take $t \in \mathcal{F}^f$, completing the proof of (2).

Assume $|Q| > 4$. Let $P_0 = N_{Q_0}(Q)$; then $P_0 \leq P$, so $t \in Z(P_0)$, and by 4.5, $|P_0 : Q| \leq 4 < |Q|$. Let $y \in N_{Q_0}(P_0)$. Then $Q^y \in \mathcal{X}(\mathcal{C})$ and $Q^y \leq P_0$, so as $|P_0 : Q| < |Q|$ we have $1 \neq Q \cap Q^y$. Hence $Q = Q^y$ by 4.3, so $y \in P_0$. Therefore $N_{Q_0}(P_0) = P_0$, so $P_0 = Q_0$ and hence $Q \trianglelefteq Q_0$. Now (3) follows from 4.5.

(4.7) *Assume $Q = Q(j)$ is of order at least 8. Then $\mathcal{C}$ is standard in $\mathcal{F}$.*

Proof. By 4.6.3 and 6.1.8.5 in [A5], we have $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{C})$. Hence by 9.2.3.8 in [A5], it suffices to show that Q is the unique maximal member of $\tilde{\mathcal{X}}(\mathcal{C})$. Thus we may assume $R \in \tilde{\mathcal{X}}(\mathcal{C})$ with $R \not\leq Q$, and it remains to produce a contradiction. Choose $t \in \mathcal{J}(\mathcal{C})$ as in 4.6.2; thus $t \in \mathcal{F}^f$ and $Q = Q_t$. By 4.6.3:

(a) $Q \trianglelefteq Q_0$, $t \in Z(Q_0)$, and $Q_0 = Q \times Z_0$.

By 4.3:

(b) $R \cap Q = 1$, so R is isomorphic to a subgroup of $Z_0 \cong \mathbf{Z}_4$.

For $r \in R$ write $r = y(r)z(r)$ with $y(r) \in Q$ and $z(r) \in Z_0$. Then

(c) $C_Q(r) = C_Q(y(r))$ is of order at least 4.

Let $\alpha \in \mathfrak{A}(R)$. Then $\mathcal{C}\alpha^*$ is a component of $N_{\mathcal{F}}(R\alpha) = \mathcal{N}$. Therefore $R\alpha \leq R_0$ a Sylow group of $C_{\mathcal{N}}(\mathcal{C}\alpha^*)$.

Next for $1 \neq x \in N_Q(R)$, we have $C_{\mathcal{C}}(z) = C_{\mathcal{C}}(z(r)) \leq C_{\mathcal{F}}(\langle x, r\rangle)$, so $C_{\mathcal{C}}(z)\alpha^* \leq C_{\mathcal{C}\alpha^*}(x\alpha)$, and hence $x\alpha \in Z_0\alpha R_0$. However $x\alpha \notin R_0$ by 4.3, so:

(d) $N_Q(R)$ is isomorphic to a subgroup of Z_0, and hence is cyclic of order at most 4.

Let $R = \langle r\rangle$ and $y = y(r)$. By (c) and (d):

(e) $C_Q(y) \cong \mathbf{Z}_4$.

Suppose first that $|R| = 2$. By (e), $Q = C_Q(y)$ is of order 4, contrary to the hypothesis of the lemma. Hence

(f) $|R| = 4$ and $r^2 \notin \mathcal{I}(\mathcal{C})$.

For if $s = r^2 \in \mathcal{I}(\mathcal{C})$, we obtain a contradiction by replacing R by $\langle s \rangle$.

(g) Q is quaternion, dihedral, or semidihedral, so $m(S) \leq 5$.

Namely by (e) and 2.1, Q is quaternion, dihedral, or semidihedral, so $m(Q) \leq 2$. Then as $m(Aut(K)) = 3$, (g) follows.

Let $\beta \in \mathfrak{A}(s\alpha)$, $\bar{s} = s\alpha\beta$, etc. Then $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{F}_{\bar{s}}}(\bar{r}))$, so there exists a component $\mathcal{D}$ of $\mathcal{F}_{\bar{s}}$ such that either $\bar{\mathcal{C}}$ is a component of $C_{\mathcal{D}}(\bar{r})$ or $\bar{\mathcal{C}}$ is diagonal in $\mathcal{D}\mathcal{D}^{\bar{r}}$. Let D be Sylow in $\mathcal{D}$; then $m(D) \leq 4$ by (g), so in the first case as $\bar{\mathcal{C}} \neq \mathcal{D}$ by (f), $K(\mathcal{D})$ is $U_4(3)$ by 1.8, contrary to 3.1.2. Therefore the second case holds; in particular $m(S_Q) = m(S) \geq m(\langle \bar{s} \rangle D D^{\bar{r}}) = 5$, as $m(Q) \leq 2$ by (g), so there exists an involution a in S_Q inducing an outer automorphism on K with $C_T(a) \cong D_8$ and $a \in A \in \mathcal{A}(S_Q)$. As $r^2 = s$, $s \in Z(S_Q)$ by (g). Let $\gamma \in \mathfrak{A}(s)$ with $s\gamma = \bar{s}$ and set $\tilde{A} = A\gamma$, etc. Then $\tilde{A} \in \mathcal{A}(S)$, so $\tilde{A}$ acts on D and $C_D(\tilde{a}) \cong C_T(a) \cong D_8$. Then $\langle \bar{s}, \tilde{a} \rangle C_{DD^{\bar{r}}}(\tilde{a})$ is of 2-rank 6, contradicting $m(S) = 5$. This final contradiction completes the proof of the lemma.

(4.8) *Assume $j \in \mathcal{J}(\mathcal{C}) \cap \mathcal{F}^f$ with $Q = Q(j)$ of order 4, and set $P_0 = N_{Q_0}(Q)$. Then $Q = Q_j \cong \mathbf{Z}_4$ and one of the following holds:*

(1) $j \in Z(Q_0)$, $Q_0 = Q \times Z_0$, and either $\mathcal{C}$ is nearly standard in $\mathcal{F}$ or there exists $R \in \tilde{\mathcal{X}}(\mathcal{C})$ with $R \cap Q = 1$ and $Q_0 = R \times Z_0$.

(2) $P_0 = C_{Q_0}(j) = Q \times Q^x \cong \mathbf{Z}_4^2$ for x an involution in $Q_0 - P_0$, and P_0 is of index 2 in $Q_0 \cong \mathbf{Z}_4$ wr $\mathbf{Z}_2$.

Proof. As $|Q| = 4$, Q is abelian, so Q centralizes j. From 4.3, $Q_j \leq Q$ and $Q \in \mathcal{X}(\mathcal{C})$ so $Q \leq Q_j$ by 6.1.16 in [A5]. Therefore $Q = Q_j$.

Suppose $j \in Z(Q_0)$; then $Q_0 = Q \times Z_0$ is of order 16. If $\mathcal{C}$ is nearly standard then the argument in paragraph two of the proof of 4.10 shows that Q is cyclic and (1) holds. So assume $\mathcal{C}$ is not nearly standard. Then the proof of 4.7 shows that (1) again holds: For example if Q is not the unique maximal member of $\tilde{\mathcal{X}}(\mathcal{C})$ then there exists $R \in \tilde{\mathcal{X}}(\mathcal{C})$ with $1 = R \cap Q$, so if $|R| = 4$ then (1) holds. So assume $R = \langle r \rangle$ is of order 2 and let $\alpha \in \mathfrak{A}(r)$. Define R_0 Sylow in $C_{\mathcal{N}}(\mathcal{C}\alpha^*)$ as in the proof of 4.7. Then $Q_0\alpha \leq R_0 Z_0 \alpha$, so as $|Q_0| = 16$ we have $|R_0 \cap Q_0\alpha| = 4$, and replacing R by $(R_0 \cap Q_0\alpha)\alpha^{-1}$, we may take $|R| = 4$, so again (1) holds.

So assume $j \notin Z(Q_0)$. As P_0 centralizes some involution $j_0 \in Q$, we may assume $P_0 = C_{Q_0}(j)$, so that $P_0 < Q_0$. Hence P_0 is of index 2 in some $P \leq Q_0$. Let $x \in P - P_0$; then $Q^x \neq Q$ and both Q and Q^x are normal in P_0. By 4.3, $Q \cap Q^x = 1$, so $P_1 = QQ^x = Q \times Q^x$, and then as Q is abelian, so is P_1. By 4.5, $P_1 = C_{Q_0}(j) = P_0$ and $Q \cong Q^x \cong P_0/Q \cong \mathbf{Z}_4$, so $P_0 \cong \mathbf{Z}_4^2$. Finally P is the wreath product of $\mathbf{Z}_4$ by $\mathbf{Z}_2$, so we may choose x to be an involution. Moreover P_0 is characteristic in P and $N_{Q_0}(P)$ permutes $\{j, jz\}$, so $Q_0 = P$. That is (2) holds in this case.

(4.9) *The 2-fusion system of $U_3(3)$ is split.*

Proof. Let $\mathcal{F}_0 = \mathcal{F}_T(K)$ be the 2-fusion system of $K \cong U_3(3)$. From section 0.13 in [A5] we must show that if $\mathcal{F}_1$ is a saturated fusion system on a 2-group S_1 such that $\mathcal{F}_0 = O^2(\mathcal{F}_1)$, Q_1 is a complement to T in S_1, and $Q_1 \cong E_4$ is tightly embedded in $\mathcal{F}_1$, then $\mathcal{F}_1 = O_2(\mathcal{F}_1) * \mathcal{F}_0$. From 2.22 in [AO], $\mathcal{F}_1 = \mathcal{F}_{S_1}(G)$ for some finite group G with $K \trianglelefteq G$. Further we may assume $G \neq O_2(G)K$, so some involution $a \in Q_1$ induces an outer automorphism on K. Let $G^* = G/O_2(G)$. By 13.7 in [A8], Q_1 is tightly embedded in G, Q_1 is faithful on K, and $Q_1 \trianglelefteq C_G(a)$. This is a contradiction as $C_{G^*}(a^*) \cong \mathbf{Z}_2 \times S_4$, so there is no 4-subgroup Q_1^* containing a^* invariant under $O^2(C_{G^*}(a^*))$.

(4.10) *Assume $\mathcal{C}$ is standard in $\mathcal{F}$ and $|Q| > 2$. Then $Q \cong \mathbf{Z}_4$ and there exists $Q \neq P \in Q^{\mathcal{F}}$ such that $Q_0 = Q \times P$.*

Proof. By 4.9, Hypothesis 15.1 of [A8] is satisfied. Hence by 15.3 in [A8], either $m(Q) = 1$ or $\Phi(Q) = 1$. Indeed if Q is quaternion then $E(\mathcal{F})$ is the 2-fusion system of $U_5(3)$ by 15.4.3 in [A8], contradicting the condition in 3.1 that $\mathcal{F}$ is of J-component type, and the fact that $U_5(3)$ is not in $\mathcal{K}_{no}$ from 3.1 in [A8], so that $E(\mathcal{F})$ is not in $\mathcal{K}_{non}$ by 1.1. Therefore if $m(Q) = 1$ we may assume that Q is cyclic.

By 15.9 in [A8], there exists $Q \neq P \in Q^{\mathcal{F}} \cap S_Q$. Choose M to tamely realized $N_{\mathcal{F}}(Q)$ as in 15.7 of [A8], and, as in 15.10 in [A8], let $H = PK \leq M$. By 15.11 and 13.7 in [A8], P is tightly embedded in H. Set $H^* = H/O_2(H)$ and let $a \in P$ be an involution. As P is tightly embedded in H, $P^* \trianglelefteq C_{H^*}(a^*)$ by 15.13.3 in [A8]. Hence as P is cyclic or $\Phi(P) = 1$, it follows from the centralizers of involutions in $Aut(K)$ that the lemma holds.

Section 5. More $K(\mathcal{C}) \cong U_3(3)$.

In this section we assume the following hypothesis:

Hypothesis 5.1. Hypothesis 4.1 holds with $Q = Q(j)$ of order 4 for some $j \in \mathcal{J}(\mathcal{C}) \cap \mathcal{F}^f$.

Notation 5.2. Continue Notation 4.2. We find in the next lemma that $Q \cong \mathbf{Z}_4$; write j_0 for a generator of Q and set $W = \langle Q^{\mathcal{F}} \rangle$ and $W_1 = \Omega_1(W)$.

Observe that $T \cong \mathbf{Z}_4$ wr $\mathbf{Z}_2$, so T has a unique abelian subgroup T_1 of index 2, and $T_1 \cong \mathbf{Z}_4^2$. Set $T_2 = O_2(C_K(z))$, and observe that $T_2 = Z_0 * Q_2$, where $Q_2 \cong Q_8$. Moreover $T_2 = \Omega_1(T)$. Therefore T_1 and T_2 are characteristic in T. Set $E = \Omega_1(T_1)$ and $S_Q = N_S(Q)$; thus $E_4 \cong E \trianglelefteq S_Q$ as T_1 char $T \trianglelefteq S_Q$.

(5.3) *(1)* $Q = Q_j \cong \mathbf{Z}_4$.

(2) Q *is tightly embedded in* $\mathcal{F}$.

