

# NON-REALIZABILITY OF THE PURE BRAID GROUP AS AREA-PRESERVING HOMEOMORPHISMS

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ABSTRACT. Let  $\text{Homeo}_+(D_n^2)$  be the group of orientation-preserving homeomorphisms of  $D^2$  fixing the boundary pointwise and  $n$  marked points as a set. Nielsen realization problem for the braid group asks whether the natural projection  $p_n : \text{Homeo}_+(D_n^2) \rightarrow B_n := \pi_0(\text{Homeo}_+(D_n^2))$  has a section over subgroups of  $B_n$ . All of the previous methods either use torsions or Thurston stability, which do not apply to the pure braid group  $PB_n$ , the subgroup of  $B_n$  that fixes  $n$  marked points pointwise. In this paper, we show that the pure braid group has no realization inside the area-preserving homeomorphisms using rotation numbers.

## 1. INTRODUCTION

Denote by  $D^2$  the 2-dimensional disk. Let  $\text{Homeo}_+(D_n^2)$  be the group of orientation-preserving homeomorphisms of  $D^2$  fixing the boundary pointwise and  $n$  marked points as a set. Denote by  $B_n := \pi_0(\text{Homeo}_+(D_n^2))$ . The Nielsen realization problem for  $B_n$  asks whether the natural projection

$$p_n : \text{Homeo}_+(D_n^2) \rightarrow B_n$$

has a section over subgroups of  $B_n$ . For the whole group  $B_n$ , this question has several previous results. Salter–Tshishiku [ST16] uses Thurston stability to show that  $B_n$  has no realization in  $\text{Diff}_+(D_n^2)$  and the author [Che19] uses “hidden torsions” and Markovic’s machinery [Mar07] to show that  $B_n$  has no realization in  $\text{Homeo}_+(D_n^2)$ . Let  $PB_n < B_n$  be the subgroup that preserves  $n$  marked points pointwise. The Nielsen realization problem for  $PB_n$  is widely open since the two methods in [ST16] and [Che19] fail to work and has no hope to repair. The following question is asked by [MT18, Question 3.12] and [ST16, Remark 1.4].

**Problem 1.1** (Realization of pure braid group). *Does  $PB_n$  have realization as diffeomorphisms or homeomorphisms? In other words, does  $p_n$  have sections over  $PB_n$ ?*

Denote by  $\text{Homeo}_+^a(D_n^2)$  the group of orientation-preserving, area-preserving homeomorphisms of  $D^2$  fixing the boundary pointwise and  $n$  marked points as a set. In this paper, we make a progress proving the following result.

**Theorem 1.2.** *The pure braid group cannot be realized as area-preserving homeomorphisms on  $D_n^2$  for  $n \geq 9$ . In other words, the natural projection  $p_n^a : \text{Homeo}_+^a(D_n^2) \rightarrow B_n$  has no sections over  $PB_n$ .*

We remark that the Nielsen realization problem is closely related to the existence of flat structures on a surface bundle. We refer the reader to [MT18] for more history and background.

**Comparing with the method in [CM19].** The novelty of this paper is to provide a different ending towards [CM19]. The original ending is to use the fact that certain Dehn twist is a product of commutator in its centralizer. However, such structure does not hold in  $PB_n$ . Instead, we prove a stronger dynamical property about Dehn twists about non-separating curves. In the beginning of Section 4, we present an outline of the proof. Since this paper has a lot of overlap with [CM19], we omit or sketch many proofs to reduce redundancy.

**Organization of the paper.**

- In Section 2, we discuss rotation numbers;
- In Section 3, we discuss the pure braid group and the minimal decomposition theory;
- In Section 4, we give an outline of the proof and finish the argument.

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## 2. ROTATION NUMBERS OF ANNULUS HOMEOMORPHISMS

In this section, we discuss the properties of rotation numbers on annuli.

**2.1. Rotation number of an area-preserving homeomorphism of an annulus.** Firstly, we define the rotation number for geometric annuli. Let

$$N = N(r) = \{w \in \mathbb{C} : \frac{1}{r} < |w| < r\}$$

be the geometric annulus in the complex plane  $\mathbb{C}$ . Denote the geometric strip in  $\mathbb{C}$  by

$$P = P(r) = \{x + iy = z \in \mathbb{C} : |y| < \frac{\log r}{2\pi}\}.$$

The map  $\pi(z) = e^{2\pi iz}$  is a holomorphic covering map  $\pi : P \rightarrow N$ . The deck transformation on  $P$  is  $T(x, y) = (x + 1, y)$ .

