

NON-REALIZABILITY OF THE PURE BRAID GROUP AS AREA-PRESERVING HOMEOMORPHISMS

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ABSTRACT. Let $\text{Homeo}_+(D_n^2)$ be the group of orientation-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. Nielsen realization problem for the braid group asks whether the natural projection $p_n : \text{Homeo}_+(D_n^2) \rightarrow B_n := \pi_0(\text{Homeo}_+(D_n^2))$ has a section over subgroups of B_n . All of the previous methods either use torsions or Thurston stability, which do not apply to the pure braid group PB_n , the subgroup of B_n that fixes n marked points pointwise. In this paper, we show that the pure braid group has no realization inside the area-preserving homeomorphisms using rotation numbers.

1. INTRODUCTION

Denote by D^2 the 2-dimensional disk. Let $\text{Homeo}_+(D_n^2)$ be the group of orientation-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. Denote by $B_n := \pi_0(\text{Homeo}_+(D_n^2))$. The Nielsen realization problem for B_n asks whether the natural projection

$$p_n : \text{Homeo}_+(D_n^2) \rightarrow B_n$$

has a section over subgroups of B_n . For the whole group B_n , this question has several previous results. Salter–Tshishiku [ST16] uses Thurston stability to show that B_n has no realization in $\text{Diff}_+(D_n^2)$ and the author [Che19] uses “hidden torsions” and Markovic’s machinery [Mar07] to show that B_n has no realization in $\text{Homeo}_+(D_n^2)$. Let $PB_n < B_n$ be the subgroup that preserves n marked points pointwise. The Nielsen realization problem for PB_n is widely open since the two methods in [ST16] and [Che19] fail to work and has no hope to repair. The following question is asked by [MT18, Question 3.12] and [ST16, Remark 1.4].

Problem 1.1 (Realization of pure braid group). *Does PB_n have realization as diffeomorphisms or homeomorphisms? In other words, does p_n have sections over PB_n ?*

Denote by $\text{Homeo}_+^a(D_n^2)$ the group of orientation-preserving, area-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. In this paper, we make a progress proving the following result.

Theorem 1.2. *The pure braid group cannot be realized as area-preserving homeomorphisms on D_n^2 for $n \geq 9$. In other words, the natural projection $p_n^a : \text{Homeo}_+^a(D_n^2) \rightarrow B_n$ has no sections over PB_n .*

We remark that the Nielsen realization problem is closely related to the existence of flat structures on a surface bundle. We refer the reader to [MT18] for more history and background.

Comparing with the method in [CM19]. The novelty of this paper is to provide a different ending towards [CM19]. The original ending is to use the fact that certain Dehn twist is a product of commutator in its centralizer. However, such structure does not hold in PB_n . Instead, we prove a stronger dynamical property about Dehn twists about non-separating curves. In the beginning of Section 4, we present an outline of the proof. Since this paper has a lot of overlap with [CM19], we omit or sketch many proofs to reduce redundancy.

Organization of the paper.

- In Section 2, we discuss rotation numbers;
- In Section 3, we discuss the pure braid group and the minimal decomposition theory;
- In Section 4, we give an outline of the proof and finish the argument.

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2. ROTATION NUMBERS OF ANNULUS HOMEOMORPHISMS

In this section, we discuss the properties of rotation numbers on annuli.

2.1. Rotation number of an area-preserving homeomorphism of an annulus. Firstly, we define the rotation number for geometric annuli. Let

$$N = N(r) = \{w \in \mathbb{C} : \frac{1}{r} < |w| < r\}$$

be the geometric annulus in the complex plane \mathbb{C} . Denote the geometric strip in \mathbb{C} by

$$P = P(r) = \{x + iy = z \in \mathbb{C} : |y| < \frac{\log r}{2\pi}\}.$$

The map $\pi(z) = e^{2\pi iz}$ is a holomorphic covering map $\pi : P \rightarrow N$. The deck transformation on P is $T(x, y) = (x + 1, y)$.