(3) $\mathcal{F}_j$ *is tamely realized by a finite group* G_j *with* $F^*(G_j) = Q \times K$*. Indeed either* $G_j = QK$ *or* $|G_j : QK| = 2$ *with* $G_j = QK\langle a\rangle$*, where* $a^2 \in Q$ *and* $C_K(a) \cong S_4$.

(4) Either $m(S) = 3$ *or* $|G_j : KQ| = 2$*,* G_j *splits over* KQ*, and* $m(S) = 4$.

(5) There exists $Q \neq P \in Q^{\mathcal{F}}$ *and for each such* $P \in \mathcal{F}_j^f$ *we have* $PQ = P \times Q = Q \times Z_0$.

(6) W *is abelian.*

(7) $W = Q \times T_1 \cong \mathbf{Z}_4^3$.

Proof. By 4.8, (1) holds and one of the three possibilities listed in that lemma holds. By 4.4, $\mathcal{C} \trianglelefteq \mathcal{F}_j$. By (1), Q is cyclic, so $Q = C_{\mathcal{F}_j}(\mathcal{C})$, so in particular $Q \trianglelefteq \mathcal{F}_j$, and hence (2) holds. Moreover $F^*(\mathcal{F}_j) = Q \times \mathcal{C}$, so by 2.22 in [AO], $\mathcal{F}_j$ is tamely realized by some finite group G_j with $F^*(G_j) = Q \times K$. Then as $Aut(K)$ is K extended by an involution a with $C_K(a) \cong S_4$, (3) holds. Then (3) implies (4).

As $Q \cong \mathbf{Z}_4$, (6) follows from (2) and 3.3.5 in [A5].

Suppose $W = Q$ and let $\mathcal{D}$ be the normal closure of Q in $\mathcal{F}$. By Theorem 4.0.2 in [A5], one of the following holds: $\mathcal{D} = Q$; $\mathcal{D}$ is the 2-fusion system of an extension of $L_2(q)$, q odd; or $\mathcal{D}$ is the 2-fusion system of $L_3(q)$ extended by a graph automorphism. In each case $O^2(C_{\mathcal{F}}(\mathcal{D})) = O^2(\mathcal{F}_j) = \mathcal{C}$, contrary to 4.1.2. Therefore $W \neq Q$, so there exists $Q \neq P \in Q^{\mathcal{F}}$, and P centralizes Q by (6). Then by (2) and 3.1.8 in [A5], $PQ = P \times Q$. Hence as $Q = C_{G_j}(K)$, P acts faithfully on K. Set $H = PK \leq G_j$. We may assume $P \in \mathcal{F}_j^f$. Then arguing as in the last two sentences of the proof of 4.10, (5) holds. The argument does not require $\mathcal{C}$ to be standard, only that (1)-(3) hold, and K has one class of involutions with representative z, and with Z_0 the unique normal $\mathbf{Z}_4$-subgroup of $C_K(z)$ and $C_{Aut(K)}(z)$

Finally $T_1 = \langle Z_0^{\mathcal{C}} \rangle$, so (5) implies (7).

(5.4) *(1) $C_S(W) = W$, so $N_{\mathcal{F}}(W)$ has a model G_W.*

(2) $|Q^{\mathcal{F}}| = 4$.

(3) $j^{\mathcal{F}} \cap W = W_1 - E$ is of order 4 and $z^{\mathcal{F}} \cap W = E^{\#}$.

(4) $Aut_{\mathcal{F}}(W_1)$ is the stabilizer in $GL(W_1)$ of E.

(5) W and E are normal in S and G_W.

(6) If $G_j = KQ$ then $|S| = 2^9$ and $W = C_{G_W}(W_1)$; otherwise $|S| = 2^{10}$.

(7) $Z = \Omega_1(Z(S))$.

(8) $\mathcal{F}_z$ is constrained, so $\mathcal{F}_z$ has a model G.

(9) $z \notin j^{\mathcal{F}}$.

(10) If $G_j \neq KQ$ then $G_j = KQ\langle a\rangle$ where a is an involution inducing an outer automorphism on K with $C_K(a) \cong S_4$, and a inverts W. Moreover $C_{G_W}(W_1) = W\langle a\rangle$.

Proof. As $W = \langle Q^{\mathcal{F}}\rangle$, W is weakly closed in S with respect to $\mathcal{F}$, so $W \trianglelefteq S$ and hence $W \in \mathcal{F}^f$. As $j \in W$, $C_S(W) = C_{S_Q}(W) = W$, using 5.3.7. Thus (1) holds and we can regard $N_{G_j}(W)$ as $C_{G_W}(j)$.

As W is abelian by 5.3.6, G_W controls fusion in W with respect to $\mathcal{F}$.

Next $N_K(W)$ has three orbits $\{j\}$, $E^{\#}$, and $jE^{\#}$ on $W_1^{\#}$ of length 1,3,3, respectively. Thus as G_W controls fusion in W, it follows from 5.3.5 that $\Theta = j^{\mathcal{F}} \cap W$ is of order 4 or 7. Suppose $|\Theta| = 7$. Then as $Aut_{G_j}(W_1) \cong S_3$, $Aut_{\mathcal{F}}(W)$ is $\mathbf{Z}_7$ extended by S_3, whereas $GL(W_1) \cong L_3(2)$ has no such subgroup. Therefore $|\Theta| = 4$, so as the map $P \mapsto i(P)$ is a bijection of $Q^{\mathcal{F}}$ with Θ, where $i(P)$ is the involution in P, it follows that (2) holds. Further $\Theta = \{j\} \cup \Delta$ where $\Delta = E^{\#}$ or $jE^{\#}$. But if $\Delta = E^{\#}$ then $\prod_{d\in\Delta} d = 1$, so as G_W induces $Sym(\Theta)$ on Θ, also $b = j \cdot jz \cdot je = 1$, for $e \in E - Z$, whereas $b = jze \neq 1$. This establishes (3), and as G_W controls fusion in W, (9) also follows.

Indeed as G_W controls fusion in W, (3) implies (4).

We've already seen that $W \trianglelefteq S$, and then also $E \trianglelefteq S$ by (4). By construction, $W \trianglelefteq G_W$ and then $E \trianglelefteq G_W$ by (4), establishing (5).

As $W \trianglelefteq S$ by (5), $|S| = |Q^{\mathcal{F}}||S_Q| = 4|S_Q|$ by (2), and then the formulas for $|S|$ in (6) follow from 5.3.3. As $C_{G_W}(W_1) = C_{N_{G_j}(W)}(W_1)$, we conclude that $C_{G_W}(W_1) = W$ when $G_j = KQ$ from 5.3.7, completing the proof of (6).

By (1), we have $Z_1 = \Omega_1(Z(S)) \leq W_1$, and then $Z_1 = Z$ by (4), establishing (7). Then (7) and 1.7 imply (8).

Assume $G_j \neq KQ$; then by 5.3.3, $G_j = KQ\langle a\rangle$ where a induces an outer automorphism on K with $C_K(a) \cong S_4$ and $a^2 \in Q$. In particular a inverts T_1 and either centralizes or inverts Q. Define P as in 5.3.5; by (3) $jz = i(P)$, so we may choose notation so that

$p_0 = j_0 z_0$ generates P. Now as a inverts T_1 we have $a \in C_S(W_1)$, so by 5.3.2, a acts on P, so $p_0^a = p_0^\epsilon$, where $\epsilon = \pm 1$. Then

$$j_0^\epsilon z_0^\epsilon = p_0^\epsilon = p_0^a = (j_0 z_0)^a = j_0^a z_0^a = j_0^a z_0^{-1},$$

so it follows that $\epsilon = -1$ and a inverts Q. Then as a inverts T_1, it follows from 5.3.7 that a inverts W. As a inverts Q and centralizes $a^2 \in Q$, it follows that $a^2 \in \langle j \rangle$. If $a^2 = 1$ then a is an involution, completing the proof of (10), so assume $a^2 = j$. Then for each $w \in W$, $(aw)^2 = a^2 w^a w = a^2 = j$. But as a inverts $W = C_{G_W}(W)$, $W\langle a \rangle \trianglelefteq G_W$, and then as $(aw)^2 = j$ for each $w \in W$ we have $j \in Z(G_W)$, contrary to (7).

(5.5) *(1) Set $G_W^! = G_W/W_1$. Then the map $\rho : w^! \mapsto w^2$ is a G_W-equivariant isomorphism from $W^!$ to W_1.*

(2) If $G_j \neq KQ$ then G_W has a subgroup G_0 of index 2 with $a \in G_W - G_0$ such that $N_{G_0}(Q) = C_{G_W}(Q)$.

Proof. Part (1) is an observation following from 5.3.7.

Suppose that $G_j \neq QK$; by 5.4.10, a inverts W, so Q is inverted in S and hence $|j_0^{\mathcal{F}}| = 2|Q^{\mathcal{F}}| = 8$. Let $W_a = W\langle a \rangle$; then $C_{G_W}(W_1) = C_{G_j}(W_1) = W_a$ by 5.4.10, and by 5.4.4, $G_W/W_a \cong S_4$ is the stabilizer in $GL(W_1)$ of E. Let $R = O_2(G_W)$ and $R_0 = (R \cap O^2(G_W))W$; in particular either $R/W = R_0/W \times W_a/W \cong E_8$, or $R = R_0$ and $R/W \cong Q_8$. In the former case as $G_W = RC_{G_W}(a)$ with $C_{G_W}(a) = C_K(a) \times \langle a \rangle$, we take $G_0 = R_0 C_K(a)$ to obtain (2). Thus we may assume the latter case holds. Hence for $r \in R_0 - W_a$, we have $r^2 = b \in aW$.

By (1) and 5.4.4, G/W_a is the stabilizer of $T_1^!$ in $GL(W^!)$. Then for $t \in T_1 - E$ there is $u \in W_1$ with $t^r = tu$. Now as b inverts W, we have $t^{-1} = t^b = (t^r)^r = (tu)^r = tuu^r$, so $t^2 = uu^r$. However R centralizes E, so $u = je$ for some $e \in E$ and $uu^r = (je)(j^r e) = jj^r$, a contradiction as we can choose t so that $t^2 \neq jj^r$. This completes the proof of (2).

Notation 5.6. Let $H = O^2(G_W)T \leq G_W$ and set $M = [O_2(G_W), O^2(G_W)]$. Let $X \in Syl_3(N_K(T_1))$, where $N_K(T_1)$ is regarded as a subgroup of G_W, and hence also a subgroup of H.

By 5.4.5, $E \trianglelefteq S$ and by 5.4.8, $\mathcal{F}_z$ is constrained with model G. Thus we can form $V = W(E) = \langle E^G \rangle \trianglelefteq G$. Set $\tilde{G} = G/Z$ and $V_H = V \cap H$.

Let $v \in Q_2 - Z_0 E$, $y \in M$ with $[y, v] \notin W$, and set $b = [y, v]$.

(5.7) *(1)* $T_2 \leq V_H$.
(2) $b \in V \cap M \leq V_H$.
(3) Either $|b| = 2$ *and* b *centralizes* v *or* $|b| = 4$ *and* b *inverts* v.

Proof. By definition of V we have $E \leq V$, and then $T_2 = \langle E^{C_K(z)} \rangle \leq V$ as $V \trianglelefteq G$. Then as $T_2 \leq T \leq H$, (1) holds.

By (1), $v \in V$, so $b = [y, v] \in V$. Similarly as $y \in M \trianglelefteq G_W$, $b = [y, v] \in M \leq H$, so $b \in V_H$, establishing (2).

As $b = [y, v]$, we have $v^{-y} = bv^{-1}$, so $|bv^{-1}| = |v| = 4$; then as $\langle v, b \rangle \leq V$ with $\Phi(V) = Z$, (3) follows.

(5.8) *(1)* $T_1 \trianglelefteq G_W$.
(2) $M/E \cong E_{16}$.
(3) $H = MXT = MN_H(X)$ *with* $T_1 \leq M = O_2(H)$, $C_M(X) = 1$, *and* $N_H(X) \cong S_3$.
(4) $H/T_1 \cong S_4$.

Proof. Set $G_W^! = G_W/W_1$ and $G_W^+ = G_W/W$. By 5.4.4, $G_W/C_{G_W}(W_1) \cong S_4$ is the stabilizer in $GL(W_1)$ of E.

If $G_j = KQ$ then $W = C_{G_W}(W_1)$ by 5.4.6, so setting $R = O_2(G_W)$ in this case, we have $G_W^+ \cong S_4$, $G_W = RN_{G_W}(X) = RXT$, and $M = [R, X]$. In this case set $G_0 = G_W$.

On the other hand if $G_j \neq KQ$ then by 5.5.2 there is G_0 of index 2 in G_W containing $C_{G_W}(Q)$ with $G_W = G_0\langle a \rangle$. Therefore $H \leq G_0$, and, setting $R = O_2(G_0)$ in this case, arguing as in the previous paragraph, and appealing to 5.4.10 to conclude $W = C_{G_0}(W_1)$, we obtain $G_0^+ \cong S_4$, $G_0 = RN_{G_0}(X) = RXT$, $M = [R, X]$, and $H = XMT$.