Denote by  $p_1 : P \rightarrow \mathbb{R}$  the projection to the  $x$ -coordinate, and by  $\text{Homeo}_+(N)$  the group of homeomorphisms of  $N$  that preserves orientation and the two ends. Fix  $f \in \text{Homeo}_+(N)$ , and  $x \in N$ , and let  $\tilde{x} \in P$  and  $\tilde{f} \in \text{Homeo}_+(P)$  denote lifts of  $x$  and  $f$  respectively. We define the translation number of the lift  $\tilde{f}$  at  $\tilde{x}$  by

$$(1) \quad \rho(\tilde{f}, \tilde{x}, P) = \lim_{n \rightarrow \infty} (p_1(\tilde{f}^n(\tilde{x})) - p_1(\tilde{x}))/n.$$

The rotation number of  $f$  at  $x$  is then defined as

$$(2) \quad \rho(f, x, N) = \rho(\tilde{f}, \tilde{x}, P) \pmod{1}.$$

The rotation number is not defined everywhere (see, e.g., [Fra03] for more background on rotation numbers). The closed annulus  $N_c$  is

$$N_c = \{\omega \in \mathbb{C} : \frac{1}{r} \leq |\omega| \leq r\},$$

For  $f \in \text{Homeo}_+(N_c)$ , the rotation and translation numbers are defined analogously.

Let  $A$  be an open annulus embedded in a Riemann surface (in particular this endows  $A$  with the complex structure). By the Riemann mapping theorem, there is a unique  $N(r) = N$  and a conformal

map  $u_A : A \rightarrow N$ . For any  $f \in \text{Homeo}_+(A)$  (the group of end-preserving homeomorphisms), we define the rotation number of  $f$  on  $A$  by

$$\rho(f, x, A) := \rho(g, u_A(x), N),$$

where  $g = u_A \circ f \circ u_A^{-1}$ .

We have the following theorems of Poincaré-Birkhoff and Handel about rotation numbers [Han90] (See also Franks [Fra03]).

**Theorem 2.1** (Properties of rotation numbers). *If  $f : N_c \rightarrow N_c$  is an orientation preserving, boundary component preserving, area-preserving homeomorphism and  $\tilde{f} : P_c \rightarrow P_c$  is any lift, then:*

- (Handel) *The translation set*

$$R(\tilde{f}) = \bigcup_{\tilde{x} \in P_c} \rho(\tilde{f}, \tilde{x}, P_c)$$

*is a closed interval.*

- (Poincaré-Birkhoff) *If  $r \in R(\tilde{f})$  is rational, then there exists a periodic orbit of  $f$  realizing the rotation number  $r \bmod 1$ .*

**2.2. Separators and its properties.** We let  $A$  continue to denote an open annulus embedded in a Riemann surface. Then  $A$  has two ends and we choose one of them to be the left end and the other one to be the right end. We call a subset  $X \subset A$  *separating* (or *essential*) if every arc  $\gamma \subset A$  which connects the two ends of  $A$  must intersect  $X$ .

**Definition 2.2** (Separator). We call a subset  $M \subset A$  a *separator* if  $M$  is compact, connected and separating.

The complement of  $M$  in  $A$  is a disjoint union of open sets. We have the following lemma.

**Lemma 2.3.** *Let  $M$  be a separator. Then there are exactly two connected components  $A_L(M)$  and  $A_R(M)$  of  $A - M$  which are open annuli homotopic to  $A$  and with the property that  $A_L(M)$  contains the left end of  $A$  and  $A_R(M)$  contains the right end of  $A$ . All other components of  $A - M$  are simply connected.*

*Proof.* We compactify the annulus  $A$  by adding points  $p_L$  and  $p_R$  to the corresponding ends of  $A$ . The compactification is a two sphere  $S^2$ . Moreover,  $M$  is a compact and connected subset of  $S^2 - \{p_L, p_R\}$ .

Now, we observe that every component of  $S^2 - M$  is simply connected. Denote by  $\Omega_L$  and  $\Omega_R$  the connected components of  $S^2 - M$  containing  $p_L$  and  $p_R$  respectively. Since  $M$  is separating we conclude that these are two different components. We define  $A_L(M) = \Omega_L - p_L$  and  $A_R(M) = \Omega_R - p_R$ . It is easy to verify that these are required annuli.  $\square$

We now prove another property of a separator. Let  $\pi : \tilde{A} \rightarrow A$  be the universal cover.

**Proposition 2.4.** *Let  $M \subset A$  be a compact domain with smooth boundary. Then  $\pi^{-1}(M)$  is connected.*

*Proof.* Since  $M$  is a compact domain with boundary which separates the two ends of  $A$ , we can find a circle  $\gamma \subset M$  which is essential in  $A$  (i.e.  $\gamma$  is a separator itself) (note that  $M$  has only finitely many boundary components). Denote by  $T$  the deck transformation of  $\tilde{A}$ . Thus, the lift  $\pi^{-1}(\gamma)$  is a  $T$ -invariant, connected subset of  $\tilde{A}$ . Let  $C$  be the component of  $\pi^{-1}(M)$  which contains  $\pi^{-1}(\gamma)$ . Then  $C$  is  $T$  invariant. We show  $\pi^{-1}(M) = C$ .