Denote by $p_1 : P \rightarrow \mathbb{R}$ the projection to the x -coordinate, and by $\text{Homeo}_+(N)$ the group of homeomorphisms of N that preserves orientation and the two ends. Fix $f \in \text{Homeo}_+(N)$, and $x \in N$, and let $\tilde{x} \in P$ and $\tilde{f} \in \text{Homeo}_+(P)$ denote lifts of x and f respectively. We define the translation number of the lift \tilde{f} at \tilde{x} by

$$(1) \quad \rho(\tilde{f}, \tilde{x}, P) = \lim_{n \rightarrow \infty} (p_1(\tilde{f}^n(\tilde{x})) - p_1(\tilde{x}))/n.$$

The rotation number of f at x is then defined as

$$(2) \quad \rho(f, x, N) = \rho(\tilde{f}, \tilde{x}, P) \pmod{1}.$$

The rotation number is not defined everywhere (see, e.g., [Fra03] for more background on rotation numbers). The closed annulus N_c is

$$N_c = \{\omega \in \mathbb{C} : \frac{1}{r} \leq |\omega| \leq r\},$$

For $f \in \text{Homeo}_+(N_c)$, the rotation and translation numbers are defined analogously.

Let A be an open annulus embedded in a Riemann surface (in particular this endows A with the complex structure). By the Riemann mapping theorem, there is a unique $N(r) = N$ and a conformal

map $u_A : A \rightarrow N$. For any $f \in \text{Homeo}_+(A)$ (the group of end-preserving homeomorphisms), we define the rotation number of f on A by

$$\rho(f, x, A) := \rho(g, u_A(x), N),$$

where $g = u_A \circ f \circ u_A^{-1}$.

We have the following theorems of Poincaré-Birkhoff and Handel about rotation numbers [Han90] (See also Franks [Fra03]).

Theorem 2.1 (Properties of rotation numbers). *If $f : N_c \rightarrow N_c$ is an orientation preserving, boundary component preserving, area-preserving homeomorphism and $\tilde{f} : P_c \rightarrow P_c$ is any lift, then:*

- (Handel) *The translation set*

$$R(\tilde{f}) = \bigcup_{\tilde{x} \in P_c} \rho(\tilde{f}, \tilde{x}, P_c)$$

is a closed interval.

- (Poincaré-Birkhoff) *If $r \in R(\tilde{f})$ is rational, then there exists a periodic orbit of f realizing the rotation number $r \bmod 1$.*

2.2. Separators and its properties. We let A continue to denote an open annulus embedded in a Riemann surface. Then A has two ends and we choose one of them to be the left end and the other one to be the right end. We call a subset $X \subset A$ *separating* (or *essential*) if every arc $\gamma \subset A$ which connects the two ends of A must intersect X .

Definition 2.2 (Separator). We call a subset $M \subset A$ a *separator* if M is compact, connected and separating.

The complement of M in A is a disjoint union of open sets. We have the following lemma.

Lemma 2.3. *Let M be a separator. Then there are exactly two connected components $A_L(M)$ and $A_R(M)$ of $A - M$ which are open annuli homotopic to A and with the property that $A_L(M)$ contains the left end of A and $A_R(M)$ contains the right end of A . All other components of $A - M$ are simply connected.*

Proof. We compactify the annulus A by adding points p_L and p_R to the corresponding ends of A . The compactification is a two sphere S^2 . Moreover, M is a compact and connected subset of $S^2 - \{p_L, p_R\}$.

Now, we observe that every component of $S^2 - M$ is simply connected. Denote by Ω_L and Ω_R the connected components of $S^2 - M$ containing p_L and p_R respectively. Since M is separating we conclude that these are two different components. We define $A_L(M) = \Omega_L - p_L$ and $A_R(M) = \Omega_R - p_R$. It is easy to verify that these are required annuli. \square

We now prove another property of a separator. Let $\pi : \tilde{A} \rightarrow A$ be the universal cover.

Proposition 2.4. *Let $M \subset A$ be a compact domain with smooth boundary. Then $\pi^{-1}(M)$ is connected.*

Proof. Since M is a compact domain with boundary which separates the two ends of A , we can find a circle $\gamma \subset M$ which is essential in A (i.e. γ is a separator itself) (note that M has only finitely many boundary components). Denote by T the deck transformation of \tilde{A} . Thus, the lift $\pi^{-1}(\gamma)$ is a T -invariant, connected subset of \tilde{A} . Let C be the component of $\pi^{-1}(M)$ which contains $\pi^{-1}(\gamma)$. Then C is T invariant. We show $\pi^{-1}(M) = C$.