Next applying 5.5.1 we obtain:

(*) $G_0^+ \cong S_4$ is the stabilizer in $GL(W^!)$ of $T_1^!$, so $T_1^! \trianglelefteq G_0^!$.

Set $G_0^* = G_0^!/T_1^!$. As X is irreducible on $T_1^!$ we have $T_1^! \leq Z(R^!)$. Then as X is irreducible on R^+ wth $C_{R^!}(X) = Q^!$, either $R^* = Q^* \times [R^*, X]$ or $R^* \cong Q_8$ with $Q^* = Z(R^*)$. In the former case $M^* = [R^*, X] \cong E_4$. In the latter case for $r \in R - W$, $r^2 \in j_0 T_1 W_1$. Then as $r^!$ centralizes $T^!$ and $r^{!2}$, it follows that $r^!$ centralizes $W^!$, contrary to (*).

Therefore $M^* \cong E_4$ and $C_{M^!}(X) = 1$. Now $b \in M - W$ with $b^2 \in Z$ by 5.7.2, so $b^! \in M^! - T_1^!$ is an involution. Then as X is transitive on $(M^!/T_1^!)^\#$, it follows that $M^! \cong E_{16}$.

By 5.4.5, $E \trianglelefteq G_W$. Set $G_W^- = G_W/E$ and $M_0 = M\langle j \rangle$. Then $M_0^-/\langle j^- \rangle \cong M^! \cong E_{16}$, with $C_{M_0}(X) = \langle j \rangle$. Therefore either $\Phi(M_0^-) = 1$, so that $M_0^- = M^- \times \langle j^- \rangle$, or there is $d \in M_0$ with $d^{-2} = j^-$. In the former case $M \cap Q = 1$, so $|M| = 2^6$ and $T_1 = W \cap M \trianglelefteq G_W$ with $M/E \cong M^! \cong E_{16}$, so the lemma holds. In the latter case, $d^2 \in jE$, so as d centralizes d^2 and E, also d centralizes W_1, contrary to $W = C_{G_0}(W_1)$. This completes the proof of the lemma.

(5.9) *(1)* $j_0^b = j_0 z_0$.

(2) b *inverts* z_0, *so* $\langle b, z_0\rangle \cong D_8$ *or* Q_8 *for* b *of order 2 or 4, respectively.*

(3) $V_H = \langle T_2, b\rangle \cong D_8 Q_8$ *or* D_8^2.

Proof. Set $G_W^+ = G_W/W$. From 5.8, $H^+ \cong S_4$ and from 5.6, $[y, v] = b$, where $v \notin O^2(H)$ and $y^+ \in O_2(G^+)$ with $b^+ \neq 1$, so $\langle y^+, v^+\rangle = S^+ \cong D_8$ with $\langle b^+\rangle = Z(S^+)$. Therefore b acts on $C_W(v) = QZ_0$, so as M/T_1 is regular on $Q^{\mathcal{F}}$, $j_0^b = j_0^\epsilon z_0$ for some $\epsilon = \pm 1$. But $[j_0, b] \in M$ as $b \in M \trianglelefteq G_W$, so as $M \cap Q = 1$ we have $\epsilon = 1$, so (1) holds.

By (1), $b^{-j_0} = z_0 b^{-1}$, so $|b| = |z_0 b^{-1}|$. Then as $\langle b, z_0\rangle \leq V_H$ by 5.7, it follows that $|b| \in \{2, 4\}$, b inverts z_0, and (2) holds.

Let $U = \langle T_2, b\rangle$; then $U \leq V_H$ by 5.7. As $\Phi(V_H) = Z$ we have $V_H \leq C_{S\cap H}(\tilde{U})$. But $S \cap H = M\langle v\rangle = U\langle t_1, y\rangle$, where $t_1 \in T_1 - Z_0E$. Finally $[y, v] = b$ and $[t_1, v] = v_1$, where $Q_2 = \langle v, v_1\rangle$, so $U = C_{S\cap H}(\tilde{U})$, and hence $U = V_H$.

Next as $\Phi(U) = Z$, we have $Z(U) = Z(C_U(Q_2))$, so as $|U| = 2^5$ it follows that $Z(U) = Z$ or $|Z(U)| = 8$ and $Z(T_2) = Z_0 \leq Z(U)$. As $b \in U$ inverts Z_0 by (2), it follows that $Z(U) = Z$ and hence U is extraspecial of order 2^5. Hence (3) holds.

(5.10) *(1)* $C_{V_H}(E) = \langle e\rangle \times F_0$, *where* $F_0 = \langle z_0, b\rangle$ *and* $e \in E - Z$.

(2) If $|b| = 2$ *then* $F_0 \cong D_8$ *and* $V_H \cong D_8^2$.

(3) If $|b| = 4$ *then* $F_0 \cong Q_8$ *and* $V_H \cong D_8 Q_8$.

Proof. Let $e \in E - Z$. By 5.9.3, $V_H \cong D_8 * F$, where $F \cong D_8$ or Q_8. Then $C_{V_H}(E) = \langle e\rangle \times F_e$, where $F_e \cong F$. But $E \leq Z(M)$, so as $F_0 = \langle z_0, b\rangle \leq M \cap V_H$, we have $F_0 \leq C_{V_H}(E)$. Further by 5.9.2, F_0 is isomorphic to D_8 or Q_8 for b of order 2 or 4, respectively, so the lemma holds.

(5.11) *(1) If* $i \in M - E$ *is an involution then* $A = \langle i^X, E\rangle \cong E_{16}$.

(2) If $E \neq \Omega_1(M)$ *then* $m(M) = 4$ *and* $G_j \neq KQ$.

(3) If $V_H \cong D_8^2$ *then* $G_j \neq KQ$.

Proof. Let $G_W^* = G_W/E$ and assume $i \in M - E$ is an involution. As $M^* \cong E_{16}$ and $C_M(X) = 1$ by 5.8, it follows that $A^{*\#} = i^{*X}$. Therefore X acts on A^* and X is transitive on $A^{*\#}$, so as $E \leq Z(M)$ we have $\Phi(A) = 1$ and hence (1) holds. Hence if $E \neq \Omega_1(M)$ then $m(M) \geq 4$, so (2) follows from 5.3.4. Finally if $V_H \cong D_8^2$ then $b \in M - E$ is an involution by 5.10, so (2) implies (3).

(5.12) *$G_j \neq KQ$.*

Proof. Assume $G_j = KQ$. Then by 5.11.2, we have $E = \Omega_1(M)$, while $V_H \cong D_8Q_8$ by 5.10 and 5.11.3.

Observe $S_H = MT$ is Sylow in H. Claim $I(S_H) \subseteq z^{\mathcal{F}}$. As $E = \Omega_1(M)$, we have $I(M) = I(E) = z^{G_W}$. Thus if $i \in I(S_H) - z^{\mathcal{F}}$ then $i \in S_H - M$. But by 5.3, $C_M(X) = 1$, so as i inverts a conjugate of X, we conclude from Exercise 2.8 in [A9] that M is transitive on $I(iM)$, so as there is an involution in $T - T_1$ and K has one class of involutions, the claim follows.

By the claim and 5.4.9, $j^{\mathcal{F}} \cap S_H = \varnothing$. Then as Q is a complement to H in G_W and Q is cyclic, it follows from Lynd transfer [Ly] that $Q \cap \mathfrak{foc}(\mathcal{F}) = 1$. Then as $M \leq O^2(M)$ and $i \in S_H - M$ is in $z^{\mathcal{F}}$, it follows that S_H is Sylow in $\mathcal{E} = O^2(\mathcal{F})$.

Let H_z be the model in G for $C_{\mathcal{E}}(z)$. Then $V_H = V \trianglelefteq H_z$ is isomorphic to D_8Q_8 and we saw during the proof of 5.9 that $C_{S_H}(\tilde{V}) = V$, so $V = O_2(H_z)$. Then $H_z^* = H_z/V \leq Out(V) = O(\tilde{V}) \cong O_4^-(2) \cong S_5$ with Sylow group $S_H^* \cong E_4$ and $O_2(H_z^*) = 1$. It follows that $H_z^* = \Omega(\tilde{V}) \cong A_5$. In particular $N_{H_z}(S_H) = S_HY$ where $Y \cong \mathbf{Z}_3$ centralizes $\tilde{E}$, and hence also E. Then Y acts on $M = C_{S_H}(E)$. As $M \in \mathcal{E}^{fc}$, $N_{\mathcal{E}}(M)$ has a model G_M, and we may take $MY = C_{G_M}(E)$ and take $X \leq G_M$. Then X acts on $C_{G_M}(E)$ so we may take Y to centralize X in G_M. Let $\mathcal{D}$ be the set of subgroups of XY of order 3; then $M = \prod_{D \in \mathcal{D}} C_M(D)$ with $E = C_M(Y)$ and $C_M(X) = 1$. Hence for some $D \in \mathcal{D} - \{X, Y\}$, we have $C_M(D) \neq 1$, so as $C_E(D) = 1$, $C_M(D) - E$ contains an involution, contradicting $E = \Omega_1(M)$.

(5.13) *(1) $V_H \trianglelefteq S$.*

(2) $C_S(\tilde{V}_H) = O_2(G) = V_HV_0$, where $V_0 = C_S(V_H) = \langle z_0j, ha_0 \rangle$, $a_0 \in aQ$, and $h \in V_H$.

(3) $V_0 \cong D_8$ or Q_8 and either $O_2(G) = C_G(\tilde{V}_H)$ or $V_0 \cong Q_8$ and $C_G(\tilde{V}_H) = V_HV_1$ where $V_1 = C_G(V_H) \cong SL_2(3)$.

(4) j_0 induces the transvection on $\tilde{V}_H$ with axis $\tilde{T}_2$ and center $\tilde{Z}_0$.

(5) j_0 centralizes jz_0 while $[a_0, j_0] = j$.

(6) $S/C_S(\tilde{V}_H) \cong D_8$.

Proof. First $V_H = V \cap H \trianglelefteq S$ as $H \trianglelefteq G_W \geq S$. Thus (1) holds. Part (4) follows from 5.9.1.

Let $R = O_2(G)$; as $V = W(E)$, we have $R \leq C_G(\tilde{V}) \leq C_G(\tilde{V}_H)$. Let $F = C_S(\tilde{V}_H)$ and $S^+ = S/F$. From 5.8 and from 5.9 and its proof, $S \cap H = V_H\langle y, t_1\rangle$ where $t_1 \in T_1 - Z_0E$, and $[y, v] = b$ and $[t_1, v] = v_1 \in Q_2 - \langle v\rangle$. In particular from these last two equalities, $\langle y, t_1\rangle^+$ is a 4-group which contains no transvection on $\tilde{V}_H$ with axis $\tilde{T}_2$.

Next j and z_0 centralizes T_2 with $[j, b] = [z_0, b] = z$ by 5.9, so $jz_0 \in V_0$ by 5.9.3. Then as $z_0 \in V_H$, we have $j \in F$.

Let $a_1 \in aQ$ and $G^*_W = G_W/E$. Then by 5.4.10, a_1 centralizes X, $\tilde{T}_2$, and T^*_1, so $C_{M^*}(a_1) = M^*$ or T^*_1 using 5.8. In the latter case $[a_1, b] \in T_1 - E$ so $a_1 \notin F$ as $b \in V_H$ by 5.9.3. Indeed a^+_1 is a transvection on $\tilde{V}_H$ with axis $\tilde{T}_2$ and center $\tilde{Z}_0$. Hence by (4), replacing a_1 by a suitable member of aQ, we may assume the former case holds.

In the former case $[b, a_1] \in E$, so as a_1 centralizes $\tilde{T}_2$, we conclude from 5.9.3 that either $a_1 \in F$ or a_1 induces a transvection on $\tilde{V}_H$ with center $\tilde{E}$, a contradiction as the center of a transvection is nonsingular. Hence $a_1 \in F$. Also $C_{M^*}(j_0a_1) = T^*_1$ and as j and a_1 are in F, also $ja_1 \in F$. On the other hand (5) follows from 5.4.10. As a Sylow 2-subgroup of $O^\epsilon_4(2)$ is isomorphic to D_8, it follows that (6) holds and we can make a choice of $a_0 \in aQ$ and $h \in V_H \cap aC_S(V_H)$ such that $F = V_HV_0$ with $V_0 = C_S(V_H) = \langle z_0j, ha_0\rangle$.Thus (2) holds.

Finally h centralizes jz_0 and a_0 inverts jz_0, so ha_0 inverts jz_0 and hence $V_0 \cong D_8$ or Q_8 is Sylow in $C_G(V_H)$ with $Z = Z(C_G(V_H))$. So as $F^*(G) = R$ it follows that either $V_0 = C_G(V_H)$ and $F = C_G(\tilde{V}_H) = R$, or $C_G(V_H) \cong SL_2(3)$. This proves (3) and completes the proof of the lemma.

(5.14) *(1)* $R = O_2(G) = V_HV_0 = C_G(\tilde{R})$.

(2) $R \cong D^3_8$.

(3) $G^+ = G/R \leq O(\tilde{R}) \cong O^+_6(2) \cong S_8$ *is faithful on* $\tilde{R}$ *with* $O_2(G^+) = 1$.