Let  $p \in M$ . Since  $M$  is a compact domain with smooth boundary, we can find an embedded closed arc  $\alpha \subset M$  which connects  $p$  and  $\gamma$ . Let  $\tilde{p}$  be a lift of  $p$  and let  $\tilde{\alpha}$  be the corresponding lift of  $\alpha$  such that  $\tilde{p}$  is one of its endpoints. Then, the other endpoint of  $\tilde{\alpha}$  is in  $\pi^{-1}(\gamma)$ , and this shows that  $\tilde{p} \in C$ . This concludes the proof. □

Now we discuss an ordering on the set of separators.

**Proposition 2.5.** *Suppose  $M_1, M_2 \subset A$  are two disjoint separators. Then either  $M_1 \subset A_L(M_2)$  or  $M_1 \subset A_R(M_2)$ . Moreover,  $M_1 \subset A_L(M_2)$  implies  $M_2 \subset A_R(M_1)$ .*

*Proof.* Since  $M_1$  is connected it follows that  $M_1$  is a subset of a connected component  $C$  of  $A - M_2$ . Since  $C$  is open, we know that there is a neighborhood  $N_1$  of  $M_1$  with smooth boundary such that  $N_1 \subset C$  (It is elementary to construct such  $N_1$ ). If  $C$  is simply connected, the cover  $\pi^{-1}(C) \rightarrow C$  is a trivial cover. Let  $\tilde{C}$  be a connected component of  $\pi^{-1}(C)$ . By Proposition 2.4, the set  $\pi^{-1}(N_1)$  is connected so it is contained in a single connected component of  $\pi^{-1}(C)$ . However, this contradicts the fact that  $\pi^{-1}(N_1)$  is also translation invariant. Thus, either  $M_1 \subset A_L(M_2)$  or  $M_1 \subset A_R(M_2)$ .

Suppose  $M_1 \subset A_L(M_2)$ . Then  $A_L(M_1) \subset A_L(M_2)$  as well. On the other hand, by the first part of the proposition we already know that either  $M_2 \subset A_L(M_1)$  or  $M_2 \subset A_R(M_1)$ . If  $M_2 \subset A_L(M_1)$ , then  $A_L(M_2) \subset A_L(M_1)$ . This shows that  $A_L(M_1) \subset A_L(M_2)$  which implies that  $M_2 \subset A_L(M_2)$ . This is absurd so we must have  $M_2 \subset A_R(M_1)$ . □

**Definition 2.6.** The inclusion  $M_1 \subset A_L(M_2)$  is denoted as  $M_1 < M_2$ .

**2.3. The rotation interval of an annular continuum and prime ends.** Let  $K \subset A$  be a separator (in literature also known as an *essential continuum*). We call  $K$  an essential *annular continuum* if  $A - K$  has exactly two components. Observe that an essential annular continuum can be expressed as a decreasing intersection of essential closed topological annuli in  $A$ .

It is possible to turn any separator  $M \subset A$  into an essential annular continuum. Let  $M$  be a separating connected set. By Lemma 2.3, we know that  $A - M$  has exactly two connected annular components  $A_L(M)$  and  $A_R(M)$ , and all other components of  $A - M$  are simply connected. We call a simply connected component of  $A - M$  a *bubble component*. Then the *annular completion*  $K(M)$  of  $M$  is defined as the union of  $M$  and the corresponding bubble components of  $A - M$ .

**Proposition 2.7.** *Let  $M \subset A$  be a separator. Then the annular completion  $K(M)$  is an annular continuum.*

*Proof.* We can again compactify  $A$  by adding the points  $p_L$  and  $p_R$ , one at each end. The compactification is the two sphere  $S^2$ . Then  $A_L(M)$  and  $A_R(M)$  are two disjoint open discs in  $S^2$ ,

and  $K(M) = S^2 - (A_L(M) \cup A_R(M))$ . But the complement of two disjoint open discs in  $S^2$  is connected. This proves the proposition. □

Now let  $f$  be a homeomorphism of  $A$  that leaves an annular continuum  $K$  invariant. If  $\mu$  is an invariant Borel probability measure supported on  $K$ , we define the  $\mu$ -rotation number

$$\sigma(f, \mu) = \int_A \phi d\mu$$

where  $\phi : A \rightarrow \mathbb{R}$  is the function which lifts to the function  $p_1 \circ f - p_1$  on  $\tilde{A}$  (recall that  $p_1 : \tilde{A} \rightarrow \mathbb{R}$  is the projection onto the first coordinate).