Let $p \in M$. Since M is a compact domain with smooth boundary, we can find an embedded closed arc $\alpha \subset M$ which connects p and γ . Let \tilde{p} be a lift of p and let $\tilde{\alpha}$ be the corresponding lift of α such that \tilde{p} is one of its endpoints. Then, the other endpoint of $\tilde{\alpha}$ is in $\pi^{-1}(\gamma)$, and this shows that $\tilde{p} \in C$. This concludes the proof. \square

Now we discuss an ordering on the set of separators.

Proposition 2.5. *Suppose $M_1, M_2 \subset A$ are two disjoint separators. Then either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$. Moreover, $M_1 \subset A_L(M_2)$ implies $M_2 \subset A_R(M_1)$.*

Proof. Since M_1 is connected it follows that M_1 is a subset of a connected component C of $A - M_2$. Since C is open, we know that there is a neighborhood N_1 of M_1 with smooth boundary such that $N_1 \subset C$ (It is elementary to construct such N_1). If C is simply connected, the cover $\pi^{-1}(C) \rightarrow C$ is a trivial cover. Let \tilde{C} be a connected component of $\pi^{-1}(C)$. By Proposition 2.4, the set $\pi^{-1}(N_1)$ is connected so it is contained in a single connected component of $\pi^{-1}(C)$. However, this contradicts the fact that $\pi^{-1}(N_1)$ is also translation invariant. Thus, either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$.

Suppose $M_1 \subset A_L(M_2)$. Then $A_L(M_1) \subset A_L(M_2)$ as well. On the other hand, by the first part of the proposition we already know that either $M_2 \subset A_L(M_1)$ or $M_2 \subset A_R(M_1)$. If $M_2 \subset A_L(M_1)$, then $A_L(M_2) \subset A_L(M_1)$. This shows that $A_L(M_1) \subset A_L(M_2)$ which implies that $M_2 \subset A_L(M_2)$. This is absurd so we must have $M_2 \subset A_R(M_1)$. \square

Definition 2.6. The inclusion $M_1 \subset A_L(M_2)$ is denoted as $M_1 < M_2$.

2.3. The rotation interval of an annular continuum and prime ends. Let $K \subset A$ be a separator (in literature also known as an *essential continuum*). We call K an *essential annular continuum* if $A - K$ has exactly two components. Observe that an essential annular continuum can be expressed as a decreasing intersection of essential closed topological annuli in A .

It is possible to turn any separator $M \subset A$ into an essential annular continuum. Let M be a separating connected set. By Lemma 2.3, we know that $A - M$ has exactly two connected annular components $A_L(M)$ and $A_R(M)$, and all other components of $A - M$ are simply connected. We call a simply connected component of $A - M$ a *bubble component*. Then the *annular completion* $K(M)$ of M is defined as the union of M and the corresponding bubble components of $A - M$.

Proposition 2.7. *Let $M \subset A$ be a separator. Then the annular completion $K(M)$ is an annular continuum.*

Proof. We can again compactify A by adding the points p_L and p_R , one at each end. The compactification is the two sphere S^2 . Then $A_L(M)$ and $A_R(M)$ are two disjoint open discs in S^2 ,

and $K(M) = S^2 - (A_L(M) \cup A_R(M))$. But the complement of two disjoint open discs in S^2 is connected. This proves the proposition. \square

Now let f be a homeomorphism of A that leaves an annular continuum K invariant. If μ is an invariant Borel probability measure supported on K , we define the μ -rotation number

$$\sigma(f, \mu) = \int_A \phi d\mu$$

where $\phi : A \rightarrow \mathbb{R}$ is the function which lifts to the function $p_1 \circ f - p_1$ on \tilde{A} (recall that $p_1 : \tilde{A} \rightarrow \mathbb{R}$ is the projection onto the first coordinate).

The set of f invariant Borel probability measures on K is a non empty, convex, and compact set (with respect to the weak topology on the space of measures). We define the *rotation interval* of K

$$\sigma(f, K) = \{\sigma(f, \mu) | \mu \in M(K)\}$$

which is a non-empty segment $[\alpha, \beta]$ of \mathbb{R} . The interval is non empty because there exists at least one f invariant measure, and it is an interval because the set of f invariant measures is convex.