(4) $S^+ \cong D_8$ *and* $W^+ \cong E_4$ *with* $S^+ = N_{G^+}(W^+)$ *and* W^+ *is quadratic on* $\tilde{R}$.

Proof. By 5.13.2, $C_S(\tilde{V}_H) = R = V_HV_0$. By 5.13.3, R is extraspecial of order 2^7, so $R \cong D^3_8$ or Q^3_8, and $C_G(\tilde{R}) = R$. Thus (1) holds.

From 5.7 and 5.13, $A = E\langle j, a_0\rangle \leq R$, so as $A \cong E_{16}$ we conclude $m(R) = 4$. Then as R is D^3_8 or Q^3_8, (2) follows.

As $R = O_2(G)$, $O_2(G^+) = 1$. As $R = C_G(\tilde{R})$ by (1), G^+ is faithful on $\tilde{R}$ and then (2) implies (3). We saw that $R = C_S(\tilde{V}_H)$, so 5.13.6 implies that $S^+ \cong D_8$. From the proof of 5.13, $W^+ = \langle j^+_0, t^+_1\rangle \cong E_4$. As W is weakly closed in S and $C_{G_W}(z) = S$,

$N_{G^+}(W^+) = S^+$. As W is an abelian normal subgroup of S, W is quadratic on $\tilde{R}$, completing the proof of (4).

(5.15) *Let $R = O_2(G)$ and $G^+ = G/R$. Then either*

(1) $G^+ \cong O_4^+(2)$ preserves a decomposition $R = R_1 * R_2$ of R with $R_1 = R \cap O^2(G) \cong Q_8^2$ and $R_2 = C_R(R_1) \cong D_8$.

(2) There exists $R_2 \trianglelefteq G$ with $R_2 \cong Q_8$.

Proof. By 5.14, $G^+ \leq O(\tilde{R})$ with $\tilde{R}$ the natural module for $O(\tilde{R})$ regarded as $Sym(\Omega)$, where $\Omega = \{1, \dots, 8\}$. Further $O_2(G^+) = 1$ and $S^+ \cong D_8$ with $E_4 \cong W^+$, $S^+ = N_{G^+}(W^+)$, and W^+ quadratic on $\tilde{R}$. We conclude from the subgroup structure of S_8 that one of the following holds:

(a) G^+ preserves the partition $\{\Theta, \Omega - \Theta\}$ of Ω, where $\Theta = \{1, 2, 3\}$, and G^+ contains an S_5-subgroup.

(b) $G^+ = N_{G^+}(Y^+)$ for $E_9 \cong Y \in Syl_3(G)$.

(c) $G^+ \cong S_5$ stabilizes a nonsingular point $\mathfrak{n}$ of $\tilde{R}$ and acts indecomposably on $\tilde{R}$ with $\mathfrak{n}^\perp/\mathfrak{n}$ the $L_2(4)$-module for G^+.

In case (a), (2) holds with R_2 the preimage in R of $\langle\Theta\rangle \leq \tilde{R}$. In case (b), (1) holds with $R_1 = [R, Y]$ and $R_2 = C_R(Y)$. Finally in case (c), W^+ is not quadratic on $\tilde{V}$, a contradiction.

Theorem 5.16. *Hypothesis 5.1 is never satisfied.*

Proof. Assume Hypothesis 5.1. By 5.15, one of the two cases in that lemma holds. In case (2) let $\Omega = \{R_2\}$. In case (1), $O^2(G) = G_1 * G_2$ with $G_i \cong SL_2(3)$ and G_1 and G_2 conjugate in S; in this case let $\Omega = \{O_2(G_1), O_2(G_2)\}$. Then in either case, $\tau = (\mathcal{F}, \Omega)$ is a quaternion fusion packet. As in [A6], let $\mathcal{F}^\circ$ be the normal closure in $\mathcal{F}$ of Ω and $\tau^\circ = (\mathcal{F}^\circ, \Omega)$, so that τ° is also a quaternion fusion packet.

As $\mathcal{C}$ is the normal closure of z in $\mathcal{F}_j$, $\mathcal{C} \leq \mathcal{F}^\circ$. Then by E-balance, $\mathcal{C} \leq E(\mathcal{F}^\circ)$.

Suppose case (1) holds. Then the 2-fusion system $\mathcal{E}_i$ of G_i is a subnormal subsystem of $\mathcal{F}_z^\circ$, so as $G_i \cong SL_2(3)$ and $\Omega = \Omega(z)$ is of order 2, we conclude from the Main Theorem of [A6] that $\mathcal{F}^\circ$ is the 2-fusion system of $L = P\Omega_n^\epsilon(3)$ for $5 \leq n \leq 7$ or $(n, \epsilon) = (8, -1)$. Let S° be Sylow in $\mathcal{F}^\circ$; as $m(S^\circ) \leq 4$, it follows that $n = 5$ or 6 and $m(S^\circ) = 4$. Then as $m(S) = 4$, $O_2(\mathcal{F}) = 1$, so $\mathcal{F}$ is realized by a subgroup Ξ of $Aut(L)$. As $\mathcal{C} \in \text{Comp}(C_{\mathcal{F}}(j))$, it follows from the 2-local structure of Ξ that L is $U_4(3)$. Then as $|S| = 2^{10} = |Aut(L)|_2$, we conclude that $\Xi = Aut(L)$. This is a contradiction as $m(\Xi) = 5$.

Therefore case (2) holds. Here as $|\Omega| = 1$ and $E(\mathcal{F}^\circ) \neq 1$, it follows from Theorem 5 in [A6] that one of cases (3)-(7) of that theorem hold. However in none of these cases is there an involution j in $Aut(E(\mathcal{F}^\circ))$ whose centralizer has a component isomorphic to $\mathcal{C}$.

Section 6. $K(\mathcal{C}) \cong U_3(3)$ with $|Q| = 2$

In this section we assume the following hypothesis:

Hypothesis 6.1. Hypothesis 4.1 holds with $Q = Q(j)$ of order 2 for some $j \in \mathcal{J}(\mathcal{C}) \cap \mathcal{F}^f$.

Notation 6.2. Continue Notation 4.2 and the notation in paragraph two of Notation 5.2. However in this section define $W = QE$.

(6.3) *(1) $\mathcal{F}_j$ is tamely realized by a finite group G_j with $F^*(G_j) = Q \times K$. Indeed either $G_j = QK$ or $|G_j : QK| = 2$ with $G_j = QK\langle a\rangle$, where $a^2 \in Q$ and $C_K(a) \cong S_4$.*

(2) Either $m(S) = 3$ or $|G_j : KQ| = 2$, G_j splits over KQ, and $m(S) = 4$.

Proof. See the proof of the corresponding parts of 5.3.

(6.4) *$j \notin Z(S)$.*

Proof. Assume otherwise and let $S_0 = \mathfrak{foc}(\mathcal{F})$. By I.7.4 in [AKO] there is $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ with S_0 Sylow in $\mathcal{F}_0$.

By hypothesis, $S = S_Q$, so $\Omega_1(Z(S)) = QZ$ with $Z = QZ \cap [S, S]$. Hence $j \notin S_0$ by I.8.5 in [AKO]. As $K = O^2(K)$ and T is Sylow in K, $T \leq S_0$. Therefore from 6.3.2 either $S_0 = T$ or $G_j = Q \times K_0$, where $K_0 = K\langle a\rangle$, and $S_0 = T\langle a\rangle$.

Assume $S_0 = T$. Then $S_0 \cong \mathbf{Z}_4$ wr $\mathbf{Z}_2$ and $\mathcal{C} = \mathcal{F}_T(K) \leq \mathcal{F}_0$ with $\mathcal{C} = \langle O^2(Aut(T_2)), O^2(Aut_{\mathcal{C}}(T_1))\rangle$ Let $\Sigma = \{T_1, T_2\}$; clain $\Sigma = \mathcal{F}_0^e$. For let $P \in \mathcal{F}_0^e$ and $\mathcal{E} = N_{\mathcal{F}_0}(P)$. If $z \in Z(\mathcal{E})$ then $P = T_2$, so we may assume $F = \langle z^{\mathcal{E}}\rangle \neq Z$. But from 5.2, $T_2 = \Omega_1(T)$, so $m(T) = 2$ and $E^{\mathcal{C}}$ is the set of 4-subgroups of T, with E the fully normalizes 4-subgroup. Hence $F = E$, so $\mathcal{E} = N_{\mathcal{C}}(E)$ and $P = T_1$, establishing the claim. By the claim $\mathcal{F}_0^e = \{T_1, T_2\}$ and $O^2(Aut_{\mathcal{F}_0}(T_i)) = O^2(Aut_{\mathcal{C}}(T_i))$, so $\mathcal{F}_0 = \mathcal{C}$, contrary to 4.1.2.

Therefore $S_0 = T\langle a\rangle$. Now $Z = \Omega_1(Z(S_0))$, so 1.7 implies that $C_{\mathcal{F}_0}(z)$ is constrained. Thus $C_{\mathcal{F}_0}(z)$ has a model L. Let $R = O_2(L)$. Now $C_{K_0}(z) \leq L$, so $R \leq O_2(K_0) = R_0 \cong Q_8^2$, with $R_0 = T_2\langle a\rangle$. Next $E \leq Z_2(S_0) \leq R$, so $T_2 = \langle E^{C_K(z)}\rangle \leq R$. Hence $R = T_2$ or R_0. But as $R_0 \leq C_{S_0}(T_2/Z)$, it is not the case that $R = T_2$, so $R = R_0$.

Next as $\mathcal{F}_0 = O^2(\mathcal{F}_0)$ and $|S_0 : T| = 2$ with $z^{\mathcal{C}}$ the set of involutions in T, by Lynd transfer there is $\alpha \in \mathfrak{A}(a)$ with $a\alpha = z$. Then $\mathcal{E} = C_{\mathcal{F}}(\langle j, a\rangle) = \langle j, a\rangle \times \mathcal{D}$, where $\mathcal{D}$ is the 2-fusion system of $D = C_K(a) \cong S_4$, so $\mathcal{E}\alpha^* = C_{\mathcal{F}}(\langle j\alpha, z\rangle = \langle j\alpha, z\rangle \times \mathcal{D}\alpha^*$. Then working in L, $C_L(j\alpha) = \langle j\alpha, z\rangle \times D_0$ with $D_0 \cong S_4$. By the Thompson $A \times B$-Lemma, $E\alpha = O_2(D_0) \leq R$. But for $X \in Syl_3(C_K(z))$, $[R, X] = Q_2 \cong Q_8$, so a Sylow 3-subgroup Y of D_0 is not contained in RX, and hence $RXY = L_1 * L_2$, where $L_i \trianglelefteq L$ with $L_i \cong SL_2(3)$.

Let $\Omega = \{O_2(L_1), O_2(L_2)\}$. Then $\tau = (\mathcal{F}_0, \Omega)$ is a quaternion fusion packet, and by the Main Theorem of [A6], $\tau^\circ = (\mathcal{F}_0^\circ, \Omega)$ is a Lie packet. Then by Theorem 5 in [A6], $\mathcal{F}_0^\circ$ is the 2-fusion system of $G_2(3)$. This is a contradiction as that system admits no involution j whose centralizer has a component isomorphic to $\mathcal{C}$.

(6.5) *(1)* $Z = \Omega_1(Z(S))$.

(2) $\mathcal{F}_z$ *is constrained, so* $\mathcal{F}_z$ *has a model* G.

(3) $z \notin j^{\mathcal{F}}$.

Proof. Part (1) follows from 6.3.1 and 6.4. Then (1) and 1.7 imply (2). As $\mathcal{C}$ is a component of $\mathcal{F}_j$, (2) implies (3).

Notation 6.6. Let $\alpha \in \mathfrak{A}(W)$ and S_W be Sylow in $N_{\mathcal{F}}(W\alpha)$. As $W \trianglelefteq S_Q$, we have $j\alpha \in \mathcal{F}^f$, but it is not clear that $T\alpha \in \mathcal{F}^f$. However in the next lemma we find that $W \in \mathcal{F}^f$, so we may take $\alpha = 1$. During parts of the proof of that lemma, we abuse notation and write W, j etc for $W\alpha, j\alpha$ etc.

(6.7) *(1)* $N_{\mathcal{F}}(W\alpha)$ *is constrained, so* $N_{\mathcal{F}}(W\alpha)$ *has a model* G_W.

(2) $j^{\mathcal{F}} \cap W = W - E$ *is of order 4 and* $z^{\mathcal{F}} \cap W = E^\#$.

(3) $Aut_{\mathcal{F}}(W)$ *is the stabilizer in* $GL(W)$ *of* E.

(4) $E\alpha$ *and* $(QT_1)\alpha$ *are normal in* G_W.

(5) If $G_j \neq KQ$ *then* $G_j = Q \times K_0$, *where* $K_0 = K\langle a\rangle \cong Aut(K) \cong G_2(2)$ *with a an involution inducing an outer automorphism on* K *with* $C_K(a) \cong S_4$, *and a inverts* T_1. *Moreover* $C_{G_W}(W\alpha) = (QT_1\langle a\rangle)\alpha$.