The set of  $f$  invariant Borel probability measures on  $K$  is a non empty, convex, and compact set (with respect to the weak topology on the space of measures). We define the *rotation interval* of  $K$

$$\sigma(f, K) = \{\sigma(f, \mu) | \mu \in M(K)\}$$

which is a non-empty segment  $[\alpha, \beta]$  of  $\mathbb{R}$ . The interval is non empty because there exists at least one  $f$  invariant measure, and it is an interval because the set of  $f$  invariant measures is convex.

The following is a classical result of Franks–Le Calvez [FC03, Corollary 3.1].

**Proposition 2.8.** *If  $\sigma(f, K) = \{\alpha\}$ , the sequence*

$$\frac{p_1 \circ f^n(x) - p_1(x)}{n}$$

*converges uniformly for  $x \in \pi^{-1}(K)$  to the constant function  $\alpha$ . This implies that points in  $K$  all have the rotation number  $\alpha$ .*

The following theorem of Franks–Le Calvez [FC03, Proposition 5.4] is a generalization of the Poincaré-Birkhoff Theorem.

**Theorem 2.9.** *If  $f$  is area-preserving and  $K$  is an annular continuum, then every rational number in  $\sigma(f, K)$  is realized by a periodic point in  $K$ .*

The theory of prime ends is an important tool in the study of 2-dimensional dynamics which can be used to transform a 2-dimensional problem into a 1-dimensional problem. Recall that we assume that  $A$  is an open annulus embedded in a Riemann surface  $S$ . Suppose that  $f$  is a homeomorphism of  $S$  which leaves  $A$  invariant. Furthermore, let  $K \subset A$  be an annular continuum and suppose that  $f$  leaves  $K$  invariant. Then both  $A_L(K)$  and  $A_R(K)$  are  $f$  invariant.

Since  $A$  is embedded in  $S$ , we can define the frontiers of  $A$ ,  $A_L(K)$ , and  $A_R(K)$ . By Carathéodory's theory of prime ends (see, e.g., [Mil06, Chapter 15]), the homeomorphism  $f$  yields an action on the frontiers of  $A_L(K)$  and  $A_R(K)$ . Consider the right hand frontier of  $A_L(K)$  (the one which is contained in  $A$ ). Then the set of prime ends on this frontier is homeomorphic to the circle, and we denote by  $f_L$  the induced homeomorphism of this circle. Likewise, the set of prime ends on left hand frontier of  $A_R(K)$  is homeomorphic to the circle, and we denote by  $f_R$  the induced homeomorphism this circle.

The rotation number of a circle homeomorphism (defined by Equation (2)), is well defined everywhere and is the same number for any point on the circle. The rotation numbers of  $f_L$  and  $f_R$  are called  $r_L$  and  $r_R$ . We refer to them as the left and right prime end rotation numbers of  $f$ . We have the following theorem of Matsumoto [Mat12].

**Theorem 2.10** (Matsumoto's theorem). *If  $K$  is an annular continuum, then its left and right prime ends rotation numbers  $r_L, r_R$  belong to the rotation interval  $\sigma(f, K)$ .*

### 3. MINIMAL DECOMPOSITIONS AND CHARACTERISTIC ANNULI

**3.1. Minimal decompositions.** We recall the theory of minimal decompositions of surface homeomorphisms. This is established in [Mar07]. Firstly we recall the upper semi-continuous decomposition of a surface; see also Markovic [Mar07, Definition 2.1]. Let  $M$  be a surface.

**Definition 3.1** (Upper semi-continuous decomposition). Let  $\mathbf{S}$  be a collection of closed, compact, connected subsets of  $M$ . We say that  $\mathbf{S}$  is an upper semi-continuous decomposition of  $M$  if the following holds:

- If  $S_1, S_2 \in \mathbf{S}$ , then  $S_1 \cap S_2 = \emptyset$ .
- If  $S \in \mathbf{S}$ , then  $E$  does not separate  $M$ ; i.e.,  $M - S$  is connected.
- We have  $M = \bigcup_{S \in \mathbf{S}} S$ .
- If  $S_n \in \mathbf{S}, n \in \mathbb{N}$  is a sequence that has the Hausdorff limit equal to  $S_0$  then there exists  $S \in \mathbf{S}$  such that  $S_0 \subset S$ .

Now we define acyclic sets on a surface.

**Definition 3.2** (Acyclic sets). Let  $S \subset M$  be a closed, connected subset of  $M$  which does not separate  $M$ . We say that  $S$  is *acyclic* if there is a simply connected open set  $U \subset M$  such that  $S \subset U$  and  $U - S$  is homeomorphic to an annulus.

The simplest examples of acyclic sets are a point, an embedded closed arc and an embedded closed disk in  $M$ . Let  $S \subset M$  be a closed, connected set that does not separate  $M$ . Then  $S$  is acyclic if and only if there is a lift of  $S$  to the universal cover  $\widetilde{M}$  of  $M$ , which is a compact subset of  $\widetilde{M}$ . The following theorem is a classical result called Moore's theorem; see, e.g., [Mar07, Theorem 2.1].