The following is a classical result of Franks–Le Calvez [FC03, Corollary 3.1].

Proposition 2.8. *If $\sigma(f, K) = \{\alpha\}$, the sequence*

$$\frac{p_1 \circ f^n(x) - p_1(x)}{n}$$

converges uniformly for $x \in \pi^{-1}(K)$ to the constant function α . This implies that points in K all have the rotation number α .

The following theorem of Franks–Le Calvez [FC03, Proposition 5.4] is a generalization of the Poincaré–Birkhoff Theorem.

Theorem 2.9. *If f is area-preserving and K is an annular continuum, then every rational number in $\sigma(f, K)$ is realized by a periodic point in K .*

The theory of prime ends is an important tool in the study of 2-dimensional dynamics which can be used to transform a 2-dimensional problem into a 1-dimensional problem. Recall that we assume that A is an open annulus embedded in a Riemann surface S . Suppose that f is a homeomorphism of S which leaves A invariant. Furthermore, let $K \subset A$ be an annular continuum and suppose that f leaves K invariant. Then both $A_L(K)$ and $A_R(K)$ are f invariant.

Since A is embedded in S , we can define the frontiers of A , $A_L(K)$, and $A_R(K)$. By Carathéodory's theory of prime ends (see, e.g., [Mil06, Chapter 15]), the homeomorphism f yields an action on the frontiers of $A_L(K)$ and $A_R(K)$. Consider the right hand frontier of $A_L(K)$ (the one which is contained in A). Then the set of prime ends on this frontier is homeomorphic to the circle, and we denote by f_L the induced homeomorphism of this circle. Likewise, the set of prime ends on left hand frontier of $A_R(K)$ is homeomorphic to the circle, and we denote by f_R the induced homeomorphism this circle.

The rotation number of a circle homeomorphism (defined by Equation (2)), is well defined everywhere and is the same number for any point on the circle. The rotation numbers of f_L and f_R are called r_L and r_R . We refer to them as the left and right prime end rotation numbers of f . We have the following theorem of Matsumoto [Mat12].

Theorem 2.10 (Matsumoto's theorem). *If K is an annular continuum, then its left and right prime ends rotation numbers r_L, r_R belong to the rotation interval $\sigma(f, K)$.*

3. MINIMAL DECOMPOSITIONS AND CHARACTERISTIC ANNULI

3.1. Minimal decompositions. We recall the theory of minimal decompositions of surface homeomorphisms. This is established in [Mar07]. Firstly we recall the upper semi-continuous decomposition of a surface; see also Markovic [Mar07, Definition 2.1]. Let M be a surface.

Definition 3.1 (Upper semi-continuous decomposition). Let \mathbf{S} be a collection of closed, compact, connected subsets of M . We say that \mathbf{S} is an upper semi-continuous decomposition of M if the following holds:

- If $S_1, S_2 \in \mathbf{S}$, then $S_1 \cap S_2 = \emptyset$.
- If $S \in \mathbf{S}$, then S does not separate M ; i.e., $M - S$ is connected.
- We have $M = \bigcup_{S \in \mathbf{S}} S$.
- If $S_n \in \mathbf{S}, n \in \mathbb{N}$ is a sequence that has the Hausdorff limit equal to S_0 then there exists $S \in \mathbf{S}$ such that $S_0 \subset S$.

Now we define acyclic sets on a surface.

Definition 3.2 (Acyclic sets). Let $S \subset M$ be a closed, connected subset of M which does not separate M . We say that S is *acyclic* if there is a simply connected open set $U \subset M$ such that $S \subset U$ and $U - S$ is homeomorphic to an annulus.

The simplest examples of acyclic sets are a point, an embedded closed arc and an embedded closed disk in M . Let $S \subset M$ be a closed, connected set that does not separate M . Then S is acyclic if and only if there is a lift of S to the universal cover \widetilde{M} of M , which is a compact subset of \widetilde{M} . The following theorem is a classical result called Moore's theorem; see, e.g., [Mar07, Theorem 2.1].