(6) If $G_j = KQ$ *then* $C_{G_W}(W\alpha) = (T_1Q)\alpha$.

(7) $W \in \mathcal{F}^f$, *so we may take* $\alpha = 1$.

Proof. In much of this proof, we adopt our convention from 6.6 of suppressing α. As $z \in W$ and $\mathcal{F}_z$ is constrained by 6.5.2, (1) holds and we can regard $N_{G_j}(W)$ as $C_{G_W}(j)$.

Next $N_K(W)$ has three orbits $\{j\}$, $E^\#$, and $jE^\#$ on $W^\#$ of length 1,3,3, respectively. Set $S_1 = N_S(S_Q)$; by 6.4 and 6.5, for $s \in S_1 - S_Q$ we have $j^s = jz$. Further s acts on the unique abelian subgroup QT_1 of S_Q of order 32, and hence also on $\Phi(QT_1) = E$, and on $\Omega_1(QT_1) = W$. Now (2) and (3) follow, and (2) implies E is normal in G_W. Further $C_{G_W}(W) = C_{G_j}(W) = QT_1$ if $G_j = KQ$ and $QT_1\langle a\rangle$ otherwise, so (4) and (6) hold in the first case, and we treat the second in the next paragraph.

Assume $G_j \neq KQ$; then by 6.3.1, $G_j = KQ\langle a\rangle$ where a induces an outer automorphism on K with $C_K(a) \cong S_4$ and $a^2 \in Q$. In particular a inverts T_1 and then (5) holds if $a^2 = 1$, as does (4) as QT_1 is the unique abelian subgroup of $QT_1\langle a\rangle$ of index 2.

So assume $a^2 = j$. Then a inverts T_1Q, so for each $d \in T_1Q$, $(ad)^2 = a^2d^ad = j$. Therefore G_W centralizes $j\alpha$, contrary to (3).

Finally we prove (7). Let $S_2 = N_S(S_1)$ and $\tilde{S} = S/Z$. Recall we showed that s acts on E and on $ZQ = \Omega_1(Z(S_Q))$. Also s acts on $O_2(O^2(C_{G_j}(z))) = Q_2$ and hence on the projection Z_0 of E on $C_{S_Q}(Q_2)$. Then $Z(\tilde{S}_1) = \tilde{Z}_0\tilde{W}$, so $W = \Omega_1(Z_0W) \trianglelefteq S_2$. Then by (2), $|S_2 : S_Q| = 4 = |j^{\mathcal{F}} \cap W| = |S_W : S_Q\alpha|$, establishing (7).

(6.8) *E is the unique normal 4-subgroup of S.*

Proof. Let $E_4 \cong U \trianglelefteq S$. By 6.5.1, $Z \leq U$ so $U = Z\langle u\rangle$ for $u \in U - Z$ and $|S : C_S(u)| = 2$ with $Z = [S, U]$. Thus as $Z \leq W$ we have $U \leq N_S(W) = S_W$ and $|S_W : C_{S_W}(u)| \leq 2$. Let $R = C_{G_W}(E)$, so that $R = O_2(G_W)$ by 6.7. By 6.7.4, $QT_1 \trianglelefteq G_W$. If $u \notin R$ then $[E, u] \neq 1$ so $|T_1Q : C_{T_1Q}(u)| > 2$, a contradiction. Thus $U \leq R$.

Let $X = \langle x\rangle \in Syl_3(G_W)$ and assume $u \notin C_R(W)$; then $[u, j] = z$, so $[u^x, j] = z^x$. Then U and U^x are normal in R with $U \cap U^x = Z \cap Z^x = 1$, so $UU^x = U \times U^x \cong E_{16}$. Therefore as $m(S) = m(S_Q)$, $G_j \neq KQ$ by 6.3.1, so a exists as in 6.7.5 and $m(S) = 4$. We may choose a to centralize X, so if $[a, U] = 1$ then a centralizes UU^x, contradicting $m(S) = 4$. Therefore $[a, u] = z$, so aj centralizes U, for the same contradiction.

Hence $u \in C_R(W)$. If $G_j \neq KQ$ then for $d \in QT_1a$, $|T_1 : C_{T_1}(d)| = 4$; therefore in any case we have $u \in T_1Q$, so $u \in \Omega_1(T_1Q) = W$. Then by 6.7.2 either $u \in j^{\mathcal{F}}$ or $U = E$. But in the former case $|S : C_S(j)| = 2$, contrary to 6.7.2. This completes the proof.

Notation 6.9. Let $H = O^2(G_W)T \leq G_W$ and set $M = [O_2(G_W), O^2(G_W)]$. Let $X \in Syl_3(N_K(T_1))$, where $N_K(T_1)$ is regarded as a subgroup of G_W, and hence also a subgroup of H.

By 6.8, $E \trianglelefteq S$ and by 6.5.2, $\mathcal{F}_z$ is constrained with model G. Thus we can form $V = W(E) = \langle E^G\rangle \trianglelefteq G$. Set $\tilde{G} = G/Z$ and $V_H = V \cap H$.

Let $v \in Q_2 - Z_0E$. By 6.7.3 there is $y \in M - C_{G_W}(W)$ with $b = [y, v] \notin C_{G_W}(W)$. Thus by parts (5) and (6) of 6.7, $b \notin WT_1$.

(6.10) *(1)* $T_2 \leq V_H$.

(2) $b \in V \cap M \leq V_H$.

(3) Either $|b| = 2$ *and* b *centralizes* v *or* $|b| = 4$ *and* b *inverts* v.

Proof. By definition of V we have $E \leq V$, and then $T_2 = \langle E^{C_K(z)} \rangle \leq V$ as $V \trianglelefteq G$. Then as $T_2 \leq T \leq H$, (1) holds.

By (1), $v \in V$, so $b = [y, v] \in V$. Similarly as $y \in M \trianglelefteq G_W$, $b = [y, v] \in M \leq H$, so $b \in V_H$, establishing (2).

As $b = [y, v]$, we have $v^{-y} = bv^{-1}$, so $|bv^{-1}| = |v| = 4$; then as $\langle v, b \rangle \leq V$ with $\Phi(V) = Z$, (3) follows.

(6.11) *(1)* $T_1 \trianglelefteq G_W$.

(2) $M/E \cong E_{16}$.

(3) $H = MXT = MN_H(X)$ *with* $T_1 \leq M = O_2(H)$, $C_M(X) = 1$, *and* $N_H(X) \cong S_3$.

(4) $H/T_1 \cong S_4$.

Proof. Set $G_W^! = G_W/W$, $G_W^+ = G_W/T_1Q$, and $G_W^* = G_W/C_{G_W}(W)$. By 6.7.3, $G_W^* \cong S_4$ is the stabilizer in $GL(W)$ of E. Set $R = O_2(G_W)$; then $G_W = RN_{G_W}(X) = RXT$, $M = [R, X]$, and $H = MXT$.

Next as as X is irreducible on R^* with $C_{R^+}(X) = 1$ or $\langle a^+ \rangle$ for $G_j = KQ$ or $G_j \neq KQ$, respectively, either $R^+ = C_{R^+}(X) \times [R^+, X]$ or $R^+ \cong Q_8$ with $\langle a^+ \rangle = Z(R^+)$. The latter is impossible as $b \in R - C_R(W)$ with $b^2 \in Z$ by 6.10.2. Therefore $M^+ = [R^+, X] \cong E_4$ and $C_{M^!}(X) = 1$. Again $b \in M - W$ with $b^2 \in Z$ by 6.10.2, so $b^! \in M^! - T_1^!$ is an involution. Then as X is transitive on $(M^!/T_1^!)^\#$, it follows that $M^! \cong E_{16}$.

By 6.7.4, $E \trianglelefteq G_W$. Set $G_W^- = G_E/E$ and $M_0 = MQ$. Then $M_0^-/Q^- \cong M^! \cong E_{16}$, with $C_{M_0}(X) = Q$. Therefore either $\Phi(M_0^-) = 1$, so that $M_0^- = M^- \times Q^-$, or there is $d \in M_0$ with $d^{-2} = j^-$. In the former case $M \cap Q = 1$, so $|M| = 2^6$ and $T_1 = T \cap M \trianglelefteq G_W$ with $M/E \cong M^! \cong E_{16}$, so the lemma holds. In the latter case, $d^2 \in jE$, so as d centralizes d^2 and E, also d centralizes W, contrary to $C_M(W) \leq T_1Q$ by 6.7. This completes the proof of the lemma.

(6.12) $V_H = \langle T_2, b \rangle$.

Proof. Let $U = \langle T_2, b \rangle$; then $U \leq V_H$ by 6.10. As $\Phi(V_H) = Z$ we have $V_H \leq C_{S \cap H}(\tilde{U})$. But $S \cap H = M\langle v \rangle = U\langle t_1, y \rangle$, where $t_1 \in T_1 - Z_0E$. Finally $[y, v] = b$ and $[t_1, v] = v_1$, where $Q_2 = \langle v, v_1 \rangle$, so $U = C_{S \cap H}(\tilde{U})$, and hence $U = V_H$. Note that as T is $\mathbf{Z}_4 wr \mathbf{Z}_2$ we have $[t_1, v] = t_1^2 t_1 t_1^v$ with $t_1 t_1^v$ generating $Z(T) = Z_0$ and $t_1^2 \in E - Z$, so $v_1 \notin Z_0$ and hence $v_1 \in Q_2$.

(6.13) *Let $V_E = C_{V_H}(E)$. Then $V_E = V \cap M = Z_0E\langle b\rangle$ and one of the following holds:*

(1) b centralizes Z_0, $V_E \cong \mathbf{Z}_4 \times E_4$, and $V_H = Q_2Z(V_H)$ with $Z(V_H) \cong \mathbf{Z}_4 \times \mathbf{Z}_2$.

(2) b inverts Z_0, $|b| = 2$, and $V_H \cong D_8^2$.

(3) b inverts Z_0, $|b| = 4$, and $V_H \cong D_8Q_8$.

Proof. From 6.12, $V_E = V \cap M = \langle b\rangle C_{T_2}(E) = Z_0E\langle b\rangle$.

Let $U = V_H$. As $\Phi(U) = Z$, we have $Z(U) = Z(C_U(Q_2))$, so as $|U| = 2^5$, it follows that $Z(U) = Z$ or $|Z(U)| = 8$ and $Z(T_2) = Z_0 \le Z(U)$. Similarly $Z(U) \le Z(V_E)$, so $Z(U) = Z$ iff b inverts Z_0, in which case U is extraspecial of order 2^5, so that U is $D_8 * F$, where $F \cong D_8$ or Q_8 and $V_E \cong \mathbf{Z}_2 \times F$. Therefore (2) or (3) holds if b inverts Z_0, so we may assume b centralizes Z_0, so that $|Z(U)| = 8$. In this case (1) holds.

Notation 6.14. Let $G_W^* = G_W/E$; by 6.11.2, $M^* \cong E_{16}$. By 6.12, $v, b \in V_H$, so $[v, b] \in Z$. Hence v^* is an involution in $G_W^* - M^*$ centralizing b^*. By Baer-Suzuki we may assume v^* inverts X^*. Then as $N_H(X) \cong S_3$ by 6.11.3, $X^*\langle v^*\rangle = N_{H^*}(X^*)$ and $B^* = \langle b^{*N_{H^*}(X^*)}\rangle \cong E_4$, and we can choose y and b so that $y^* \in B^*$. Write B for the preimage of B^* in M. Note that $X^*\langle v^*\rangle$ induces $Sym(B^{*\#}) \cong S_3$ on $B^{*\#}$, so there is a conjugate i of v under X with cycle (b^*, y^*). Pick a generator x of X with $z^x = [j, y]$ and let $d = z_0^x$; thus $d^2 = z^x$ and i has cycle (z, z^x) on $E^{\#}$.

(6.15) *(1) $B \cong E_{16}$ or $\mathbf{Z}_4^2$ for $|b| = 2$ or 4, respectively.*

(2) $[b, y] = 1 = [d, z_0]$.

(3) $X^*\langle v^*\rangle$ induces $Sym(T_1^{*\#})$ on $T_1^{*\#}$ and i^* has cycle (z_0^*, d^*).

(4) $\{b^*y^*\} = B^{*\#} - \{b^*, y^*\}$, $\{zz^x\} = E^{\#} - \{z, z^x\}$, and $\{z_0^*d^*\} = T_1^{*\#} - \{z_0^*, d^*\}$.

Proof. If b is an involution then as X is transitive on $B^{*\#}$ and $E \le Z(B)$, it follows that $B \cong E_{16}$. If $|b| = 4$ then $E = \Omega_1(B)$, so $B \cong \mathbf{Z}_4^2$ by Lemma 4 in Higman [Hi]. This proves (1). Then B is abelian by (1), and T_1 is abelian so (2) follows. By 6.11, $T_1 \trianglelefteq G_W$ and $C_M(X) = 1$, so (3) follows using the fact that the map $t^* \mapsto t^2$ is a G_W-equivariant isomorphism of T_1^* with E. Part (4) is trivial.

(6.16) *One of the following holds:*

(1) $M \cong \mathbf{Z}_4^2 \times E_4$ is abelian.