**Theorem 3.3** (Moore's theorem). *Let  $M$  be a surface and  $\mathbf{S}$  be an upper semi-continuous decomposition of  $M$  so that every element of  $\mathbf{S}$  is acyclic. Then there is a continuous map  $\phi : M \rightarrow M$  that is homotopic to the identity map on  $M$  and such that for every  $p \in M$ , we have  $\phi^{-1}(p) \in \mathbf{S}$ . Moreover  $\mathbf{S} = \{\phi^{-1}(p) | p \in M\}$ .*

We call the map  $M \rightarrow M / \sim$  the *Moore map* where  $x \sim y$  if and only if  $x, y \in S$  for some  $S \in \mathbf{S}$ . The following definition is [Mar07, Definition 3.1]

**Definition 3.4** (Admissible decomposition). Let  $\mathbf{S}$  be an upper semi-continuous decomposition of  $M$ . Let  $G$  be a subgroup of  $\text{Homeo}(M)$ . We say that  $\mathbf{S}$  is admissible for the group  $G$  if the following holds:

- Each  $f \in G$  preserves setwise every element of  $\mathbf{S}$ .
- Let  $S \in \mathbf{S}$ . Then every point, in every frontier component of the surface  $M - S$  is a limit of points from  $M - S$  which belong to acyclic elements of  $\mathbf{S}$ .

If  $G$  is a cyclic group generated by a homeomorphism  $f : M \rightarrow M$  we say that  $\mathbf{S}$  is an admissible decomposition of  $f$ .

An admissible decomposition for  $G < \text{Homeo}(M)$  is called *minimal* if it is contained in every admissible decomposition for  $G$ . We have the following theorem [Mar07, Theorem 3.1].

**Theorem 3.5** (Existence of minimal decompositions). *Every group  $G < \text{Homeo}(M)$  has a unique minimal decomposition.*

Denote by  $\mathbf{A}(G)$  the sub collection of acyclic sets from  $\mathbf{S}(G)$ . By a mild abuse of notation, we occasionally refer to  $\mathbf{A}(G)$  as a subset of  $S_g$  (the union of all sets from  $\mathbf{A}(G)$ ). To distinguish the two notions we do the following. When we refer to  $\mathbf{A}(G)$  as a collection then we consider it as the collection of acyclic sets. When we refer to as a set (or a subsurface of  $S_g$ ) we have in mind the other meaning.

We have the following result [Mar07, Proposition 2.1].

**Proposition 3.6.** *Every connected component of  $\mathbf{A}(G)$  (as a subset of  $S_g$ ) is a subsurface of  $M$  with finitely many ends.*

**Lemma 3.7.** *For  $H < G < \text{Homeo}(M)$ , we have that  $\mathbf{A}(G) \subset \mathbf{A}(H)$ .*

*Proof.* The inclusion  $\mathbf{A}(G) \subset \mathbf{A}(H)$  is because that the minimal decomposition of  $G$  is also an admissible decomposition of  $H$  and the minimal decomposition of  $H$  is finer than that of  $G$ .  $\square$

**3.2. Lifting through hyper-elliptic branched cover.** Denote by  $S_{g;n,b}$  the surface of genus  $g$  with  $b$  boundary components and  $n$  marked points. To make the analysis easier, we take the following hyper-elliptic  $\mathbb{Z}/2$  branched covers

$$\pi_n : S = S_{\frac{n-1}{2};n,1} \rightarrow S_{0;n,1} \text{ for } n \text{ odd or } \pi_n : S = S_{\frac{n}{2}-1;n,2} \rightarrow S_{0;n} \text{ for } n \text{ even.}$$

The cover is shown by the following figures. The hyperelliptic involution on  $S$  is denoted by  $\tau$ .

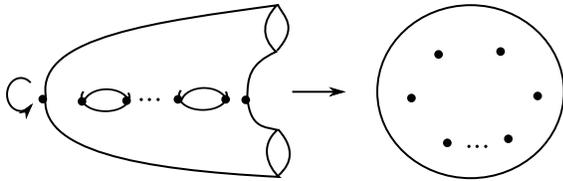


FIGURE 1.  $n$  even

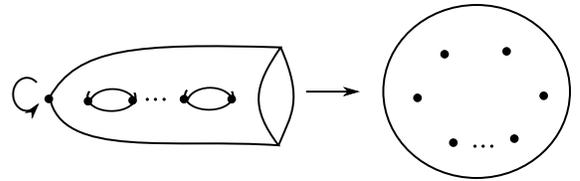


FIGURE 2.  $n$  odd

Denote by  $\widetilde{PB}_n$  the lifts of mapping classes under  $\pi_n$ , where it satisfies the following

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \widetilde{PB}_n \xrightarrow{L} PB_n \rightarrow 1.$$

Let  $c$  be a simple closed curve on  $S_{0;n,1}$  and denote by  $T_c$  the Dehn twist about  $c$ . For every simple closed curve  $c$  on  $S_{0;n,1}$ , we have the following easy fact about its preimage under  $\pi_n$ .