Theorem 3.3 (Moore's theorem). *Let M be a surface and \mathbf{S} be an upper semi-continuous decomposition of M so that every element of \mathbf{S} is acyclic. Then there is a continuous map $\phi : M \rightarrow M$ that is homotopic to the identity map on M and such that for every $p \in M$, we have $\phi^{-1}(p) \in \mathbf{S}$. Moreover $\mathbf{S} = \{\phi^{-1}(p) | p \in M\}$.*

We call the map $M \rightarrow M / \sim$ the *Moore map* where $x \sim y$ if and only if $x, y \in S$ for some $S \in \mathbf{S}$. The following definition is [Mar07, Definition 3.1]

Definition 3.4 (Admissible decomposition). Let \mathbf{S} be an upper semi-continuous decomposition of M . Let G be a subgroup of $\text{Homeo}(M)$. We say that \mathbf{S} is admissible for the group G if the following holds:

- Each $f \in G$ preserves setwise every element of \mathbf{S} .
- Let $S \in \mathbf{S}$. Then every point, in every frontier component of the surface $M - S$ is a limit of points from $M - S$ which belong to acyclic elements of \mathbf{S} .

If G is a cyclic group generated by a homeomorphism $f : M \rightarrow M$ we say that \mathbf{S} is an admissible decomposition of f .

An admissible decomposition for $G < \text{Homeo}(M)$ is called *minimal* if it is contained in every admissible decomposition for G . We have the following theorem [Mar07, Theorem 3.1].

Theorem 3.5 (Existence of minimal decompositions). *Every group $G < \text{Homeo}(M)$ has a unique minimal decomposition.*

Denote by $\mathbf{A}(G)$ the sub collection of acyclic sets from $\mathbf{S}(G)$. By a mild abuse of notation, we occasionally refer to $\mathbf{A}(G)$ as a subset of S_g (the union of all sets from $\mathbf{A}(G)$). To distinguish the two notions we do the following. When we refer to $\mathbf{A}(G)$ as a collection then we consider it as the collection of acyclic sets. When we refer to as a set (or a subsurface of S_g) we have in mind the other meaning.

We have the following result [Mar07, Proposition 2.1].

Proposition 3.6. *Every connected component of $\mathbf{A}(G)$ (as a subset of S_g) is a subsurface of M with finitely many ends.*

Lemma 3.7. *For $H < G < \text{Homeo}(M)$, we have that $\mathbf{A}(G) \subset \mathbf{A}(H)$.*

Proof. The inclusion $\mathbf{A}(G) \subset \mathbf{A}(H)$ is because that the minimal decomposition of G is also an admissible decomposition of H and the minimal decomposition of H is finer than that of G . \square

3.2. Lifting through hyper-elliptic branched cover. Denote by $S_{g;n,b}$ the surface of genus g with b boundary components and n marked points. To make the analysis easier, we take the following hyper-elliptic $\mathbb{Z}/2$ branched covers

$$\pi_n : S = S_{\frac{n-1}{2};n,1} \rightarrow S_{0;n,1} \text{ for } n \text{ odd or } \pi_n : S = S_{\frac{n}{2}-1;n,2} \rightarrow S_{0;n} \text{ for } n \text{ even.}$$

The cover is shown by the following figures. The hyperelliptic involution on S is denoted by τ .

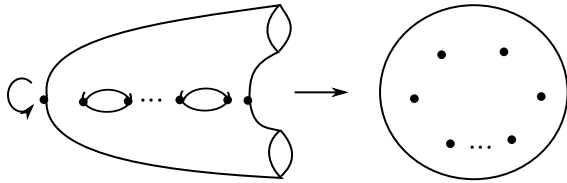


FIGURE 1. n even

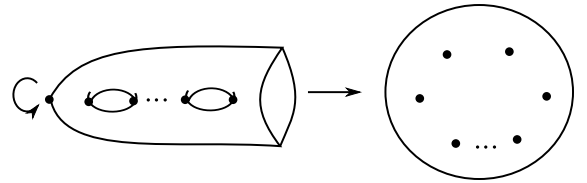


FIGURE 2. n odd

Denote by \widetilde{PB}_n the lifts of mapping classes under π_n , where it satisfies the following

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \widetilde{PB}_n \xrightarrow{L} PB_n \rightarrow 1.$$

Let c be a simple closed curve on $S_{0;n,1}$ and denote by T_c the Dehn twist about c . For every simple closed curve c on $S_{0;n,1}$, we have the following easy fact about its preimage under π_n .