(2) $[b, z_0] = [y, d] = 1$ and $[b, d] = zz^x = [y, z_0]$.

(3) b inverts z_0, y inverts d, and $[b, d] = zz^x = [y, z_0]$. If $|b| = 4$ then $|bd| = 2$.

Proof. First assume 6.13.1 holds; then $[b, z_0] = 1$, so applying the symmetry i of 6.14 and 6.15, also $[y, d] = 1$. By 6.15.2, $[b, y] = 1 = [z_0, d]$. Thus if $[b, d] = 1$, then as $M = \langle b, y, z_0, d\rangle$, it follows that M is abelian, so that (1) holds.

So assume $[b, d] \neq 1$. Applying the symmetry x and appealing to 6.15.4, $1 = [by, z_0 d] = [b, d][y, z_0]$, so $1 \neq [b, d] = [y, z_0]$. As $[b, d]^i = [y, z_0] = [b, d]$, applying the symmetry i, we conclude that $[b, d] = zz^x$, so that (2) holds.

Next assume 6.13.1 does not hold, so that b inverts z_0 and $\langle b, z_0\rangle \cong D_8$ or Q_8 by 6.13. Suppose that $[b, d] = 1$. Then applying i, also $[y, z_0] = 1$, so using 6.15.2, $M = \langle z_0, b\rangle \times \langle d, y\rangle$. But now the Krull-Schmidt Theorem contradicts the action of X on M.

Therefore $[b, d] \neq 1$. Hence from 6.15.4 and the symmetry x, we have

$$zz^x = [z_0 d, by] = [z_0, b][z_0, y][d, b][d, y] = zz^x[z_0, y][d, b],$$

so that $[z_0, y] = [d, b] \neq 1$, and then as above we have $[d, b] = zz^x$. Thus the first part of (3) holds. Finally suppose that $|b| = 4$. Then $(bd)^2 = b^2 d^2 [b, d] = zz^x \cdot zz^x = 1$, so indeed bd is an involution, completing the proof that (3) holds.

Notation 6.17. For $u \in M - E$ set $M(u) = \langle u^X, E\rangle$. Given an involution $s \in S_W$, set $M_s = C_M(s)$.

(6.18) *(1) If $u \in M - E$ then $M(u) \cong E_{16}$, $\mathbf{Z}_4^2$ for $|u| = 2, 4$, respectively.*

(2) $m(M) = 4$, $G_j \neq KQ$, and a is an involution.

Proof. Let $u \in M - E$. As $M^* \cong E_{16}$ and $C_M(X) = 1$ by 6.11, it follows that $M(u)^{*\#} = u^{*X}$. Therefore X acts on $M(u)$ and X is transitive on $M(u)^{*\#}$, so if u is an involution then as $E \leq Z(M)$ we have $\Phi(M(u)) = 1$ and hence (1) holds in this case. Similarly if $|u| = 4$ then $E = \Omega_1(M(u))$, so by Lemma 4 in [Hi] we have $M(u) \cong \mathbf{Z}_4^2$, completing the proof of (1).

If $E \neq \Omega_1(M)$ then $m(M) \geq 4$ by (1), so (2) follows from 6.3.2 in this case. Finally from 6.16, $E \neq \Omega_1(M)$; for example in cases (1) and (2) of 6.16, b or bz_0 is an involution by 6.13, while in case (3) either b or bd is an involution by 6.16.

(6.19) *(1) a inverts T_1 and centralizes $\tilde{T}_2$.*

(2) a inverts Z_0 and $z_0 j$.

(3) Either a centralizes M^* or $C_{M^*}(a) = [M^*, a] = T_1^*$.

(4) If $[M^*, a] = T_1^*$ then b inverts z_0 and $V_H \cong D_8^2$ or $D_8 Q_8$.

(5) If $[M^*, a] = T_1^*$ then a induces a transvection on $\tilde{V}_H$ with center $\tilde{Z}_0$ and axis $\tilde{T}_2$.

Proof. Part (1) follows from 6.7.5. Then (1) implies (2). As some $a_0 \in aT_1$ centralizes X and $C_M(X) = 1$, part (3) follows from (1).

Assume $[M^*, a] = T_1^*$; then $[b, a] \in T_1 - E$. By (1) and (2), a centralizes E and acts on Z_0E, so a act on $V_E = Z_0E\langle b\rangle$ by 6.13, and then as $[b, a] \in T_1 - E$ we have $[b, a] \in z_0E$. But then if b centralizes Z_0, if follows that $|b| \neq |b^a|$, a contradiction; for if $|b| = 2$ then $(b^a)^2 = (bz_0e)^2 = z_0^2 = z$, so $|b^a| = 4$, while if $|b| = 4$ then $(b^a)^2 = b^2z = 1$ as $b \in V_H$ by 6.12 and $\Phi(V_H) = Z$. Now (4) follows from 6.13. Further by (1), a induces a transvection on $\tilde{V}_H$ with axis $\tilde{T}_2$. Then as V_H is extraspecial, the center of a is $\tilde{T}_2^\perp = \tilde{Z}_0$, so (5) holds.

(6.20) *(1) b inverts z_0 and $V_H \cong D_8^2$ or D_8Q_8.*
(2) If $V_H \cong D_8^2$ then $[M^*, a] = T_1^*$.

Proof. Suppose first that a centralizes M^* and let $U = \langle j, a\rangle$. Then U centralizes $F = N_H(X) \cong S_3$ and $F^* = X^*\langle v^*\rangle$. As $C_M(X) = 1$, F^* has exactly three 2-dimensional irreducibles on M^*. As U centralizes M^* and E and as $T_1 = C_M(j)$ is inverted by a, $\{M_u^* : u \in U^\#\}$ is the set of these irreducibles. Then $M_u = M(c)$ for $c \in M_u - E$, so $M_u \cong E_{16}$ or $\mathbf{Z}_4^2$ by 6.18.1, and as $m(S) = 4$, the latter holds.

We show that, unless $V_H \cong D_8Q_8$, F acts on $M(i)$ for some involution $i \in M - E$. But by 6.18.1, $M(i) \cong E_{16}$, so $M(i) \neq M_u$ for any $u \in U^\#$, a contradiction as M_u^*, $u \in U^\#$, is the set of F-irreducibles. It suffices to choose $i^* \in C_{M^*}(v^*)$. But as $Z \leq E$, v^* centralizes V_E^* by 6.12, so it suffices to observe that when V_H is not D_8Q_8 there is an involution in $V_E - E$ by 6.13.

We've shown that if a centralizes M^* then $V_H \cong D_8Q_8$. In that case the lemma follows from 6.13, so by 6.19.3 we may assume $[M^*, a^*] = T_1^*$. Then (1) follows from 6.19.4 and (2) follows from 6.19.3.

(6.21) *(1) jz_0 centralizes V_H.*
(2) $V_H \trianglelefteq S_W$.
(3) $V_1 = C_G(\tilde{Q}\tilde{T}_2) = V_H\langle a, j\rangle$.
(4) $Y \in Syl_3(C_K(z))$ acts on V_1 with $Q_2 = [V_1, Y]$.
(5) Y acts on V_H with $Q_2 = [V_H, Y]$.
(6) If a centralizes M^* then $[a, b] \in Z$, so a centralizes $\tilde{V}_H$.

Proof. By 6.20.1, b inverts z_0, so as $[b, j] = z$ from 6.9, jz_0 centralizes b. Then (1) follows from 6.12 as j and z_0 centralize T_2.

Claim S_W acts on H. For from 6.7, $S_W = MS_Q$ and from 6.11, $H = MXT$. From 6.7, $M \trianglelefteq S_W$, while $[S_Q, XT] \leq XT$, so the claim holds.

As $V_H = V \cap H$ and S_W acts on V and H, (2) holds.

Next $V_1 = C_{V_1}(j)\langle b\rangle$ and $V_H = \langle T_2, b\rangle$ by 6.12, so to prove (3) it suffices to show $C_{V_1}(j) = T_2\langle a, j\rangle = V_2$. Now V_2 centralizes j, and $S_Q = V_2\langle d\rangle$ with $[\tilde{Q}_2, d] \neq 1$, so $C_{V_1}(j) \leq V_2$. Also $V_H \leq V_1$, so it remain to show $a \in V_1$. But from the structure of $Aut(K)$, a centralizes $\tilde{T}_2$, completing the proof of (3).

As Y acts on QT_2, Y acts on V_1. Further from the structure of $Aut(K)$, Y centralizes a suitable $a_0 \in aT_2$, so Y centralizes $\langle a_0, j, z_0\rangle = V_3$. Then as $|V_1 : Q_2V_3| = 2$ and Y acts on V_1, it follows that $Q_2 = [V_1, Y]$, establishing (4). Then (4) implies (5).

Assume a centralizes M^*. Then $[y, a] = e \in E$. Now a centralizes $\tilde{v}$ by (3), so by 8.5.4 in [FGT],

$$\tilde{b}^a = [\tilde{y}^a, \tilde{v}^a] = [\tilde{y}\tilde{e}, \tilde{v}] = [\tilde{y}, \tilde{v}]^{\tilde{e}}[\tilde{e}, \tilde{v}] = [\tilde{y}, \tilde{v}] = \tilde{b},$$

proving (6).

(6.22) *Let $S_H = S \cap H$, $L = \langle O^2(C_K(z)), M\rangle \leq G$ and $j_0 = jz_0$. Then*

(1) $S_H = T_1V_H\langle y\rangle$ with T_1V_H centralizing j_0 and y inverting j_0.

(2) L acts on V_H and $\langle j_0\rangle$.

(3) $Aut_L(\tilde{V}_H) = \Omega(\tilde{V}_H)$.

(4) If $[M^*, a] = T_1^*$ then a induces a transvection on $\tilde{V}_H$, so $C_{LS_W}(\tilde{V}_H) = V_H\langle j_0\rangle$. Otherwise $C_{LS_W}(\tilde{V}_H) = V_H\langle a, j\rangle$.

Proof. First M acts on V_H by 6.21.2. Also by 6.11.3, $S_H = MT$, while $M = T_1\langle b, y\rangle$ from 6.14, $V_H = \langle T_2, b\rangle$ by 6.12, and $T = T_1T_2$, so $S_H = T_1V_H\langle y\rangle$ with T_1V_H centralizing j_0 by 6.21.1, $[y, z_0] = zz^x$ by 6.16.3, and $[y, j] = z^x$ by 6.14. Therefore y inverts j_0, S_H acts on $\langle j_0\rangle$, and (1) holds.

Next $C_K(z)$ acts on V_H by 6.21.5, and centralizes j_0. Thus (2) holds.

Let $\Delta = Aut_L(\tilde{V}_H)$; then $\Delta \leq Out(V_H) = O(\tilde{V}_H) \cong O_4^{\epsilon}(2)$ by 6.20.1. Further $[M, V_E^*] = 1$ as M^* is abelian and $V_E \leq M$ by 6.13, so $Aut_M(\tilde{V}_H)$ centralizes $1 < \tilde{E} < \tilde{V}_E < \tilde{V}_H$, and hence $Aut_M(\tilde{V}_H) \leq \Omega(\tilde{V}_H)$, and of course $O^2(Aut_L(\tilde{V}_H)) \leq \Omega(\tilde{V}_H)$, so $\Delta \leq \Omega(\tilde{V}_H)$. Indeed $Aut_M(\tilde{V}_H) \cong M/V_E \cong E_4$ from the proof of 6.12, so as a Sylow 2-subgroup of $\Omega(\tilde{V}_H)$ is of order 4, it follows that $Aut_M(\tilde{V}_H)$ is Sylow in $\Omega(\tilde{V}_H)$. Then as y does not act on $\tilde{E}$, (3) follows, using 6.21.5 when $V_H \cong D_8^2$.

Next $C_L(\tilde{V}_H) = V_HC_L(V_H)$ and $C_L(V_H)$ centralizes Z_0E and acts on $\langle j_0\rangle$, so $[C_L(V_H), j] \leq Z$ and hence $C_L(V_H) \leq N_G(W) = S_W$.

By 6.21.3, $C_{LS_W}(\tilde{V}_H) \leq V_H\langle j, a\rangle$, with equality if a centralizes M^* by 6.21.6. On the other hand if $[M^*, a] = T_1^*$ then a induces an outer automorphism of Δ by 6.19.5, so in any event (4) holds.

(6.23) *Assume $V_H \cong D_8Q_8$. Then*

(1) a centralizes M^*.

(2) $C_{S_W}(V_H) = \langle j_0, a_0\rangle$ where $j_0 = jz_0$ and $\{a_0\} = aV_H \cap C_S(V_H)$.

(3) $\langle j_0, a_0\rangle \cong Q_8$.

Proof. Let $V_2 = \langle j_0\rangle$. Assume first that (1) fails. By 6.22.4, $C_L(\tilde{V}_H) \leq V_HV_2$ so by 6.21.1, $C_L(\tilde{V}_H)$ centralizes j_0. As V_2 is cyclic and L acts on V_2 by 6.22.2, $O^2(L)$ centralizes V_2. Then as $O^2(Aut_L(\tilde{V}_H)) = Aut_L(\tilde{V}_H)$ by 6.22.3, and as $C_L(\tilde{V}_H)$ centralizes j_0, it follows that L centralizes V_2. This is a contradiction as y inverts V_2 by 6.22.1. Thus (1) holds.