- Fact 3.8.** (1) If  $c$  bounds odd number of points, then the lift is a single curve  $c'$ . The preimage of  $T_c^2$  under  $L$  are  $T_{c'}$  and  $T_{c'\tau}$ .
- (2) If  $c$  bounds even number of points, then the lift is two curves  $c_1, c_2$ . The preimage of  $T_c$  under  $L$  are  $T_{c_1}T_{c_2}$  and  $T_{c_1}T_{c_2}\tau$ . In particular, if  $c$  bounds 2 points, then  $c_1 = c_2$ .

From the above fact, we know that if  $c$  bounds 2 points and  $c_1 = c_2$  are the lifts, we have that  $T_{c_1}^2 \in \widetilde{PB}_n$ . We have the following.

**Fact 3.9.** If  $\alpha$  is a nonseparating simple closed curve that is invariant under  $\tau$ , then a square of the Dehn twist about  $c$  is in  $\widetilde{PB}_n$ . We call such element an *invariant Dehn twist square*.

Let  $b$  be the curve in  $D_n^2$  bounding 5 points  $P_1, \dots, P_5$ . The lift of  $b$  under the cover  $\pi_n$  is a curve  $c$  bounding a genus 2 subsurface as the following figure.

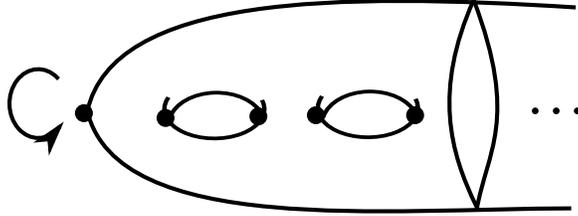


FIGURE 3. the curve  $c$  bounding a genus 2 surface is the lift of a curve bounding 5 points

If a curve  $\alpha$  is on the genus 2 subsurface of  $S$  that is cut out by  $c$ , then we call the invariant Dehn twist square about  $\alpha$  a *left invariant Dehn twist square*. We have the following important relations in  $\widetilde{PB}_n$ .

**Proposition 3.10.** *The element  $T_c \in \widetilde{PB}_n$  is a product of left invariant Dehn twist squares in  $\widetilde{PB}_n$ .*

*Proof.* We have the basic fact that  $T_b$  is generated by Dehn twists about curves in the interior of  $b$  bounding 2 points; see, e.g., [FM12, Chapter 9]. Take a lift of all of the elements, we obtain that a product of squares of Dehn twists about nonseparating curves that are disjoint from  $c$  and on the left of  $c$  in  $\widetilde{PB}_n$ . After taking the square of the equation, we obtain the proposition.  $\square$

**3.3. Characteristic annuli.** From now on, we work with the assumption that there exists a realization of the pure braid group

$$\mathcal{E}' : PB_n \rightarrow \text{Homeo}_+^a(D_n^2).$$

Lifting by the hyperelliptic involution, we obtain a new realization

$$\mathcal{E} : \widetilde{PB}_n \rightarrow \text{Homeo}_+^a(S_g)^\tau$$

where the image lies in the centralizer of the hyper-elliptic involution  $\tau$ . We now only work with the new realization  $\mathcal{E}$ .

For an element  $f \in \widetilde{PB}_n$ , or a subgroup  $F < \widetilde{PB}_n$ , we shorten  $\mathbf{A}(\mathcal{E}(f))$  as  $\mathbf{A}(f)$ , and  $\mathbf{A}(\mathcal{E}(F))$  as  $\mathbf{A}(F)$ , to denote the corresponding collections of acyclic components. Denote by  $S$  the hyper-elliptic cover we defined in Section 3.2. Recall that  $c \subset S$  is a separating curve that is invariant under  $\tau$  that divides  $S$  into subsurfaces  $S_L$  of genus 2 and  $S_R = S - S_L$  (see more about  $c$  in the previous section). We know that  $T_c \in \widetilde{PB}_n$ . We have the following theorem about the minimal decomposition of  $\mathcal{E}(T_c)$ .