- Fact 3.8.** (1) If c bounds odd number of points, then the lift is a single curve c' . The preimage of T_c^2 under L are $T_{c'}$ and $T_{c'}\tau$.
- (2) If c bounds even number of points, then the lift is two curves c_1, c_2 . The preimage of T_c under L are $T_{c_1}T_{c_2}$ and $T_{c_1}T_{c_2}\tau$. In particular, if c bounds 2 points, then $c_1 = c_2$.

From the above fact, we know that if c bounds 2 points and $c_1 = c_2$ are the lifts, we have that $T_{c_1}^2 \in \widetilde{PB}_n$. We have the following.

Fact 3.9. If α is a nonseparating simple closed curve that is invariant under τ , then a square of the Dehn twist about c is in \widetilde{PB}_n . We call such element an *invariant Dehn twist square*.

Let b be the curve in D_n^2 bounding 5 points P_1, \dots, P_5 . The lift of b under the cover π_n is a curve c bounding a genus 2 subsurface as the following figure.

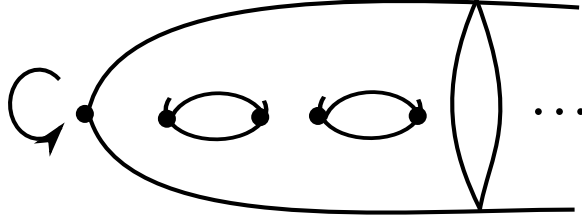


FIGURE 3. the curve c bounding a genus 2 surface is the lift of a curve bounding 5 points

If a curve α is on the genus 2 subsurface of S that is cut out by c , then we call the invariant Dehn twist square about α a *left invariant Dehn twist square*. We have the following important relations in \widetilde{PB}_n .

Proposition 3.10. *The element $T_c \in \widetilde{PB}_n$ is a product of left invariant Dehn twist squares in \widetilde{PB}_n .*

Proof. We have the basic fact that T_b is generated by Dehn twists about curves in the interior of b bounding 2 points; see, e.g., [FM12, Chapter 9]. Take a lift of all of the elements, we obtain that a product of squares of Dehn twists about nonseparating curves that are disjoint from c and on the left of c in \widetilde{PB}_n . After taking the square of the equation, we obtain the proposition. \square

3.3. Characteristic annuli. From now on, we work with the assumption that there exists a realization of the pure braid group

$$\mathcal{E}' : PB_n \rightarrow \text{Homeo}_+^a(D_n^2).$$

Lifting by the hyperelliptic involution, we obtain a new realization

$$\mathcal{E} : \widetilde{PB}_n \rightarrow \text{Homeo}_+^a(S_g)^\tau$$

where the image lies in the centralizer of the hyper-elliptic involution τ . We now only work with the new realization \mathcal{E} .

For an element $f \in \widetilde{PB}_n$, or a subgroup $F < \widetilde{PB}_n$, we shorten $\mathbf{A}(\mathcal{E}(f))$ as $\mathbf{A}(f)$, and $\mathbf{A}(\mathcal{E}(F))$ as $\mathbf{A}(F)$, to denote the corresponding collections of acyclic components. Denote by S the hyper-elliptic cover we defined in Section 3.2. Recall that $c \subset S$ is a separating curve that is invariant under τ that divides S into subsurfaces S_L of genus 2 and $S_R = S - S_L$ (see more about c in the previous section). We know that $T_c \in \widetilde{PB}_n$. We have the following theorem about the minimal decomposition of $\mathcal{E}(T_c)$.