Part (2) follows from (1) and parts (1) and (4) of 6.22.

Next $a_0 = af$ for some $f \in V_H$, so $[a_0, j_0] = [af, j_0] = [a, j_0][f, j_0] = [a, j_0] = z$, as V_H centralizes j_0. Thus $U = \langle a_0, j_0\rangle \cong D_8$ or Q_8, so $V_HU \cong D_8^2Q_8$ or D_8^3. As $E_{16} \cong \langle a, j, E\rangle \leq V_HU$, we have $m(V_HU) = 4$, so (3) follows.

Notation 6.24. Set $j_0 = z_0j$, $G_0 = C_G(V_H)$, $R = O_2(G)$, and $R_0 = C_R(V_H)$. Let $S_0 = C_S(V_H)$.

(6.25) *Let $U_0 = C_{S_W}(V_H)$. Then*

(1) $C_{G_0}(j_0) = \langle j_0\rangle$.

(2) If $V_H \cong D_8^2$ then $G_0 = U_0 = \langle j_0\rangle$.

(3) If $V_H \cong D_8Q_8$ then $U_0 = \langle j_0, a_0\rangle \cong Q_8$ and either

(a) $G_0 = U_0 \cong Q_8$ or $G_0 \cong SL_2(3)$, or

(b) G_0 is quaternion or semidihedral of order at least 16, or $O^2(G_0) \cong SL_2(3)$ and $|G_0 : O^2(G_0)| = 2$ with S_0 quaternion or semidihedral.

(4) $R = V_HR_0$, and if V_H is D_8^2 then $R_0 = U_0$, while if V_H is D_8Q_8 then either $R_0 = S_0 = U_0$ or S_0 is quaternion or semidihedral of order at least 16 and $|S_0 : R_0| \leq 2$.

(5) $V = V_HV_0$, where $V_0 = C_V(V_H)$ and one of the following holds:

(i) $V = V_H$.

(ii) $V_0 \cong \mathbf{Z}_4$.

(iii) $V_H \cong D_8Q_8$ and either $V_0 = S_0 = U_0 \cong Q_8$ or $V_0 = R_0$ is of index 2 in S_0 and isomorphic to Q_8 or D_8.

Proof. If V_H is D_8^2 then $U_0 = \langle j_0\rangle$ by 6.20.2, 6.21.1, and 6.22.4. If $V_H \cong D_8Q_8$ then $U_0 = \langle a_0, j_0\rangle \cong Q_8$ by 6.23.

As $z_0 \in V_H$ and $j_0 = jz_0$, it follows that $C_{G_0}(j) = C_{G_0}(j_0)$. But $C_{G_0}(j) = C_G(QV_H) \leq C_G(QT_2) = Z_0Q$, and then using 6.21.1, $C_{Z_0Q}(V_H) = \langle j_0\rangle$, as $z_0 \notin Z(V_H)$, establishing (1).

Suppose $g \in G_0$ inverts j_0. Then from 6.12, g centralizes Z_0E, so $z = [j_0, g] = [j, g]$, and hence $g \in N_G(W) = S_W$ by 6.7. Therefore $g \in C_{S_W}(V_H) = U_0$. In particular if $V_H \cong D_8^2$ then $N_{G_0}(\langle j_0\rangle = \langle j_0\rangle$ and $G_0 = O(G_0)\langle j_0\rangle$ by the Z^*-Theorem, so (2) follows as $F^*(G) = R$. So assume for the moment that V_H is D_8Q_8. Then $U_0 = N_{G_0}(\langle j_0\rangle) \cong Q_8$. By (1) and 2.1, S_0 satisfies one of the conclusions of (3). Then as $F^*(G) = R$, (3) follows.

From 6.9, $\tilde{V} \leq Z(\tilde{R})$, so $R = V_HR_0$. As $F^*(G) = R$, $C_G(\tilde{R}) \leq R$, so $C_{G_0}(\tilde{R}_0) \leq R_0$. Also $R_0 \trianglelefteq G_0$. Now (4) follows from these remarks.

By (4), $V = V_HV_0$ with $\tilde{V}_0 \leq Z(\tilde{R}_0)$, $V_0 \trianglelefteq G_0$, and $\Phi(\tilde{V}_0) = 1$. Then (5) follows from (1)-(4).

(6.26) *Assume $V_H \cong D_8^2$ and let $\mathcal{D} = O^2(\mathcal{F})$. Then*

(1) $S = S_W$.

(2) $\mathcal{D} = F^*(\mathcal{F})$ is the 2-fusion system of $D \cong U_4(3)$ with S_H Sylow in D.

(3) $\mathcal{F}$ is the 2-fusion system of $D_0 \leq Aut(D)$ with $D_0/D \cong S/S_H \cong E_4$.

Proof. By parts (2) and (4) of 6.25, $R = V_HR_0$ and $R_0 = \langle j_0\rangle \cong \mathbf{Z}_4$, so $R \cong D_8^2 * \mathbf{Z}_4$. As $F^*(G) = R$, $C_G(\tilde{R}) = R$. Therefore $G/R \leq O(\tilde{R}) \cong S_6$.

By 6.20.2, $[M^*, a] = T_1^*$, so by 6.19.5, a induces a transvection on $\tilde{V}_H$ with center $\tilde{Z}_0$ and axis $\tilde{T}_2$. Let $L_W = LS_W$. By 6.22 and as a induces a transvection on $\tilde{V}_H$, $C_{L_W}(V_H) = R_0$, $C_{L_W}(\tilde{V}_H) = V_HR_0 = R$, and $L_W/R = O(\tilde{V}_H) \cong O_4^+(2)$. Therefore $L_0 = O^2(L_W) = L_1 * L_2$ where $L_i \cong SL_2(3)$ and S_W is transitive on $\{L_1, L_2\}$.

By 6.25.5, $V = V_H$ or $V = R$. Suppose first that $V = R$. Then as L_W is irreducible on $\tilde{V}_H$ and $V = \langle E^G\rangle$, it follows that G is indecomposable on $\tilde{R}$. Hence as $G/R \leq O(\tilde{R}) \cong S_6$ and $L_W/R \cong O_4^+(2)$ is maximal in S_6, we conclude that $G/R = O(\tilde{R})$, so in particular G is transitive on $I(R) - \{z\}$. This contradicts 6.5.3 as $j \in I(R)$ and $E^\# \subseteq I(R)$.

Therefore $V = V_H$, so V and $R_0 = C_R(V)$ are normal in G. Then as $L_W/R = Aut_G(\tilde{V})$ and $O^2(G)$ centralizes R_0, it follows that $L_0 = O^2(G)$. Hence $SL_2(3) \cong L_i$ is subnormal in G for $i = 1, 2$. Set $P_i = O_2(L_i)$, $\Omega = \{P_1, P_2\}$, and $\tau = (\mathcal{F}, \Omega)$. As L_i is subnormal in G, τ is a quaternion fusion packet. Let $\mathcal{F}^\circ$ be the normal closure of Ω in $\mathcal{F}$; by the Main Theorem of [A6], $\tau^\circ = (\mathcal{F}^\circ, \Omega)$ is the Lie packet of a simple group D of Lie type over the field of order 3. As $\Omega = \Omega(z)$ is of order 2 with S_W transitive on Ω, $D = P\Omega_n^\epsilon(3)$ for some $5 \leq n \leq 8$.

We may take $Q_2 = P_2$, so $\mathcal{C}$ is contained in the normal closure $\mathcal{F}^\circ$ of Q_2, so $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}^\circ}(j))$. It follows that D is $P\Omega_6^-(3) \cong U_4(3)$. As Q is of order 2, $C_S(\mathcal{F}^\circ) = 1$, so $\mathcal{F}^\circ = F^*(\mathcal{F})$. Then as $Out(D) \cong D_8$ is a 2-group, $\mathcal{F}^\circ = O^2(\mathcal{F}) = \mathcal{D}$.

Let S° be Sylow in $\mathcal{F}^\circ$; then $|S^\circ| = 2^7 = |S_H|$. But $S_H = V_H M$ with $V_H \leq L_0$ and $M \leq O^2(G_W)$, so $S_H \leq O^2(\mathcal{F})$ and hence $S_H = S^\circ$.

By 2.22 in [AO], $\mathcal{F} = \mathcal{F}_S(D_0)$ for some $D_0 \leq Aut(D)$. If $D_0 = Aut(D)$ then $m(S) = 5$, contrary to 6.3.2. Therefore $|S| < 2^{10}$, so as $|S_W| = 2^9$ we conclude that $S = S_W$. This completes the proof of the theorem.

In light of 6.20.1 and 6.26, for the next few lemmas we assume:

Hypothesis 6.27. Hypothesis 6.1 holds with $V_H \cong D_8Q_8$.

(6.28) $V \neq V_H$.

Proof. Assume otherwise; then $V = V_H$ and G_0 and R_0 are normal in G. By 6.25.4 either $R_0 \cong Q_8$ or S_0 is quaternion or semidihedral of order at least 16 and $|S_0 : R_0| \leq 2$. In the latter case by 6.25.3, either $G_0 = S_0$ or $R_0 \cong Q_8$. However if $G_0 = S_0$ then $R_0 = G_0$ as $G_0 \trianglelefteq G$. Then either R_0 is quaternion or R_0 is semidihedral and has a unique quaternion subgroup of index 2, which is then normal in G.

We've shown that G has a normal quaternion subgroup P contained in G_0. Set $\Omega = \{P\}$ and $\tau = (\mathcal{F}, \Omega)$; then τ is a quaternion fusion packet, as is $\tau^\circ = (\mathcal{F}^\circ, \Omega)$, where $\mathcal{F}^\circ$ is the normal closure of P in $\mathcal{F}$. Then $\mathcal{F}^\circ$ is described in Theorem 5 of [A6]. As $V = \langle z^{\mathcal{F}} \cap V\rangle$, V is contained in the Sylow group S° of $\mathcal{F}^\circ$, so $V \leq C_{S_0}(P)$. Therefore $m(S^\circ) \geq m(VP) = 4$, whereas $m(S^\circ) \leq 3$ in each of the conclusions of Theorem 5 in [A6].

(6.29) V_0 *is not* $\mathbf{Z}_4$.

Proof. Assume otherwise; then $V = V_H V_0 \cong Q_8^2 * \mathbf{Z}_4$, so $Aut_G(\tilde{V}) \leq O(\tilde{V}) = \mathfrak{S} \cong S_6$. We view $\mathfrak{S}$ as $Sym(\Omega)$, where $\Omega = \{1, \dots, 6\}$ and view $\tilde{V}$ as the core of the permutation module for $\mathfrak{S}$ on Ω. Thus $e_\theta \in \tilde{V}$ is an involution iff $|\theta| \equiv 0 \mod 4$.

Set $G_1 = C_G(\tilde{V})$ and $G^+ = G/G_1$. Thus $G^+ = Aut_G(\tilde{V})$. Let $\mathfrak{A} = Alt(\Omega) \leq \mathfrak{S}$. By 6.22.3, $L^+ = \Omega(\tilde{V}_H)$ is maximal in $\mathfrak{A}$ and irreducible on $\tilde{V}_H$. As $V = \langle E^G\rangle$ and L^+ is irreducible on $\tilde{V}_H$, it follows that G^+ is indecomposable on $\tilde{V}$, so as L^+ is maximal in $\mathfrak{A}$ we have $\mathfrak{A} \leq G^+$; that is $G^+ = \mathfrak{S}$ or $\mathfrak{A}$.

Next $\tilde{V}_0 \leq Z(\tilde{G})$, so $G_1 = C_G(\tilde{V}_H) = V_H G_0$. Then $C_{G_1}(V_0) = V_H V_1$, where $V_1 = C_{G_0}(V_0)$ is cyclic by 6.25.3. Hence $\Omega_2(C_{G_1}(V_0)) = V$.

As $\mathbf{Z}_4 \cong V_0 \trianglelefteq G$, V_0 is tightly embedded in $\mathcal{F}$, so $\mathfrak{W} = \langle V_0^{\mathcal{F}}\rangle$ is abelian by 3.3.5 in [A5]. Observe that $\mathfrak{W} \cap G_1 \leq \Omega_2(C_{G_1}(V_0)) = V$.

For $V' \in V_0^{\mathcal{F}}$ containing $i \in z^{\mathcal{F}}$, write V_i for V'; thus $V_i \leq \mathfrak{W}$. In particular for $e \in E - Z$, e centralizes V_0 and $Aut_{N_{\mathcal{F}}(C_S(E))}(E)$ is transitive on $E^{\#}$, so $V_e \leq \mathfrak{W}$. If $V_e^+ = 1$ then $V_e \leq V_H G_0$, so $e \in \Phi(V_e) \leq \Phi(V_H)G_0 = G_0$, a contradiction. Therefore $V_e^+ \neq 1$.