**Theorem 3.11.** *The set  $\mathbf{A}(T_c)$  has a component  $\mathbf{L}(c)$  which is homotopic to  $S_L$  and a component  $\mathbf{R}(c)$  homotopic to  $S_R$ .*

*Proof sketch.* The proof is the same as the proof of [CM19, Theorem 4.1]. We use the fact that there are pseudo-Anosov elements on the left and on the right of  $c$  in  $\widetilde{PB}_n$ . In this theorem, we need  $n \geq 9$ .  $\square$

For the rest of paper, denote by

$$\mathbf{B} := S - \mathbf{L}(c) - \mathbf{R}(c).$$

Let  $p_L : \mathbf{L}(c) \rightarrow \mathbf{L}(c)/\sim$  and  $p_R : \mathbf{R}(c) \rightarrow \mathbf{R}(c)/\sim$  be the Moore maps of  $\mathbf{L}(c)$  and  $\mathbf{R}(c)$  corresponding to the decomposition  $\mathbf{S}(c)$ . Let  $\mathbf{L} \subset \mathbf{L}(c)/\sim$  be an open annulus bounded by the end of  $\mathbf{L}(c)'$  on one side, and by a simple closed curve on the other. The open annulus  $\mathbf{R} \subset \mathbf{R}(c)/\sim$  is defined similarly. We have the following definition (see [Mar07, Chapter 5]).

**Definition 3.12.** An annulus of the form  $A = p_L^{-1}(\mathbf{L}) \cup \mathbf{B} \cup p_R^{-1}(\mathbf{R})$  is called a *characteristic annulus*.

Denote by  $f = \mathcal{E}(T_c)$ . Every characteristic annulus is invariant under  $f$ . We observe that  $\mathbf{B}$  is a separator in  $A$ , that is,  $\mathbf{B}$  is an essential, compact, and connected subset of  $A$ . Note that a characteristic annulus  $A$  is invariant under  $f$ , but it may not be invariant under homeomorphisms which are lifts (with respect to  $\mathcal{E}$ ) of other elements from  $\widetilde{PB}_n$ . However,  $\mathbf{B}$  is invariant under these lifts of elements from the image under  $\mathcal{E}$  of the centralizer of  $T_c$  in  $\widetilde{PB}_n$ . As we see from the next lemma, the dynamical information about  $f$  is contained in  $\mathbf{B}$ .

**Lemma 3.13.** *Fix a characteristic annulus  $A$ . Then*

- (1) *every number  $0 < r < 1$  appears as the rotation number  $\rho(f, x, A)$ , for some  $x \in A$ ,*
- (2) *if  $0 < \rho(f, x, A) < 1$ , then  $x \in B$ .*

The proof of the above lemma can be seen in [CM19, Lemma 4.5]. The reason is that  $f$  is homotopic to a Dehn twist and that the realization is area-preserving.

#### 4. THE PROOF OF THEOREM 1.2

In this section, we discuss the proof of Theorem 1.2. We now discuss the main strategy.

**4.1. Outline of the proof.** Recall that  $c$  is a separating simple closed curve that divides the surface  $S$  (the hyper-elliptic cover of  $S_{0;1,n}$ ) into a genus 2 subsurface and the rest. Fix a characteristic annulus  $A$ . Let  $E_r$  be the set of points in  $A$  that have rotation numbers equal to  $r$  under  $\mathcal{E}(T_c)$ . Lemma 3.13 states that the set  $E_r$  is not empty when  $0 < r < 1$ .

The key observation of the proof lies in the analysis of connected components of  $E_r$ . Let  $E$  be a component of  $E_r$ . We show the following:

- (1)  $E$  is  $\mathcal{E}(h)$ -invariant for  $h$  a left invariant Dehn twist square,
- (2)  $\overline{E}$  is a separator in  $A$ ,
- (3) if  $E$  contains a periodic orbit, then  $E$  contains a separator.

Denote by  $K(\overline{E})$  the annular completion of  $\overline{E}$ , and let  $\rho(\mathcal{E}(T_c), K(\overline{E}))$  be the rotation interval of  $K(\overline{E})$ . We claim that  $\rho(\mathcal{E}(T_c), K(\overline{E})) = \{r\}$ . First of all, we know that  $r \in \rho(\mathcal{E}(T_c), K(\overline{E}))$ . If  $\rho(\mathcal{E}(T_c), K(\overline{E})) \neq \{r\}$ , then  $\rho(\mathcal{E}(T_c), K(\overline{E}))$  contains infinitely many rational numbers. By Theorem 2.9, there exist three periodic points  $x_1, x_2, x_3 \in K(\overline{E})$  with different rational rotation numbers  $r_1, r_2, r_3$ . Let  $F_i$  denote the connected component of  $E_{r_i}$  containing  $r_i$ , and let  $M_i \subset F_i$  be a separator.

By Proposition 2.5, there is an ordering on disjoint separators. Without loss of generality, we assume that  $M_1 < M_2 < M_3$ . Based on a discussion about the position  $E$  with respect to  $M_i$ 's, we obtain a contradiction. Thus,  $\rho(\mathcal{E}(T_c), K(\overline{E}))$  is the singleton  $\{r\}$ .