Theorem 3.11. *The set $\mathbf{A}(T_c)$ has a component $\mathbf{L}(c)$ which is homotopic to S_L and a component $\mathbf{R}(c)$ homotopic to S_R .*

Proof sketch. The proof is the same as the proof of [CM19, Theorem 4.1]. We use the fact that there are pseudo-Anosov elements on the left and on the right of c in \widetilde{PB}_n . In this theorem, we need $n \geq 9$. \square

For the rest of paper, denote by

$$\mathbf{B} := S - \mathbf{L}(c) - \mathbf{R}(c).$$

Let $p_L : \mathbf{L}(c) \rightarrow \mathbf{L}(c)/\sim$ and $p_R : \mathbf{R}(c) \rightarrow \mathbf{R}(c)/\sim$ be the Moore maps of $\mathbf{L}(c)$ and $\mathbf{R}(c)$ corresponding to the decomposition $\mathbf{S}(c)$. Let $\mathbf{L} \subset \mathbf{L}(c)/\sim$ be an open annulus bounded by the end of $\mathbf{L}(c)'$ on one side, and by a simple closed curve on the other. The open annulus $\mathbf{R} \subset \mathbf{R}(c)/\sim$ is defined similarly. We have the following definition (see [Mar07, Chapter 5]).

Definition 3.12. An annulus of the form $A = p_L^{-1}(\mathbf{L}) \cup \mathbf{B} \cup p_R^{-1}(\mathbf{R})$ is called a *characteristic annulus*.

Denote by $f = \mathcal{E}(T_c)$. Every characteristic annulus is invariant under f . We observe that \mathbf{B} is a separator in A , that is, \mathbf{B} is an essential, compact, and connected subset of A . Note that a characteristic annulus A is invariant under f , but it may not be invariant under homeomorphisms which are lifts (with respect to \mathcal{E}) of other elements from \widetilde{PB}_n . However, \mathbf{B} is invariant under these lifts of elements from the image under \mathcal{E} of the centralizer of T_c in \widetilde{PB}_n . As we see from the next lemma, the dynamical information about f is contained in \mathbf{B} .

Lemma 3.13. *Fix a characteristic annulus A . Then*

- (1) *every number $0 < r < 1$ appears as the rotation number $\rho(f, x, A)$, for some $x \in A$,*
- (2) *if $0 < \rho(f, x, A) < 1$, then $x \in \mathbf{B}$.*

The proof of the above lemma can be seen in [CM19, Lemma 4.5]. The reason is that f is homotopic to a Dehn twist and that the realization is area-preserving.

4. THE PROOF OF THEOREM 1.2

In this section, we discuss the proof of Theorem 1.2. We now discuss the main strategy.

4.1. Outline of the proof. Recall that c is a separating simple closed curve that divides the surface S (the hyper-elliptic cover of $S_{0;1,n}$) into a genus 2 subsurface and the rest. Fix a characteristic annulus A . Let E_r be the set of points in A that have rotation numbers equal to r under $\mathcal{E}(T_c)$. Lemma 3.13 states that the set E_r is not empty when $0 < r < 1$.

The key observation of the proof lies in the analysis of connected components of E_r . Let E be a component of E_r . We show the following:

- (1) E is $\mathcal{E}(h)$ -invariant for h a left invariant Dehn twist square,
- (2) \overline{E} is a separator in A ,
- (3) if E contains a periodic orbit, then E contains a separator.

Denote by $K(\overline{E})$ the annular completion of \overline{E} , and let $\rho(\mathcal{E}(T_c), K(\overline{E}))$ be the rotation interval of $K(\overline{E})$. We claim that $\rho(\mathcal{E}(T_c), K(\overline{E})) = \{r\}$. First of all, we know that $r \in \rho(\mathcal{E}(T_c), K(\overline{E}))$. If $\rho(\mathcal{E}(T_c), K(\overline{E})) \neq \{r\}$, then $\rho(\mathcal{E}(T_c), K(\overline{E}))$ contains infinitely many rational numbers. By Theorem 2.9, there exist three periodic points $x_1, x_2, x_3 \in K(\overline{E})$ with different rational rotation numbers r_1, r_2, r_3 . Let F_i denote the connected component of E_{r_i} containing r_i , and let $M_i \subset F_i$ be a separator.

By Proposition 2.5, there is an ordering on disjoint separators. Without loss of generality, we assume that $M_1 < M_2 < M_3$. Based on a discussion about the position E with respect to M_i 's, we obtain a contradiction. Thus, $\rho(\mathcal{E}(T_c), K(E))$ is the singleton $\{r\}$.

We know from Theorem 2.10 that the left and right prime ends rotation numbers of $K(\overline{E})$ are both r . But in the group of circle homeomorphisms, the centralizer of an irrational rotation is essentially $SO(2)$.