We may choose notation so that $\tilde{e} = e_{1,2,3,4}$ and S^+ stabilizes $\Lambda = \{\{1,2\},\{3,4\},\{5,6\}\}$. ∎ Now $O^2(C_{G^+}(\tilde{e}))$ centralizes V_e^+, so $V_e^+ = \langle(5,6)\rangle$. In particular $G^+ = \mathfrak{S}$ and $V_f \leq W$ for $f \in \mathfrak{F}$ consisting of those f with $\tilde{f}$ equal to e, e_{1256}, e_{3456}, and V_f^+ is generated by $(5,6)$, $(3,4)$, $(1,2)$, respectively; this follows since $V_f \in V_e^{\mathcal{F}}$ is contained in S. Let $\mathfrak{W}_F = \langle V_f : f \in \mathfrak{F}\rangle$ and $F = \langle \mathfrak{F}\rangle$; then $\mathfrak{W}_F \leq \mathfrak{W}$. But $C_{\mathfrak{S}}(\mathfrak{W}_F^+) = \mathfrak{W}_F^+$, so $\mathfrak{W} \leq \mathfrak{W}_F C_{G_1}(V_0 F)$. Then as $V_0\langle F\rangle$ is selfcentralizing in V, $\mathfrak{W} \cap V = V_0 F$ and $\mathfrak{W} = \mathfrak{W}_F V_0$. Also $C_S(\mathfrak{W}) = V_1\mathfrak{W}$, and as $\mathfrak{W}$ is weakly closed in S, $e \in z^{Aut_{\mathcal{F}}(C_S(\mathfrak{W}))}$. But if $V_1 \neq V_0$ then $z \in \Phi^2(V_1\mathfrak{W}) = \Phi^2(V_1)$, so $V_1 = V_0$. Hence $R = V_H R_0$ with $R_0 = U_0 = \langle j_0, a_0\rangle \cong Q_8$ by 6.23 and 6.25.

If $V_0 = \langle j_0\rangle$ then $j \in V$, a contradiction as G is transitive on $I(V) - \{z\}$ and $e \in z^{\mathcal{F}} \neq j^{\mathcal{F}}$. Thus we may take $V_0 = \langle a_0\rangle$.

If $U_0 \trianglelefteq G$ then as $U_0 \cong Q_8$, $\tau = (\mathcal{F}, \Gamma)$ is a quaternion fusion packet, where $\Gamma = U_0^{\mathcal{F}}$. This is contrary to Theorem 1 in [A6] and the structure of G. Therefore G is indecomposable on $\tilde{R}$ and we may view $\tilde{R}$ as the 6-dimensional permutation module on Ω for $G^+ = Sym(\Omega)$. But then $\tilde{j}$ is of weight 1, 3, or 5 in $\tilde{R}$, so $|C_G(j)|$ is divisible by 5 or 9, a contradiction.

(6.30) *Assume V_0 is isomorphic to Q_8 or D_8. Then*

(1) $j \notin \mathfrak{foc}(\mathcal{F})$.

(2) G is $V \cong D_8^3$ or $D_8^2 Q_8$ extended by $O(\tilde{V})$.

(3) j induces a transvection on $\tilde{V}$.

Proof. Assume otherwise; then $V = V_H V_0 \cong D_8^3$ or $D_8^2 Q_8$, for V_0 isomorphic to Q_8 or D_8, respectively. Thus $Aut_G(\tilde{V}) \leq O(\tilde{V}) \cong O_6^+(2)$ or $O_6^-(2)$, respectively.

Let $G_1 = C_G(\tilde{V})$ and $G^+ = G/G_1$, so that $G^+ = Aut_G(\tilde{V})$. Thus $G_1 = C_G(\tilde{V}_H) \cap C_G(\tilde{V}_0)$ with $C_G(\tilde{V}_H) = V_H G_0$ and $C_{V_H G_0}(\tilde{V}_0) = V_H C_{G_0}(\tilde{V}_0) = V_H V_0 = V$ by 6.25.5. Moreover $R = V_H R_0$ by 6.25.4, so $V = R$ by 6.25.5. Also by 6.25.5, either $V_0 = U_0 \cong Q_8$ or $|S_0 : V_0| = 2$ and $s_0 \in S_0 - V_0$ induces a transvection on $\tilde{V}$ with center $[\tilde{V}_0, s_0]$.

Claim G^+ is irreducible on $\tilde{V}$. For L^+ is irreducible on $\tilde{V}_H$, so if $1 \neq B < \tilde{V}$ is G^+-invariant then either $\tilde{V}_H \leq B$ or $B \leq \tilde{V}_0$. As $\tilde{E} \leq \tilde{V}_H$ and $V = \langle E^G\rangle$ the first case is out, so the second holds. Then G^+ acts on $B^\perp$ with $\tilde{V}_H \leq B^\perp$, for the same contradiction. The claim is established.

Next as L^+ is imprimitive on $\tilde{V}_H$, we conclude G^+ is imprimitive on $\tilde{V}_H$. But if $V_0 \neq U_0$ then s_0^+ induces a transvection on $\tilde{V}$, so $G^+ = O(\tilde{V})$ by [Mc]. Therefore G is transitive on $I(V) - \{z\}$, so $j \notin V$ by 6.5.3, and hence $j_0 \notin V_0$. Hence j_0^+ induces a transvection on $\tilde{V}$, so also j induces a transvection on $\tilde{V}$, so that $j \notin O^2(G)$. But then $j \notin \mathfrak{foc}(\mathcal{F})$ by Lyons tranfer for fusion systems 2.3 in [A8]. Hence the lemma holds in this case.

Thus we may assume $V_0 = U_0$, so $V \cong D_8^3$ and hence $G^+ \leq O(\tilde{V}) \cong O_6^+(2) \cong S_8$. We can view $\tilde{V}$ as the 6-dimensional section of the permutation module for $\mathfrak{S} = Sym(\Omega)$, where $\Omega = \{1, \dots, 8\}$. Then $G^+ \leq \mathfrak{S}$ is irreducible on $\tilde{V}$ with L^+ contained in the pointwise stabilizer in $\mathfrak{S}$ of $\theta = \{1, 2, 3\}$. As G^+ is irreducible on $\tilde{V}$ either G^+ is primitive on Ω or $F^*(G^+) = A_7$. In the former case as L^+ contains a 3-cycle, $F^*(G^+) = Alt(\Omega)$. But then in either case, G is transitive on $I(V) - \{z\}$ for our usual contradiction.

(6.31) *V_0 is not Q_8 or D_8.*

Proof. Assume otherwise. As $\mathcal{F}_z$ is constrained by 6.5.2, $N_{\mathcal{F}}(E)$ is also constrained, and hence has a model G_E. Proceeding in the spirit of 8.15 in [A9], we roughly determine the structure of G_E.

As $E \trianglelefteq G_W$ by 6.7.3, we may regard G_W as $N_{G_E}(W)$. Similarly we identify $N_G(E)$ with $C_{G_E}(z)$. In particular, $X \leq G_E$. Let $x \in X^\#$ and $\Xi = \langle V, V^x \rangle \leq G_E$. As $T_2 \leq V$ and $X \leq \langle T_2, T_2^x \rangle \leq G_W$, also $X \leq \Xi$. Let $B = C_G(E)$, $V_z = C_V(E)$, $P_0 = V_z V_z^x$, and $P = O_2(G_E)$. As X is irreducible on E, we have $P = O_2(B)$. Set $G_E^* = G_E/E$ and $G^+ = G/V$; then $G^+ \cong O_6^\epsilon(2)$ by 6.30.2.

As $V \trianglelefteq G$, we have $V_z \trianglelefteq B$, so V_z^x and P_0 are normal in B. Hence $P_0 \leq P$. Next $[V, E] = Z$, so $[V^x, E] = Z^x$ and hence $Aut_\Xi(E) = GL(E)$, so $G_E = B\Xi$. Also $[V, B] \leq V \cap B = V_z$, so $[\Xi, B] \leq P_0 \leq \Xi$ and hence $\Xi \trianglelefteq G_E$. As $N_G(E)$ is irreducible on V_z/E, it follows that $V_z \cap V_z^x = E$, so $P_0^* = V_z^* \times V_z^{x*}$. Then as $|O_2(C_{G^+}(\tilde{E}))| = 16 = |V_z^{x*}|$, it follows that $P_0 = P$. As $P^* = V_z^* \times V_z^{x*}$, we have $C_P(X) = 1$. Then $N_\Xi(X)$ is a complement to P in Ξ. Indeed as $\Xi \trianglelefteq G_E$, $N_{G_E}(X)$ is a complement to P in G_E by a Frattini argument. Also $N_{G_E}(X) = X_0 \times B_0$, where $X_0 = X\langle u \rangle$ for some involution $u \in V$ inverting X, and $B_0 = C_B(X) \cong O_4^\epsilon(2)$ is a Levi factor for the parabolic $C_{G^+}(\tilde{E})$ of $G^+ \cong O_6^\epsilon(2)$.

Observe that $j \in C_B(X) = B_0$, while by 6.30.3, $j+$ is a transvection in G^+ on $\tilde{V}$; therefore j is a transvection in B_0, so $C_{B_0}(j) = \langle j \rangle \times B_j$ with $B_j \cong S_3$. Then B_j centralizes $E\langle j \rangle = W$, a contradiction as $C_{G_W}(W)$ is a 2-group by 6.7.5.

Theorem 6.32. *Assume Hypothesis 6.1. Then* $j \notin \mathfrak{foc}(\mathcal{F})$ *and* $\mathcal{F}$ *is the 2-fusion system of* $D_0 \leq Aut(D)$ *with* $D \cong U_4(3)$ *and* $D/D_0 \cong E_4$*.*

Proof. By 6.20.1, $V_H \cong D_8^2$ or D_8Q_8. In the first case the theorem holds by 6.26, so we may assume the second case holds. Thus Hypothesis 6.27 is satisfied.

By 6.25.5, one of the three cases in that lemma holds. Cases (i) and (ii) are out by 6.28 and 6.29, while case (iii) is not satisfied by 6.31.

We close this section with a proof of the Main Theorem. So assume the hypothesis of the Main Theorem. Observe that this hypothesis implies Hypothesis 3.1. In addition we may assume further that conclusion (1) of the Main Theorem does not hold, so $\mathcal{C} \notin \text{Comp}(\mathcal{F})$. Let T be Sylow in $\mathcal{C}$; replacing $\mathcal{C}$ by $\mathcal{C}\gamma^*$ for $\gamma \in \mathfrak{A}(T)$, we may assume $T \in \mathcal{F}^f$. Therefore Hypothesis 4.1 is satisfied. By 4.4, we may assume $j \in \mathcal{J}(\mathcal{C})$ is fully centralized. Let $Q = Q(j)$ be as in 4.3.1.

If $|Q| > 4$ then $\mathcal{C}$ is standard in $\mathcal{F}$ by 4.7; but then 4.10 supplies a contradiction. Therefore $|Q| \leq 4$. Suppose $|Q| = 4$; then Hypothesis 5.1 is satisfied, and now Theorem 5.16 supplies a contradiction. This leaves the case $|Q| = 2$, where Hypothesis 6.1 holds. But now conclusion (2) of the Main Theorem holds by Theorem 6.32, completing the proof of the Main Theorem.

References

[A1] M. Aschbacher, *The Tits group as a standard subgroup*, Math. Zeit. **181** (1982), 229-252.

[A2] M. Aschbacher, *Generation of fusion systems of characteristic 2-type*, Invent. Math. **180** (2010), 225-299.

[A3] M. Aschbacher, *The generalized Fitting subsystem of a fusion system*, Memoirs AMS **209** (2011), 1–110.

[A4] M. Aschbacher, *Finite simple groups and fusion systems*, Groups St Andrews 2017 in Birmingham, London Math. Soc. Lecture Note Series 455, Cambridge Univ. Press, 2019, pp. 1–15.

[A5] M. Aschbacher, *On fusion systems of component type*, Memoirs AMS **257** (2019), 1–182.

[A6] M. Aschbacher, *Quaternion fusion packets*, preprint.

[A7] M. Aschbacher, *Walter's theorem for fusion systems*, preprint.

[A8] M. Aschbacher, *Fusion systems with 2-small components*, preprint.

[A9] M. Aschbacher, *Sporadic Groups*, Cambridge University Press, 1994.

[AKO] M. Aschbacher, R. Kessar, and B. Oliver, *Fusion Systems in Algebra and Topology*, Cambridge University Press, 2011.

[AO] M. Aschbacher and B. Oliver, *Fusion systems*, Bull. AMS **53** (2016), 555-615.

[FGT] M. Aschbacher, *Finite Group Theory*, Cambridge Univ. Press, 1986.

[GLS3] D. Gorenstein, R. Lyons, and R. Solomon, *The Classification of the Finite Simple Groups, Number 3*, Mathematical Surveys and Monographs, vol. 40, AMS, 1998.

[Hi] G. Higman, *Suzuki 2-groups*, Illionis J. Math. **7** (1963), 79–96.

[Ly] J. Lynd, *The Thompson-Lyons transfer lemma for fusion systems*, Bull. LMS **46** (2014), 1276–1282.

[Mc] J. McLaughlin, *Some subgroups of $SL_n(F_2)$*, Illinois J. Math. **13** (1969), 108–115.

PASADENA, CALIFORNIA 91125