We know from Theorem 2.10 that the left and right prime ends rotation numbers of  $K(\overline{E})$  are both  $r$ . But in the group of circle homeomorphisms, the centralizer of an irrational rotation is essentially  $SO(2)$ .

We then show a new ingredient of the proof: the rotation numbers of the realization of a left invariant Dehn twist square on the set of prime ends of  $K(\overline{E})$  are all 0. This contradicts the fact that  $T_c$  is a product left invariant Dehn twist squares as in Proposition 3.10.

**4.2. The set  $E_r$ .** Once again we use abbreviation  $f = \mathcal{E}(T_c)$ . For a characteristic annulus  $A$ , we let

$$E_r = \{x \in A : \rho(f, x, A) = r\}.$$

By Lemma 3.13, if  $0 < r < 1$ , we know that  $E_r$  is nonempty and  $E_r \subset \mathbf{B}$ .

Next, we have the following key lemmas which corresponds to [CM19, Lemma 5.1, 5.3, 5.4].

**Lemma 4.1.** *Fix  $0 < r < 1$ , and let  $E$  denote a connected component of  $E_r$ . Fix a left invariant Dehn twist square  $h$  in  $\widetilde{PB}_n$ . For  $x \in E$ , let  $C(x) \in \mathbf{A}(h)$  be the corresponding acyclic set. Then  $C(x) \subset E$ . In particular,  $E$  is  $\mathcal{E}(C(T_c))$ -invariant.*

**Lemma 4.2.** *The closed set  $\overline{E}$  is a separator (as defined in Section 2).*

**Lemma 4.3.** *Let  $x$  be a periodic orbit of  $f$  such that  $\rho(f, x, A) = p/q$  and  $0 < p/q < 1$ . Then, the connected component  $E$  of  $E_{p/q}$  which contains  $x$ , also contains a separator (as a subset).*

Fix an irrational number  $r \in (0, 1)$ . By Lemma 3.13, we know that  $E_r$  is not empty. Let  $E$  be a connected component of  $E_r$ . By Lemma 4.1, we know that  $E$  is invariant under  $\mathcal{E}(C(T_c))$ .

By Lemma 4.2, we know that  $\overline{E}$  is a separator. The annular completions  $K(\overline{E})$  of  $\overline{E}$  is also  $\mathcal{E}(C(T_c))$ -invariant since the definition is canonical. The following claim is at the heart of the entire construction.

**Claim 4.4.** *Let  $r_L$  and  $r_R$  be the left and right prime ends rotation numbers of  $f$  on  $K(\overline{E})$ . Then  $r_L = r_R = r$ .*

*Remark.* We refer the reader to [CM19, Claim 5.2] for the proof. The only property we use about  $\widetilde{PB}_n$  is Proposition 3.10.

**4.3. Finishing the proof.** We need to show a new property of a left invariant Dehn twist square  $h \in \widetilde{PB}_n$ .

**Theorem 4.5.** *The action of  $\mathcal{E}(T_b^2)$  on the set of prime ends of  $K(\overline{E})$  has rotation number 0.*

*Proof.* Now we consider the rotation set of  $\mathcal{E}(T_b^2)$  on  $K(\overline{E})$ . We claim that the rotation set satisfies

$$\sigma(\mathcal{E}(T_b^2), K(\overline{E})) = \{0\}.$$

The reason is that if not, then it is a nontrivial closed interval. By Theorem 2.9, rational rotation numbers are realized by periodic orbit. However  $K(\overline{E}) \subset B$ , that means every point for  $x \in K(\overline{E}) \subset B$ , there exists  $C(x) \in \mathbf{A}(T_b^2)$  such that  $C(x) \subset B$  by Lemma 4.1. However  $C(x)$  is acyclic and fixed by  $\mathcal{E}(T_b^2)$ . Therefore, we know that the rotation number of  $\mathcal{E}(T_b^2)$  on points in  $C(x)$  is zero, which is a contradiction. Then by Theorem 2.10, we know that the rotation number of the action of  $\mathcal{E}(T_b^2)$  on the set of prime ends is also zero.  $\square$

We now finish the proof.

*Proof.* Since the rotation number of  $\mathcal{E}(T_c)$  on the prime ends of  $K(\overline{E})$  is an irrational number  $r$ , then it is semiconjugate to an irrational rotation. Then up to the same semiconjugation, the image of the centralizer of  $T_c$  under  $\mathcal{E}$  is  $SO(2)$ . The image of each element is determined by its rotation number. However,  $\mathcal{E}(T_c)$  is a product of  $\mathcal{E}(T_b^2)$  for  $b$  nonseparating and invariant under  $\tau$  by Proposition 3.10. By Lemma 4.5, we know that the rotation number of  $\mathcal{E}(T_b^2)$  is zero. Thus their product should also have 0 rotation number. This contradicts the fact that the rotation number of  $\mathcal{E}(T_c)$  is  $r$ , which is nonzero.  $\square$

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