We then show a new ingredient of the proof: the rotation numbers of the realization of a left invariant Dehn twist square on the set of prime ends of $K(\overline{E})$ are all 0. This contradicts the fact that T_c is a product left invariant Dehn twist squares as in Proposition 3.10.

4.2. The set E_r . Once again we use abbreviation $f = \mathcal{E}(T_c)$. For a characteristic annulus A , we let

$$E_r = \{x \in A : \rho(f, x, A) = r\}.$$

By Lemma 3.13, if $0 < r < 1$, we know that E_r is nonempty and $E_r \subset \mathbf{B}$.

Next, we have the following key lemmas which corresponds to [CM19, Lemma 5.1, 5.3, 5.4].

Lemma 4.1. *Fix $0 < r < 1$, and let E denote a connected component of E_r . Fix a left invariant Dehn twist square h in \widetilde{PB}_n . For $x \in E$, let $C(x) \in \mathbf{A}(h)$ be the corresponding acyclic set. Then $C(x) \subset E$. In particular, E is $\mathcal{E}(C(T_c))$ -invariant.*

Lemma 4.2. *The closed set \overline{E} is a separator (as defined in Section 2).*

Lemma 4.3. *Let x be a periodic orbit of f such that $\rho(f, x, A) = p/q$ and $0 < p/q < 1$. Then, the connected component E of $E_{p/q}$ which contains x , also contains a separator (as a subset).*

Fix an irrational number $r \in (0, 1)$. By Lemma 3.13, we know that E_r is not empty. Let E be a connected component of E_r . By Lemma 4.1, we know that E is invariant under $\mathcal{E}(C(T_c))$.

By Lemma 4.2, we know that \overline{E} is a separator. The annular completions $K(\overline{E})$ of \overline{E} is also $\mathcal{E}(C(T_c))$ -invariant since the definition is canonical. The following claim is at the heart of the entire construction.

Claim 4.4. *Let r_L and r_R be the left and right prime ends rotation numbers of f on $K(\overline{E})$. Then $r_L = r_R = r$.*

Remark. We refer the reader to [CM19, Claim 5.2] for the proof. The only property we use about \widetilde{PB}_n is Proposition 3.10.

4.3. Finishing the proof. We need to show a new property of a left invariant Dehn twist square $h \in \widetilde{PB}_n$.

Theorem 4.5. *The action of $\mathcal{E}(T_b^2)$ on the set of prime ends of $K(\overline{E})$ has rotation number 0.*

Proof. Now we consider the rotation set of $\mathcal{E}(T_b^2)$ on $K(\overline{E})$. We claim that the rotation set satisfies

$$\sigma(\mathcal{E}(T_b), K(\overline{E})) = \{0\}.$$

The reason is that if not, then it is a nontrivial closed interval. By Theorem 2.9, rational rotation numbers are realized by periodic orbit. However $K(\overline{E}) \subset B$, that means every point for $x \in K(\overline{E}) \subset B$, there exists $C(x) \in \mathbf{A}(T_b^2)$ such that $C(x) \subset B$ by Lemma 4.1. However $C(x)$ is acyclic and fixed by $\mathcal{E}(T_b^2)$. Therefore, we know that the rotation number of $\mathcal{E}(T_b^2)$ on points in $C(x)$ is zero, which is a contradiction. Then by Theorem 2.10, we know that the rotation number of the action of $\mathcal{E}(T_b^2)$ on the set of prime ends is also zero. \square

We now finish the proof.

Proof. Since the rotation number of $\mathcal{E}(T_c)$ on the prime ends of $K(\overline{E})$ is an irrational number r , then it is semiconjugate to an irrational rotation. Then up to the same semiconjugation, the image of the centralizer of T_c under \mathcal{E} is $SO(2)$. The image of each element is determined by its rotation number. However, $\mathcal{E}(T_c)$ is a product of $\mathcal{E}(T_b^2)$ for b nonseparating and invariant under τ by Proposition 3.10. By Lemma 4.5, we know that the rotation number of $\mathcal{E}(T_b^2)$ is zero. Thus their product should also have 0 rotation number. This contradicts the fact that the rotation number of $\mathcal{E}(T_c)$ is r , which is nonzero. \square

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